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OPTIMAL SCALING OF BALLS AND POLYHEDRA*

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The concern is with solving as linear or convex quadratic programs special cases of the optimal containment and meet problems. The optimal containment or meet problem is that of finding the smallest scale of a set for which some translation contains a set or meets each element in a collection of sets, respectively. These sets are unions or intersections of cells where a cell is either a closed polyhedral convex set or a closed solid ball.

Key words: Linear Programming, Quadratic Programming, Optimal Scaling, Cells, Balls, Polyhedral, Meet, Containment.

1. Introduction

By a cell we mean either a nonempty closed polyhedral convex set (not necessarily bounded or solid) or a nonempty closed solid ball. Our concern is with solving as linear or convex quadratic programs special cases of the following two problems.

Optimal containment problem (OCP). Let $\mathscr{X}$ be a finite union of cells and $\mathscr{Y}$ a finite intersection of cells. Find the smallest positive scale $s\mathscr{Y}$ of $\mathscr{Y}$ for which some translate $s\mathscr{Y}+t$ contains $\mathscr{X}$.

$$\text{OCP}\begin{cases}\underset{s,t}{\text{infimum}} & s,\\ \text{subject to} & s\mathscr{Y}+t\supseteq\mathscr{X},\quad s>0.\end{cases}$$

Optimal meet problem (OMP). Let $\mathscr{X}$ and $\mathscr{Y}_q$ for $q=1,\dots,p$, be each an intersection of cells. Find the smallest positive scale $s\mathscr{X}$ of $\mathscr{X}$ for which some translate $s\mathscr{X}+t$ meets every $\mathscr{Y}_q$ for $q=1,\dots,p$.

$$\text{OMP}\begin{cases}\underset{s,t}{\text{infimum}} & s,\\ \text{subject to} & (s\mathscr{X}+t)\cap\mathscr{Y}_q\neq\emptyset,\quad q=1,\dots,p,\quad s>0.\end{cases}$$

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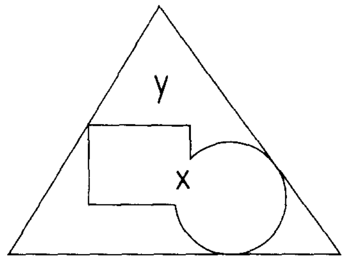

Fig. 1. OCP.

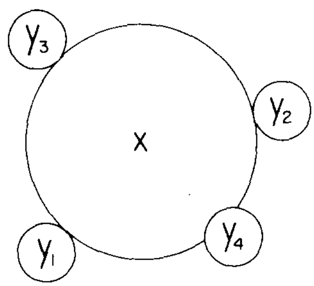

Fig. 2. OMP.

The general OCP and OMP are well beyond our reach but serve as useful overviews. Depending upon the composition of the $\mathscr{X}$'s and $\mathscr{Y}$'s as unions and intersections of cells and the representation of the cells we can or cannot formulate the problems as linear or convex quadratic programs.

Our initial interest in OCP and OMP originated with 'engineering design' through interior solution concepts for convex sets, see van der Vet [9] and Director and Hachtel [1]; also see Eaves and Freund [2].

2. Preliminaries

Most of the notation we use is standard. Let $\mathbb{R}^n$ be n-dimensional Euclidean space. By $\|\cdot\|$ we mean the Euclidean norm. Let $x \cdot y$ and $x \circ y$ represent inner and outer products, respectively. Let $e = (1, 1, \ldots, 1)$ where its length is dictated by context. For the empty set, $\emptyset$, define $\inf \emptyset = +\infty$ as usual, but define $\sup \emptyset = 0$ for the purposes of this presentation. By a convex program we mean a program of the form

$$\text{P}\begin{cases} \underset{x}{\text{minimum}} & f(x) \\ \text{subject to} & g_i(x) \le 0, \quad i = 1, \ldots, m, \end{cases}$$

where all f, g_i are convex functions, and m is finite. If, in addition, each g_i is affine, $f(x) = xQx + q \cdot x$, and Q is positive semi-definite, then we call P a quadratic program. Furthermore, if Q is zero, then we call P a linear program.

Let $\mathscr{X}$ be a set in $\mathbb{R}^n$. We denote by $\operatorname{tng}(\mathscr{X})$ the smallest vector subspace of $\mathbb{R}^n$ for which some translate contains $\mathscr{X}$. We denote by $\operatorname{rec}(\mathscr{X})$ the recession set of $\mathscr{X}$, that is, the set

$$\{z \in \mathbb{R}^n \mid \exists\, x \in \mathscr{X} \text{ such that } x + \alpha z \in \mathscr{X} \text{ for any } \alpha \geq 0\}.$$

We also make use of the following variation of Farkas' lemma.

