Mathematical Programming 1 (1971) 275–290. North-Holland Publishing Company

AN "OUT-OF-KILTER" ALGORITHM FOR SOLVING MINIMUM COST POTENTIAL PROBLEMS

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Received 15 October 1970
Revised manuscript received 13 March 1971

An algorithm is given for solving the optimum potential problem, which is the dual of the classical "out-of-kilter" algorithm for flow problems. Moreover, a new proof of finiteness is provided, which holds even for non-rational data; it applies to all the algorithms of network theory which include a labeling process.

1. Introduction

In 1961, D.R. Fulkerson published his "out-of-kilter" algorithm for constructing an optimum flow in a transportation network. To a great extent, this method may be regarded as the culmination of the theory of network flow; for, besides its own very convenient features – monotone process, possibility of starting with a non-feasible flow, or of altering some data during the computation – it synthesizes most of the already existing algorithms.

On the other hand, the dual problem, i.e. the optimum tension problem, seems to have been largely overlooked. For this problem, we know of no algorithm corresponding to "out-of-kilter", and the present paper aims at filling this gap.

The method we describe in the sequel is almost exactly the dual of the one given by Fulkerson: the only differences between them are the following:

(i) Slightly modifying the rules for altering the potential permits one to detect the absence of a finite optimum;

(ii) Introducing a total order on the set of arcs and referring to this order during the labeling process enables us to give a new proof for the finiteness of the algorithm, including the case in which the data are arbitrary real numbers.

2. Definitions and notations

Here $G = (X, U)$ will denote a graph with a set of vertices X and a set of arcs U. For sake of convenience, u_{xy} will represent an arc which has initial vertex x and terminal vertex y.

We assume that the reader is acquainted with the concepts of chain, path, cycle, and circuit.

Let μ be a cycle on which we have chosen an orientation: μ^+ (resp. μ^-) will be the set of the forward (resp. reverse) arcs of μ.

Let M be a subset of X, i.e. a set of vertices: by definition, the *cocycle* $\omega(M)$ is the set of those arcs which have one end in M and the other in $X - M$.

Again $\omega^+(M)$ (resp. $\omega^-(M)$) will denote the class of those arcs of $\omega(M)$ which go from M to $X - M$ (resp. from $X - M$ to M).

A *flow* is a function from U to the real line $\mathbb{R}$ which satisfies Kirchhoff's law for currents, i.e. which is conservative at each node.

If μ is an oriented cycle, φ^μ will be the unit flow carried by μ, whose components are:

$$\varphi^\mu_{xy} = \begin{cases} 1 & \text{if } u_{xy} \in \mu^+ \\ -1 & \text{if } u_{xy} \in \mu^- \\ 0 & \text{if } u_{xy} \notin \mu \,. \end{cases}$$

A *potential* is an arbitrary function from X to $\mathbb{R}$.

If M is a subset of X, π^M will be the unit potential carried by M, whose components are:

$$\pi^M_x = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{if } x \notin M \,. \end{cases}$$

A *tension* is a function from U to $\mathbb{R}$ which satisfies Kirchhoff's law for voltages, i.e. which is a potential difference.

If M is a subset of X, θ^M will be the unit tension carried by $\omega(M)$, whose components are:

$$\theta_{xy}^M = \begin{cases} -1 & \text{if } u_{xy} \in \omega^+(M) \\ 1 & \text{if } u_{xy} \in \omega^-(M) \\ 0 & \text{if } u_{xy} \notin \omega(M)\,. \end{cases}$$

Observe that θ^M derives from potential π^M.

If M reduces to a singleton $\{x\}$, we shall simplify the notation and write:

$$\omega_x, \omega_x^+, ..., \pi^x, \theta^x$$

instead of:

$$\omega(\{x\}), \omega^+(\{x\}), ..., \pi^{\{x\}}, \theta^{\{x\}}\,.$$

Sometimes, it will be convenient to give an expression like

$$\sum_{u_{xy} \in U} c_{xy}\, \theta_{xy}$$

the form $\langle c, \theta \rangle$ of a scalar product, in which case the (obvious) range of summation remains implicit.

Finally, let E be a subset of $\mathbb{R}$, and $\inf_E = \inf (x/x \in E)$: by convention, we shall take $\inf_\phi = +\infty$.

3. Statement of the problem

3.1. Let $G = (X, U)$ be a directed graph which is supposed to be finite, connected, and without loops or multiple arcs.

For each arc u_{xy} of G, three numbers are given:

$$t_{xy} \in \mathbb{R} \cup \{-\infty\} \tag{1}$$

$$T_{xy} \in \mathbb{R} \cup \{+\infty\} \tag{2}$$

$$c_{xy} \in \mathbb{R}. \tag{3}$$

The problem is to find in G some tension $\hat{\theta}$ which minimizes the total cost

$$\langle c, \hat{\theta} \rangle = \sum_{U} c_{xy} \hat{\theta}_{xy} \tag{4}$$

subject to the constraints

$$t_{xy} \leqslant \hat{\theta}_{xy} \leqslant T_{xy} \qquad (u_{xy} \in U). \tag{5}$$

If a feasible tension exists, this trivially implies:

$$t_{xy} \leqslant T_{xy} \qquad (u_{xy} \in U). \tag{6}$$

Henceforth, we suppose these conditions to be fulfilled, and we denote by Δ_{xy} the closed interval $[t_{xy}, T_{xy}]$ in which θ_{xy} must lie.

On the other hand, the assumption that G has no multiple arcs is by no means restrictive. In fact, let the nodes x and y be related by p distinct arcs $u_{xy}^1, u_{xy}^2, ..., u_{xy}^p$, all oriented from x to y; obviously, these arcs will all bear the same tension θ_{xy}. Now:

(i) Either $\bigcap_{k=1}^{p} \Delta_{xy}^k = \phi$: then G does not carry any feasible tension;

(ii) Or $\bigcap_{k=1}^{p} \Delta_{xy}^k \neq \phi$: then we can equally well replace this set of arcs by a single arc u_{xy} with prescribed interval $\Delta_{xy} = \bigcap_{k=1}^{p} \Delta_{xy}^k$ and cost $c_{xy} = \sum_{k=1}^{p} c_{xy}^k$.

A similar argument shows that the assumption we made about loops entails no loss of generality.

3.2. In order to state that θ is a tension, the most convenient way is to introduce as usual a potential $\pi = (\pi_x)_{x \in X}$ related to θ by:

$$\theta_{xy} = \pi_y - \pi_x \qquad (u_{xy} \in U). \tag{7}$$

Then we must accordingly transform the objective function:

$$\begin{aligned}\langle c, \theta \rangle &= \sum_{u_{xy} \in U} c_{xy} (\pi_y - \pi_x) \\ &= \sum_{x \in x} \left[\sum_{u_{yx} \in \omega_x^-} c_{yx} - \sum_{u_{xy} \in \omega_x^+} c_{xy} \right] \pi_x \\ &= \sum_{x \in X} k_x \pi_x = \langle k, \pi \rangle \end{aligned} \tag{8}$$

with:

$$k_x = \sum_{u_{yx} \in \omega_x^-} c_{yx} - \sum_{u_{xy} \in \omega_x^+} c_{xy} = \langle c, \theta^x \rangle . \tag{9}$$