Lemma. *Suppose that the system of inequalities*

$$Ax \leq b$$

has a solution and that every solution satisfies $cx \leq d$. *Then there exists* $\lambda \geq 0$ *such that* $\lambda A = c$ *and* $\lambda b \leq d$.

The manner in which cells are represented is crucial to our formulations. We assume all cells are in $\mathbb{R}^n$. We define an H-cell to be a cell of the form

$$\{x \mid Ax \leq b\}$$

as it is represented by half-spaces; it is assumed (A, b) is given. A cell of the form

$$\{x \mid x = U\lambda + V\mu \colon e\lambda = 1, \lambda \geq 0, \mu \geq 0\}$$

is defined to be a W-cell as it is a weighting of points; it is assumed (U, V) is given. Of course, every H-cell can be represented as a W-cell and vice versa. However, we shall suppose, and typically rightly so, that the computational burden of the conversion is prohibitive, see Mattheiss and Rubin [8] for H to W. Thus we shall regard H- and W-cells as quite distinct. A B-cell is defined to be the ball

$$\{x \mid \|c - x\| \leq r\};$$

it is assumed that the center c and radius $r \geq 0$ are given.

Let $\mathscr{X}$ be a cell, and let $s > 0$ be a scale of $\mathscr{X}$ and t a translate. If $\mathscr{X}$ is an H-cell, $\{x \mid Ax \leq b\}$, then $s\mathscr{X} + t = \{x \mid Ax \leq bs + At\}$. If $\mathscr{X}$ is a B-cell, $\{x \mid \|c - x\| \leq r\}$, $s\mathscr{X} + t = \{x \mid \|(sc + t) - x\| \leq sr\}$. And if $\mathscr{X}$ is a W-cell, $\{x \mid x = U\lambda + V\mu,\ e\lambda = 1,\ \lambda \geq 0,\ \mu \geq 0\}$, then

$$s\mathscr{X} + t = \{x \mid x = (sU + t \circ e)\lambda + V\mu,\ e\lambda = 1,\ \lambda \geq 0,\ \mu \geq 0\}$$

or equivalently

$$s\mathscr{X} + t = \{x \mid x = t + U\lambda + V\mu,\ e\lambda = s,\ \lambda \geq 0,\ \mu \geq 0\}.$$

To describe special cases of OCP and OMP we shall use notation as, for example, $(\mathrm{HB}_1, \mathrm{WB}_=)$ which denotes that $\mathscr{X}$ is composed of any finite number of

H-cells and one B-cell, and that $\mathcal{Y}$ or each $\mathcal{Y}_q$ are composed of any number of W-cells and B-cells but all B-cells have the same radius. If B is not subscripted by ' = ' or '1', then any finite number may be employed and the radii may vary. Thus, again, for example, (HWB, HWB) describes the most general case of OMP or OCP.

Consider the following three programs wherein the vectors w_i are given and fixed.

$$(\mathrm{Q1})\begin{cases} v_1 = \underset{s,\,t}{\text{infimum}} & s, \\ \text{subject to} & \|w_i - (sc+t)\| + d \le sr, \quad i = 1, \dots, m. \end{cases}$$

$$(\mathrm{Q2})\begin{cases} v_2 = \underset{f,\,x}{\text{infimum}} & f, \\ \text{subject to} & \|w_i - x\|^2 \le f, \quad i = 1, \dots, m. \end{cases}$$

$$(\mathrm{Q3})\begin{cases} v_3 = \underset{a,\,x}{\text{infimum}} & x \cdot x - a, \\ \text{subject to} & w_i \cdot w_i - 2w_i \cdot x + a \le 0, \quad i = 1, \dots, m. \end{cases}$$

Assuming r is non-zero, we show that solving any one yields solutions to the other two. Let $\sqrt{\cdot}$ denote nonnegative square root.