Now the problem reads as follows:

$$\text{Min} \sum_X k_x \pi_x$$

subject to

$$\pi_y - \pi_x \geqslant t_{xy} \qquad (u_{xy} \in U)$$

$$\pi_x - \pi_y \geqslant -T_{xy} \qquad (u_{xy} \in U)$$

$$(\pi_x) \qquad (x \in X) .$$

4. Characterization of the optimal solutions

4.1. To characterize the optimal solutions of the problem, the simplest way is probably to introduce the dual problem and the complementary slackness conditions. We thus obtain:

$$[\text{Min} \sum_X k_x \pi_x] \qquad\qquad [\text{Max} \sum_U (r_{xy} t_{xy} - s_{xy} T_{xy})]$$

$$(S) \qquad \begin{array}{c|c|l} \theta_{xy} = \pi_y - \pi_x \geqslant t_{xy} & u_{xy} \in U & r_{xy} \geqslant 0 \\ -\theta_{xy} = \pi_x - \pi_y \geqslant -T_{xy} & u_{xy} \in U & s_{xy} \geqslant 0 \\ (\pi_x) & x \in X & \langle r - s, \theta^x \rangle = k_x \end{array}$$

$$(\theta_{xy} - t_{xy}) r_{xy} = 0 \qquad (u_{xy} \in U)$$

$$(T_{xy} - \theta_{xy}) s_{xy} = 0 \qquad (u_{xy} \in U)\,.$$

It is a well-known fact that the potential π will be optimal if and only if we can find r and s such that π, r and s satisfy the system (S).

4.2. Since flow and tension are dual concepts, we may expect that this system can be transformed so as to express the dual constraints and the complementary slackness conditions in terms of flows. The system (S) is in fact equivalent to the following system (S'):

$$(S') \qquad \begin{array}{c|c|l} \theta_{xy} = \pi_y - \pi_x \geqslant t_{xy} & u_{xy} \in U & (\varphi_{xy}) \\ -\theta_{xy} = \pi_x - \pi_y \geqslant -T_{xy} & u_{xy} \in U & \langle \varphi, \theta^x \rangle = 0 \\ (\pi_x) & x \in X & \end{array}$$

$$\bar{c}_{xy} = c_{xy} + \varphi_{xy} \qquad (u_{xy} \in U)$$

$$(\theta_{xy} - t_{xy}) \bar{c}_{xy} \leqslant 0 \qquad (u_{xy} \in U)$$

$$(T_{xy} - \theta_{xy}) \bar{c}_{xy} \geqslant 0 \qquad (u_{xy} \in U)\,.$$

The relations:

$$\langle \varphi, \theta^x \rangle = 0 \qquad (x \in X) \qquad (10)$$

mean that φ is indeed a flow in G.

On the other hand, the equivalence between (S) and (S') can be proved as follows:

Proposition 1

If (π, r, s) is a solution of (S), then (π, φ) with $\varphi = r - s - c$ is a solution of (S')

Proof.

(i) $\langle \varphi, \theta^x \rangle = \langle r - s, \theta^x \rangle - \langle c, \theta^x \rangle = k_x - k_x = 0$

(ii) $(\theta_{xy} - t_{xy})\, \bar{c}_{xy} = (\theta_{xy} - t_{xy})(r_{xy} - s_{xy}) = -(\theta_{xy} - t_{xy})\, s_{xy} \leqslant 0$

(iii) $(T_{xy} - \theta_{xy})\, \bar{c}_{xy} = (T_{xy} - \theta_{xy})(r_{xy} - s_{xy}) = (T_{xy} - \theta_{xy})\, r_{xy} \geqslant 0$

Proposition 2

If (π, φ) is a solution of (S'), then (π, r, s) with $r = \sup(\bar{c}, 0)$ and $s = \sup(-\bar{c}, 0)$ is a solution of (S)

Proof.

(i) $r_{xy} \geqslant 0$ and $s_{xy} \geqslant 0$

(ii) $\langle r - s, \theta^x \rangle = \langle r, \theta^x \rangle - \langle s, \theta^x \rangle$

$$\begin{aligned} &= \sup(\langle \bar{c}, \theta^x \rangle, 0) - \sup(-\langle \bar{c}, \theta^x \rangle, 0) \\ &= \sup(k_x, 0) - \sup(-k_x, 0) \\ &= k_x + 0 = k_x \end{aligned}$$

(iii) $(\theta_{xy} - t_{xy})\, r_{xy} = \sup((\theta_{xy} - t_{xy})\, \bar{c}_{xy}, 0) = 0$

(iv) $(T_{xy} - \theta_{xy})\, s_{xy} = \sup(-(T_{xy} - \theta_{xy})\, \bar{c}_{xy}, 0) = 0$.

Finally, it is to be observed that for any tension θ we have

$$\langle \bar{c}, \theta \rangle = \langle c, \theta \rangle + \langle \varphi, \theta \rangle = \langle c, \theta \rangle \tag{11}$$

since φ and θ are orthogonal.

4.3. Assume that the potential π and the flow φ are given. Then any arc u_{xy} of G lies in one of the mutually exclusive nine states which are listed and depicted below:

$$\begin{array}{lll} \alpha & : \bar{c}_{xy} > 0 & \text{and } \theta_{xy} = t_{xy} \\ \alpha_1 & : \bar{c}_{xy} > 0 & \text{and } \theta_{xy} < t_{xy} \\ \alpha_2 & : \bar{c}_{xy} > 0 & \text{and } \theta_{xy} > t_{xy} \\ \beta & : \bar{c}_{xy} = 0 & \text{and } t_{xy} \leqslant \theta_{xy} \leqslant T_{xy} \\ \beta_1 & : \bar{c}_{xy} = 0 & \text{and } \theta_{xy} < t_{xy} \\ \beta_2 & : \bar{c}_{xy} = 0 & \text{and } \theta_{xy} > T_{xy} \\ \gamma & : \bar{c}_{xy} < 0 & \text{and } \theta_{xy} = T_{xy} \\ \gamma_1 & : \bar{c}_{xy} < 0 & \text{and } \theta_{xy} < T_{xy} \\ \gamma_2 & : \bar{c}_{xy} < 0 & \text{and } \theta_{xy} > T_{xy} . \end{array}$$

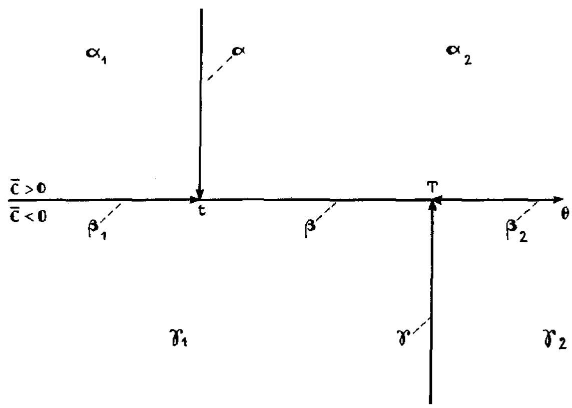

Fig. 1.

The arc u_{xy} will be said "almost-in-kilter" if the tension θ satisfies: $t_{xy} \leqslant \theta_{xy} \leqslant T_{xy}$

It will be said "in-kilter" if it lies in states α, β, or γ, and "out-of-kilter" otherwise.

According to the preceding section, the tension θ will be optimal if and only if we can find a flow φ such that all the arcs of G are "in-kilter".

5. Statement of the algorithm

(A). *Initialization*

a. Select an arbitrary starting potential, for instance $\pi = 0$

b. Select an arbitrary starting flow, for instance $\varphi = 0$, and take

$$\bar{c} = c + \varphi$$

c. Define an arbitrary total order **O** on the set U of the arcs of G.

(B). *Choice of the target-arc*

a. a_1 If every arc is "in-kilter", the present tension is optimal: STOP

a_2 Otherwise, let u_{pq} be the **O**-smallest arc which is "out-of-kilter"

(C). *Labeling process*

At all times, let M be the set of the labeled nodes, and z the labeled node from which one is trying to extend the labeling.