Equivalence of (*Q*1) *and* (*Q*2): If $(\bar{s}, \bar{t})$ and (s, t) are feasible for (Q1) with $\bar{s} < s$ (respectively: $\bar{s} \le s$), then $(\bar{f}, \bar{x}) = ((\bar{s}r - d)^2, \bar{s}c + \bar{t})$ and $(f, x) = ((sr - d)^2, sc + t)$ are feasible for (Q2) with $\bar{f} < f$ (respectively: $\bar{f} \le f$). If $(\bar{f}, \bar{x})$ and (f, x) are feasible for (Q2) with $\bar{f} < f$ (respectively: $\bar{f} \le f$), then $(\bar{s}, \bar{t}) = ((\sqrt{\bar{f}} + d)/r, \bar{x} - \bar{s}c)$ and $(s, t) = ((\sqrt{f} + d)/r, x - sc)$ are feasible for (Q1) with $\bar{s} < s$ (respectively: $\bar{s} \le s$).

Equivalence of (Q2) *and* (Q3): If $(\bar{f}, \bar{x})$ and (f, x) are feasible for (Q2) with $\bar{f} < f$ (respectively: $\bar{f} \le f$), then $(\bar{a}, \bar{x}) = (\bar{x} \cdot \bar{x} - f, \bar{x})$ and $(a, x) = (x \cdot x - f, x)$ are feasible for (Q3) with $\bar{x} \cdot \bar{x} - \bar{a} = \bar{f} < x \cdot x - a = f$ (respectively: $\bar{x} \cdot \bar{x} - \bar{a} = \bar{f} \le x \cdot x - a = f$). If $(\bar{a}, \bar{x})$ and (a, x) are feasible for (Q3) with $\bar{x} \cdot \bar{x} - \bar{a} < x \cdot x - a$ (respectively: $\bar{x} \cdot \bar{x} - \bar{a} \le x \cdot x - a$), then $(\bar{f}, \bar{x}) = (\bar{x} \cdot \bar{x} - \bar{a}, \bar{x})$ and $(f, x) = (x \cdot x - a, x)$ are feasible for (Q2) with $\bar{f} = \bar{x} \cdot \bar{x} - \bar{a} < f = x \cdot x - a$ (respectively: $\bar{f} = \bar{x} \cdot \bar{x} - \bar{a} \le f = x \cdot x - a$).

We thus have the following result which shall be used in our study of OCP and OMP.

Lemma 2.1 (Equivalence of (Q1) and (Q3)). *For r non-zero*

(i) *If* (s, t) *is feasible or optimal for* (Q1), *then*

$$(a, x) = ((sc+t) \cdot (sc+t) - (sr-d)^2, sc+t)$$

is feasible or optimal for (Q3), *respectively.*

(ii) *If* (a, x) *is feasible or optimal for* (Q3), *then*

$$(s, t) = ((d + \sqrt{x \cdot x - a})/r,\ x - (d + \sqrt{x \cdot x - a})c/r)$$

is feasible or optimal for (Q1), *respectively.*

Consequently (Q1) can be solved via the quadratic program (Q3). Note that (Q3), and hence (Q1), always has a unique optimal solution.

3. The optimal containment problem (OCP)

Let $\mathscr{X}$ be a finite union of cells and $\mathscr{Y}$ be a finite intersection of cells. The optimal containment problem can be written as:

$$\text{OCP1}\begin{cases} z_1 = \underset{s,t}{\text{supremum}} & s, \\ \text{subject to} & s\mathscr{X} + t \subseteq \mathscr{Y}, \quad s > 0 \end{cases}$$

or as

$$\text{OCP2}\begin{cases} z_2 = \underset{s,t}{\text{infimum}} & s, \\ \text{subject to} & \mathscr{X} \subseteq s\mathscr{Y} + t, \quad s > 0. \end{cases}$$

Our first results concern the equivalence of OCP1 and OCP2.

Lemma 3.1 (Equivalence of OCP1 and OCP2).

(i) (Solutions) (s, t) *is a feasible or optimal solution to* OCP1 *if and only if* $(1/s, -t/s)$ *is a feasible or optimal solution to* OCP2.

(ii) (Feasibility) *The following are equivalent*:

(a) OCP1 *is feasible*;

(b) OCP2 *is feasible*;

(c) $\text{tng}(\mathscr{X}) \subseteq \text{tng}(\mathscr{Y})$ *and* $\text{rec}(\mathscr{X}) \subseteq \text{rec}(\mathscr{Y})$.

(iii) (Attainment) *The following are equivalent*:

(a) $0 < z_1 < +\infty$;

(b) $0 < z_2 < +\infty$;

(c) OCP1 *has an optimum*;

(d) OCP2 *has an optimum.*

(iv) (Non-attainment) *The following are equivalent*:

(a) $z_1 = +\infty$;

(b) $z_2 = 0$;

(c) $\mathscr{X} + t \subseteq \text{rec}(\mathscr{Y})$ *for some translate* t.