The arc u_{xy} is said to be "labeling" if it lies in one of the following four states:

$$
\begin{array}{lll}
(1')\ x = z \in M, y \notin M, \bar{c}_{xy} \geqslant 0 & \text{and} & \theta_{xy} \leqslant t_{xy} \\
(1'')\ x = z \in M, y \notin M, \bar{c}_{xy} < 0 & \text{and} & \theta_{xy} \leqslant T_{xy} \\
(2')\ x \notin M, y = z \in M, \bar{c}_{xy} \leqslant 0 & \text{and} & \theta_{xy} \geqslant T_{xy} \\
(2'')\ x \notin M, y = z \in M, \bar{c}_{xy} > 0 & \text{and} & \theta_{xy} \geqslant t_{xy}\ .
\end{array} \tag{12}
$$

a. a_1 If u_{pq} lies in state α_1, β_1 or γ_1, give q the label (−), take $z = q$ and go to b

a_2 If u_{pq} lies in state α_2, β_2 or γ_2, give p the label (−), take $z = p$ and go to b

b. b_1 If the star-cocycle ω_z contains labeling arcs, let u_{zy} or u_{yz} be the $\mathbb{O}$ -smallest of them: give y the label (z), take $z = y$ and go to c

b_2 Otherwise, go to d

c. c_1 If z is that end of u_{pq} which has not been labeled at step a, go to (E).

c_2 Otherwise, go to b

d. d_1 If z has the label (−), go to (D).

d_2 If z has the label (x), take $z = x$ and go to b.

(D). *Non-breakthrough and change of potential*

a. Define

$$
\begin{array}{lll}
P'_+ & = & \{u_{xy} / u_{xy} \in \omega^+(M) \ \text{and}\ \theta_{xy} \leqslant T_{xy}\} \\
P''_+ & = & \{u_{xy} / u_{xy} \in \omega^+(M) \ \text{and}\ \theta_{xy} > T_{xy}\} \\
P'_- & = & \{u_{xy} / u_{xy} \in \omega^-(M) \ \text{and}\ \theta_{xy} \geqslant t_{xy}\} \\
P''_- & = & \{u_{xy} / u_{xy} \in \omega^-(M) \ \text{and}\ \theta_{xy} < t_{xy}\}
\end{array} \tag{13}
$$

b. Compute

$$
\begin{aligned}
\delta'_+ &= \inf\,(\theta_{xy} - t_{xy}\,/u_{xy} \in P'_+)\\
\delta''_+ &= \inf\,(\theta_{xy} - T_{xy}\,/u_{xy} \in P''_+)\\
\delta'_- &= \inf\,(T_{xy} - \theta_{xy}\,/u_{xy} \in P'_-)\\
\delta''_- &= \inf\,(t_{xy} - \theta_{xy}\,/u_{xy} \in P''_-)\\
\delta &= \inf\,(\delta'_+, \delta''_+, \delta'_-, \delta''_-)
\end{aligned}
\tag{14}
$$

c. c_1 If $\delta = +\infty$, then the network does not carry any finite optimal tension: STOP

c_2 If $\delta < +\infty$, then add $\delta\pi^M$ to the potential π, $\delta\theta^M$ to the tension θ, erase the labels and go to (B).

(E). *Breakthrough and change of flow*

Let $\mu = \mu^+ \cup \mu^-$ be the cycle we found at step (C), oriented according to the chronological order of the labeling process.

a. Define

$$
\begin{aligned}
F_+ &= \{u_{xy}\,/u_{xy} \in \mu^+,\ t_{xy} \neq T_{xy}, \bar{c}_{xy} < 0 \ \text{ and } \ \theta_{xy} \geqslant t_{xy}\}\\
F_- &= \{u_{xy}\,/u_{xy} \in \mu^-,\ t_{xy} \neq T_{xy}, \bar{c}_{xy} > 0 \ \text{ and } \ \theta_{xy} \leqslant T_{xy}\}
\end{aligned}
\tag{15}
$$

b. Compute

$$
\begin{aligned}
\epsilon_+ &= \inf\,(-\bar{c}_{xy}\,/u_{xy} \in F_+)\\
\epsilon_- &= \inf\,(\ \bar{c}_{xy}\,/u_{xy} \in F_-)\\
\epsilon &= \inf\,(\epsilon_+, \epsilon_-)
\end{aligned}
\tag{16}
$$

c. c_1 If $\epsilon = +\infty$, then the network does not carry any feasible tension: STOP

c_2 If $\epsilon < +\infty$, then add $\epsilon\varphi^\mu$ to the flow φ and to the cost $\bar{c}$, erase the labels and go to (B).