For a specific realization of OCP1 or OCP2, $\mathscr{X}$ and $\mathscr{Y}$ will be given in the forms

$$\mathscr{X} = \Big[\bigcup_{h\in\alpha} \mathrm{H}_h\Big] \cup \Big[\bigcup_{i\in\beta} \mathrm{W}_i\Big] \cup \Big[\bigcup_{j\in\gamma} \mathrm{B}_j\Big]$$

$$\mathscr{Y} = \Big[\bigcap_{k\in\rho} \mathrm{H}_k\Big] \cap \Big[\bigcap_{l\in\sigma} \mathrm{W}_l\Big] \cap \Big[\bigcap_{m\in\tau} \mathrm{B}_m\Big],$$

where $\mathrm{H}_{(\cdot)} = \{x \mid A_{(\cdot)}x \le b_{(\cdot)}\}$, $\mathrm{W}_{(\cdot)} = \{x \mid x = U_{(\cdot)}\lambda + V_{(\cdot)}\mu,\ e\lambda = 1,\ \lambda \ge 0,\ \mu \ge 0\}$, and $\mathrm{B}_{(\cdot)} = \{x \mid \|c_{(\cdot)} - x\| \le r_{(\cdot)}\}$, For a given set $\mathrm{H}_{(\cdot)} = \{x \mid A_{(\cdot)}x \le b_{(\cdot)}\}$, we define $a_{(\cdot)}$ to be the column vector whose qth component is the (Euclidean) norm of the qth row of $A_{(\cdot)}$.

We begin with case (HWB, H) of OCP which corresponds to $\sigma = \emptyset$ and $\tau = \emptyset$, that is, $\mathscr{X}$ is a union of any finite number of H-, W-, and B-cells and $\mathscr{Y}$ is an intersection of H-cells.

Case (HWB, H) *of* OCP *is a linear program*

We treat the optimal containment problem (HWB, H) through OCP2. The formulation as a linear program is

$$\begin{aligned} z_2 = \underset{s,\,t,\,\Lambda}{\text{minimum}} \quad & s, \\ \text{subject to} \quad & \Lambda_{hk}A_h = A_k, && h \in \alpha, k \in \rho, \\ & \Lambda_{hk}b_h \le b_k s + A_k t, && h \in \alpha, k \in \rho, \\ & A_k U_i \le b_k s + A_k t, && i \in \beta, k \in \rho, \\ & A_k V_i \le 0, && i \in \beta, k \in \rho, \\ & A_k c_j + a_k r_j \le b_k s + A_k t, && j \in \gamma, k \in \rho, \\ & \Lambda_{hk} \ge 0, && h \in \alpha, k \in \rho, \\ & s \ge 0. \end{aligned}$$

Note that case (B_1, H) of OCP, a special case of (HWB, H), is the well-known problem of finding the largest ball inscribed in an H-cell and has been part of the folklore of linear programming for over a decade.

Case (W, HW) *of* OCP *is a linear program*

Formulated through OCP1, we have:

$$\begin{aligned} z_1 = \underset{s,\,t,\,\Lambda,\,\pi,\,\Omega}{\text{maximum}} \quad & s \\ \text{subject to} \quad & sU_i + t \circ e = U_l\Lambda_{il} + V_l\pi_{il}, && i \in \beta, l \in \sigma, \\ & V_i = V_l\Omega_{il}, && i \in \beta, l \in \sigma, \\ & A_k(sU_i + t \circ e) \le b_k \circ e, && i \in \beta, k \in \rho, \\ & A_k(V_i) \le 0, && i \in \beta, k \in \rho, \\ & e\Lambda_{il} = e,\ \Lambda_{il} \ge 0,\ \pi_{il} \ge 0,\ \Omega_{il} \ge 0, && i \in \beta, l \in \sigma, \\ & s \ge 0. \end{aligned}$$

Case $(B_1, B_=)$ *of* OCP *is a quadratic program*

Let (c, r) be the given center and radius of the ball $\mathscr{X}$. Treating the optimal containment problem through OCP1, we have:

$$z_1 = \underset{s,t}{\text{maximum}} \quad s,$$

$$\text{subject to} \quad \|c_m - (sc + t)\| + sr \leq r_m, \quad m \in \tau,$$
$$s \geq 0.$$

If $\mathscr{Y} \neq \emptyset$, i.e., the intersection of the B_m is not empty, then the constraint $s \geq 0$ is superfluous, and can be dropped. Since all r_m are equal, the above program is seen to be an instance of (Q1) and hence can be solved via the quadratic program (Q3). Note that if the optimal solution to the program (Q3) returns a negative value of s, then $\mathscr{Y} = \emptyset$. A variation of this problem was first shown to be a quadratic program by Gale [4].

Case $(B_=, B_1)$ *of* OCP *is a quadratic program*

Here we let (c, r) be the given center and radius of the ball $\mathscr{Y}$. Formulated through OCP2, the optimal containment problem is written as

$$z_2 = \underset{s,t}{\text{minimum}} \quad s,$$

$$\text{subject to} \quad \|c_j - (sc + t)\| + r_j \leq sr, \quad j \in \gamma,$$
$$s \geq 0.$$

Note that the constraint $s \geq 0$ is superfluous, and can be omitted. As this program is a realization of (Q1), it is solvable as the quadratic program (Q3) for $r > 0$.

The special case of $(B_=, B_1)$ where all $r_j = 0$ is the problem of finding the smallest ball covering the points c_j, $j \in \gamma$ and has been treated by Elzinga and Hearn [3] and Kuhn [7].

The case (W, B_1) *of* OCP *is a quadratic program*

(W, B_1) of OCP is a special case of $(B_=, B_1)$ just discussed when $\text{rec}(\mathscr{X}) = \{\emptyset\}$ (else OCP is infeasible), since all $V_i = 0$ and each column of the U_i can be considered as the center of a B-cell with radius zero. Thus (W, B_1) of OCP is solvable through the quadratic program (Q3).

Other cases of OCP

Cases (WB, HB) and (W, HWB) of OCP can be formulated as convex programs using the logic already employed; however, we have been unable to formulate either case as a quadratic or linear program. As regards all other cases of OCP, we are convinced that their formulation as a convex program, much less a quadratic or linear program, cannot be accomplished. The reason for this is that the problem of testing $\mathscr{X} \subseteq \mathscr{Y}$, where either (i) $\mathscr{X}$ is an H-cell and $\mathscr{Y}$ is a

W-cell, (ii) $\mathscr{X}$ is an H-cell and $\mathscr{Y}$ is a B-cell, or (iii) $\mathscr{X}$ is a B-cell and $\mathscr{Y}$ is a W-cell, appears to be intractable without conversion of the polyhedra from H-cell to W-cell or vice versa.

4. The Optimal Meet Problem (OMP)

Let $\mathscr{X}$ and $\mathscr{Y}_q$, $q = 1, \dots, p$, each be a finite intersection of cells. The optimal meet problem can be written as:

$$\text{OMP1}\begin{cases} v_1 = \underset{s,t}{\text{infimum}} & s, \\ \text{subject to} & (s\mathscr{X} + t) \cap \mathscr{Y}_q \neq \emptyset, \quad q = 1, \dots, p, \\ & s > 0 \end{cases}$$

or

$$\text{OMP2}\begin{cases} v_2 = \underset{s,t}{\text{supremum}} & s, \\ \text{subject to} & \mathscr{X} \cap (s\mathscr{Y}_q + t) \neq \emptyset, \quad q = 1, \dots, p, \\ & s > 0. \end{cases}$$

The following result concern the equivalence of OMP1 and OMP2.

Lemma 4.1 (Equivalence of OMP1 and OMP2).

(i) (Solutions) (s, t) *is a feasible or optimal solution to* OMP1 *if and only if* $(1/s, -t/s)$ *is a feasible or optimal solution to* OMP2.

(ii) (Feasibility) *The following are equivalent*:

(*a*) OMP1 *is feasible*;

(b) OMP2 *is feasible*;

(c) *For some translate* t, $(t + \text{tng}(\mathscr{X})) \cap \mathscr{Y}_q \neq \emptyset$, $q = 1, \dots, p$.

(iii) (Attainment) *The following are equivalent*:

(a) $0 < v_1 < \infty$;

(b) $0 < v_2 < \infty$;

(c) OMP1 *has an optimum*;

(d) OMP2 *has an optimum.*

(iv) (Non-attainment) *The following are equivalent*:

(a) $v_1 = 0$;

(b) $v_2 = +\infty$;

(c) *For some translate* t, $(\text{rec}(\mathscr{X}) + t) \cap \mathscr{Y}_q \neq \emptyset$, $q = 1, \dots, p$.