6. Proof of the validity of the algorithm

Proposition 3

If a non-breakthrough occurs with $\delta = +\infty$, then the network does not carry any finite optimal tension

Proof

The assumptions (1) and (2), the labeling rules (12) and the rules (13) and (14) for changing the potential imply:

$$u_{xy} \in \omega^+(M) \Rightarrow [-\infty = t_{xy} < \theta_{xy} \leqslant T_{xy} \quad \text{and} \quad \bar{c}_{xy} \geqslant 0]$$
$$u_{xy} \in \omega^-(M) \Rightarrow [t_{xy} \leqslant \theta_{xy} < T_{xy} = +\infty \quad \text{and} \quad \bar{c}_{xy} \leqslant 0]. \tag{17}$$

Moreover, the cocycle $\omega(M)$ contains the target-arc u_{pq}, which is "out-of-kilter"; hence $\bar{c}_{pq} \neq 0$.

It follows that the unit tension θ^M carried by $\omega(M)$ satisfies:

$$\langle \bar{c}, \theta^M \rangle < 0. \tag{18}$$

Now let $\hat{\theta}$ be a finite optimal tension. From (17) and (18), it is clear that the tension $\theta' = \hat{\theta} + \theta^M$ is feasible and has a cost $\langle c, \theta' \rangle$ strictly less than the cost $\langle \bar{c}, \hat{\theta} \rangle$ of $\hat{\theta}$: this contradiction establishes the above result.

Proposition 4

If a breakthrough occurs with $\epsilon = +\infty$, then the network does not carry any feasible tension

Proof

The labeling rules (12) and the rules (15) and (16) for changing the flow imply:

$$u_{xy} \in \mu^+ \Rightarrow [[\theta_{xy} < t_{xy}] \text{ or } [\theta_{xy} = t_{xy} \text{ and } \bar{c}_{xy} \geqslant 0]]$$
$$u_{xy} \in \mu^- \Rightarrow [[\theta_{xy} > T_{xy}] \text{ or } [\theta_{xy} = T_{xy} \text{ and } \bar{c}_{xy} \leqslant 0]]. \tag{19}$$

Since θ is a tension, we obtain by adding these inequalities:

$$\sum_{\mu^-} T_{xy} \leqslant \sum_{\mu^-} \theta_{xy} = \sum_{\mu^+} \theta_{xy} \leqslant \sum_{\mu^+} t_{xy}. \tag{20}$$

Moreover, the cycle μ contains the target-arc u_{pq}, which is "out-of-kilter": hence at least one of the above two inequalities is strict. We have thus found a cycle μ for which

$$\sum_{\mu^-}' T_{xy} < \sum_{\mu^+} t_{xy} \tag{21}$$

and the necessary condition for the existence of a feasible tension (Ghouila-Houri, [3]) is not fulfilled.

Now we must prove that the proposed algorithm always terminates in a finite number of steps.

Lemma

Any arc which is "in-kilter" (resp: "almost-in-kilter") at some stage of the computation remains "in-kilter" (resp: "almost-in-kilter") during the subsequent steps.

Proof

(Obvious)

Proposition 5

An infinite sequence of consecutive non-breakthroughs cannot occur.