For a specific realization of OMP1 or OMP2, $\mathscr{X}$ and $\mathscr{Y}_q$ will be given in the form:

$$\mathscr{X} = \Big[\bigcap_{h\in\alpha} \text{H}_h\Big] \cap \Big[\bigcap_{i\in\beta} \text{W}_i\Big] \cap \Big[\bigcap_{j\in\gamma} \text{B}_j\Big],$$

$$\mathscr{Y}_q = \Big[\bigcap_{k_q\in\rho_q} \text{H}_{k_q}\Big] \cap \Big[\bigcap_{l_q\in\sigma_q} \text{W}_{l_q}\Big] \cap \Big[\bigcap_{m_q\in\tau_q} \text{B}_{m_q}\Big], \quad q = 1, \dots, p$$

where $H_{(\cdot)}$, $W_{(\cdot)}$ and $B_{(\cdot)}$ are as in Section 3. Our solvable cases are as follows.

Case (HW, HW) *of* OMP *is a linear program*

Treated through OMP1, case (HW, HW) is the linear program:

$$v_1 = \underset{s,\,t,\,x,\,\lambda,\,\mu}{\text{minimum}} \quad s$$

$$\begin{aligned} \text{subject to} \quad & A_h x_q \leq b_h s + A_h t, && h \in \alpha,\ q = 1, \ldots, p, \\ & A_{k_q} x_q \leq b_{k_q}, && k_q \in \rho_q,\ q = 1, \ldots, p, \\ & x_q = t + U_i \lambda_{iq} + V_i \mu_{iq}, && i \in \beta,\ q = 1, \ldots, p, \\ & x_q = U_{l_q} \lambda_{l_q} + V_{l_q} \mu_{l_q}, && l_q \in \sigma_q,\ q = 1, \ldots, p, \\ & e\lambda_{iq} = s,\ \lambda_{iq} \geq 0,\ \mu_{iq} \geq 0, && i \in \beta,\ q = 1, \ldots, p, \\ & e\lambda_{l_q} = 1,\ \lambda_{l_q} \geq 0,\ \mu_{l_q} \geq 0, && l_q \in \sigma_q,\ q = 1, \ldots, p, \\ & s \geq 0. \end{aligned}$$

Case $(B_1, B_=)$ *of* OMP *is a quadratic program*

This is the case where each $\mathscr{Y}_q$ is a ball of given center c_q and radius r_q with $r_1 = \cdots = r_p \geq 0$. Let $\mathscr{X}$ be the ball with center c and positive radius r, and we proceed through OMP1. The formulation is

$$v_1 = \underset{s,\,t}{\text{minimum}} \quad s,$$

$$\begin{aligned} \text{subject to} \quad & \|c_q - (sc + t)\| \leq sr + r_q, \quad q = 1, \ldots, p. \\ & s \geq 0. \end{aligned}$$

If $\bigcap_{q=1}^{p} \mathscr{Y}_q = \emptyset$ (otherwise OMP does not attain its minimum), then the constraint $s \geq 0$ is superfluous and can be omitted. Furthermore, since all r_q are equal, the above program is an instance of (Q1) and hence can be solved via the quadratic program (Q3).

Other cases of OMP

Note that the most general case of OMP, namely (HWB, HWB), can be formulated as a convex program using the logic employed herein. However, we have been unable to formulate any case of OMP other than the above two cases as a quadratic or linear program.

5. Remark

Our final remark concerns the interrelated issues of computational complexity, the conversion of H- to W-cells and vice versa, and our division of problems solvable as quadratic or linear programs from other convex programs. In [5] and

[6], it is shown that linear and (convex) quadratic programs are solvable in polynomial time. The conversion of an H- to W-cell, or vice versa, is an exponential problem. To see this, consider the sets $\mathscr{C} \triangleq \{x \in R^n \mid \|x\|_\infty \leq 1\}$ and $\mathscr{D} \triangleq \{x \in R^n \mid \|x\|_1 \leq 1\}$. $\mathscr{C}$, as an H-cell, can be represented by $2n$ halfspaces, but as a W-cell it requires the enumeration of at least 2^n (extreme) points. $\mathscr{D}$, as a W-cell, can be represented by $2n$ (extreme) points, but as an H-cell it requires the enumeration of at least 2^n halfspaces. We thus see that our distinction of H-cells and W-cells as different entities is consistent from the standpoint of the solvability of the quadratic and linear programs contained herein.

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