Proof

Let U_δ be the set of those arcs which give the infimum δ; three mutually exclusive cases are to be considered:

(i) We have $\delta = +\infty$
Then the algorithm terminates;

(ii) An arc belonging to U_δ becomes "in-kilter" or "almost-in-kilter"
According to the lemma, this case can occur only a finite number of times since G is a finite network;

(iii) Each arc of U_δ was (and remains) "in-kilter"
Then we have

$$\begin{aligned} u_{xy} \in U_\delta \cap \omega^+(M) &\Rightarrow [\theta_{xy} = t_{xy} + \delta \text{ and } \bar{c}_{xy} = 0] \\ u_{xy} \in U_\delta \cap \omega^-(M) &\Rightarrow [\theta_{xy} = T_{xy} - \delta \text{ and } \bar{c}_{xy} = 0] \,. \end{aligned} \tag{22}$$

Now let u_{vw} be the first arc of U_δ encountered during the labeling process: according to the labeling rules (12) and to (22), u_{vw}, which was non-labeling, will become labeling for the next process. On the other hand, each labeling arc will remain labeling, since the corresponding components of θ are not altered by the change of potential.

The new set of labeled nodes will thus strictly include the old one, and, again because of the finiteness of G, this completes the proof.

Proposition 6

An infinite sequence of consecutive breakthroughs cannot occur.

Proof

Let U_ϵ be the set of those arcs which give the infimum ϵ; three mutually exclusive cases are to be considered:

(i) We have $\epsilon = +\infty$
Then the algorithm terminates;

(ii) An arc belonging to U_ϵ becomes "in-kilter"
According to the lemma, this case can occur only a finite number of times since G is a finite network;

(iii) Each arc of U_ϵ was (and remains) "in-kilter"
Then we have:

$$\begin{aligned} u_{xy} \in U_\epsilon \cap \mu^+ &\Rightarrow [t_{xy} < \theta_{xy} = T_{xy} \text{ and } \bar{c}_{xy} = -\epsilon] \\ u_{xy} \in U_\epsilon \cap \mu^- &\Rightarrow [t_{xy} = \theta_{xy} < T_{xy} \text{ and } \bar{c}_{xy} = \epsilon]\,. \end{aligned} \tag{23}$$

Now let u_{vw} be the first arc of U_ϵ encountered during the labeling process: according to the labeling rules (12) and to (23), u_{vw}, which was labeling, will become non-labeling for the next process. On the other hand, each non-labeling arc will remain non-labeling, since the corresponding components of φ are not altered by the change of flow.

If the next labeling process leads again to a breakthrough, the cycle μ' thus found will be strictly greater than μ in the lexicographical order induced by the total order $\mathbb{O}$: this completes the proof since G has but a finite number of elementary cycles.

The above two propositions are not sufficient to ensure the finiteness of the algorithm. We still have to show that an infinite sequence, consisting of a finite number of breakthroughs, followed by a finite number of non-breakthroughs, itself followed by a finite number of breakthroughs, and so on ... cannot occur. In order to study this case, it will be convenient to introduce some auxiliary networks, which are quite similar to the so-called "incremental network" used by several authors for network flow problems.

More precisely, for each target-arc u_{pq} and each flow φ, we shall construct a network $G'_{pq}(\varphi)$ defined as follows:
(i) $G'_{pq}(\varphi)$ has the same set of vertices X as G
(ii) Let θ^o be the tension carried by G when u_{pq} has been selected for the first time, and let u_{xy} be an arc of G: u_{xy} induces in $G'_{pq}(\varphi)$ two arcs, u'_{xy} from x to y and u''_{yx} from y to x, whose respective lengths l'_{xy} and l''_{yx} are to be read in the table below:

If $\bar{c}_{xy} > 0$ and $\theta^o_{xy} = t_{xy}$	then $l'_{xy} = 0$	, $l''_{yx} = 0$
If $\bar{c}_{xy} > 0$ and $\theta^o_{xy} < t_{xy}$	then $l'_{xy} = t_{xy} - \theta^o_{xy}$	, $l''_{yx} = 0$
If $\bar{c}_{xy} > 0$ and $\theta^o_{xy} > t_{xy}$	then $l'_{xy} = 0$	, $l''_{yx} = \theta^o_{xy} - t_{xy}$
If $\bar{c}_{xy} = 0$ and $t_{xy} \leqslant \theta^o_{xy} \leqslant T_{xy}$	then $l'_{xy} = T_{xy} - \theta^o_{xy}$	, $l''_{yx} = \theta^o_{xy} - t_{xy}$
If $\bar{c}_{xy} = 0$ and $\theta^o_{xy} < t_{xy}$	then $l'_{xy} = T_{xy} - \theta^o_{xy}$	, $l''_{yx} = 0$
If $\bar{c}_{xy} = 0$ and $\theta^o_{xy} > T_{xy}$	then $l'_{xy} = 0$	, $l''_{yx} = \theta^o_{xy} - t_{xy}$
If $\bar{c}_{xy} < 0$ and $\theta^o_{xy} = T_{xy}$	then $l'_{xy} = 0$	, $l''_{yx} = 0$
If $\bar{c}_{xy} < 0$ and $\theta^o_{xy} < T_{xy}$	then $l'_{xy} = T_{xy} - \theta^o_{xy}$	, $l''_{yx} = 0$
If $\bar{c}_{xy} < 0$ and $\theta^o_{xy} > T_{xy}$	then $l'_{xy} = 0$	, $l''_{yx} = \theta^o_{xy} - T_{xy}$.

These networks could be used to compute separately the alterations to be made on θ and φ. Nevertheless, we introduce them only to make the proof of the next Proposition easier.

Now let us say that the network G is "partially saturated" when a breakthrough is about to follow a non-breakthrough with the same target-arc.

Proposition 7

The number of partial saturations for a given target-arc is necessarily finite.

Proof

Let u_{pq} be the given target-arc, θ^o the tension in G when u_{pq} has just been selected for the first time, θ^i and φ^i the tension and the flow at the i-th partial saturation S^i relative to u_{pq}, and let us assume for instance that the state of u_{pq} is such that one tries to increase θ_{pq}.

At the i-th partial saturation S^i, the increase $(\theta^i_{pq} - \theta^o_{pq})$ given to θ_{pq} is clearly equal to the length λ^i_{pq} of a shortest path going from p to q in the network $G'_{pq}(\varphi^i)$.

Now let i and j be two integers with $j > i$. The partial saturations S^i and S^j are separated by at least one non-breakthrough: hence $\lambda^i_{pq} < \lambda^j_{pq}$ and consequently $G'_{pq}(\varphi^i) \neq G'_{pq}(\varphi^j)$. But the number of auxiliary networks associated with a given target-arc is obviously finite, and this completes the proof.

Finally, we may state the following:

Theorem

The "out-of-kilter" algorithm for tension problems terminates in a finite number of steps.

Proof

This theorem is a straightforward consequence of propositions 3 through 7.

We may notice that the above arguments do not make use of "slack numbers" as in the original paper by Fulkerson [2].

Nevertheless, if the data are integers or rational numbers, introducing such slack numbers permits one to simplify rather considerably several proofs, especially when the starting tension is assumed to be feasible.

7. The lexicographical labeling: a general tool in network theory

Using a lexicographical labeling process is the major difference between our algorithm and the classical "out-of-kilter" given in [2] and [1]. Only this new device enables us to prove the finiteness of the procedure even if the data are arbitrary real numbers. But this rather simple trick can be introduced with the same success into all other algorithms which solve network problems by means of a labeling process, in particular into the original "out-of-kilter". Thus, for instance, the counter-example given on p. 21 of [1] does not hold for a lexicographical max-flow algorithm.

In fact, this new result is not surprising: for it is well known that the simplex algorithm, at least in its lexicographical form, always terminates in a finite number of steps even if the data are not rational numbers. It is intellectually satisfying to verify that a similar method applied to a simpler case is not less powerful.

Moreover, it must be observed that this improvement is not at all expensive, for the programmer who writes down a code which contains a labeling process must make a choice and give the computer a nonambiguous list of instructions: but the lexicographical method is precisely one of the most natural ways to perform this process. That is why we believe that most of the existing codes are – unintentionally – of the lexicographical type.

References

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