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THE CUTTING STOCK PROBLEM AND INTEGER ROUNDING

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An integer programming problem is said to have the integer round-up property if its optimal value is given by the least integer greater than or equal to the optimal value of its linear programming relaxation. In this paper we prove that certain classes of cutting stock problems have the integer round-up property. The proof of these results relies upon the decomposition properties of certain knapsack polyhedra.

Key words: Cutting Stock Problem, Knapsack Polyhedron, Integer Rounding.

1. Introduction

A knapsack problem is an integer programming problem of the form

$$\begin{array}{ll} \max & c \cdot x \\ \text{subject to} & a \cdot x \leq b, \\ & x \geq 0, \text{ integral,} \end{array} \tag{KP}$$

where b is a positive integer and a and c are n-vectors consisting of positive integers. The *knapsack polyhedron* (denoted by P in this paper) is the convex hull of the feasible solutions of (KP); that is, the convex hull of the set $\{x \in \mathbb{Z}_+^n \mid ax \leq b\}$. To the maximization problem (KP) corresponds a covering problem defined as follows: let M be the matrix whose rows are the maximal elements of $\{x \in \mathbb{Z}_+^n \mid ax \leq b\}$. Then the covering problem

$$\begin{array}{ll} \min & 1 \cdot y \\ \text{s.t.} & yM \geq w, \\ & y \geq 0, \ y \text{ integral,} \end{array} \tag{CS}$$

where $w \in \mathbb{Z}_+^n$, is known as the *cutting stock problem.* When $a_i \leq b$ for $i = 1, 2, \ldots, n$ the matrix M does not have any zero column, and thus (CS) is feasible for all right-hand sides $w \in \mathbb{Z}_+^n$. Since (CS) is bounded from below, it follows that (CS) has an optimal solution. Henceforth we shall assume that $a_i \leq b$ for $i = 1, 2, \ldots, n$.

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It has been observed that for cutting stock problems which are encountered in practice, the difference between the optimal value of (CS) and that of its linear programming relaxation is small. In particular, it is the case that for many instances of the cutting stock problem, the optimal value of the problem is given by the least integer greater than or equal to the optimal value of its linear programming relaxation (see Berge and Johnson (1976)). In this paper we shall prove that certain classes of cutting stock problems have the 'rounding property'. More specifically, we say that (CS) has the integer round-up (IRU) property if s_w^* (the optimal value of (CS)) is equal to $\lceil r_w^* \rceil$, where r_w^* is the optimal value of the linear relaxation of (CS) and $\lceil r_w^* \rceil$ denotes the least integer greater than or equal to r_w^*. On the other hand, let P be any polyhedron contained within the nonnegative orthant and k be any positive integer. Let also kP be defined as $\{kx \mid x \in P\}$. We say that P has the integral decomposition property if for any positive integer k and any integral y belonging to kP, y can be expressed as the sum of k vectors, each of which is integral and belongs to P.

Let P now denote the knapsack polyhedron. P is clearly a lower comprehensive polyhedron, and it follows from Theorem 1 in Baum and Trotter (1981) that problem (CS) has the IRU property if and only if P, the corresponding knapsack polyhedron, has the integral decomposition property. In the rest of the paper, we shall use this theorem to establish that certain classes of cutting stock problems have the integer round-up property. In Section 2 we state and prove some lemmas which will be useful in the sequel. Each of Sections 3, 4 and 5 is devoted to a class of cutting stock problems for which the IRU property holds.

Because of the close relationship between instances of the knapsack problem and instances of the cutting stock problem, we shall denote by $(a_1, a_2, \ldots, a_n : b)$ the cutting stock problem arising from the knapsack relation $ax \leq b$; in other words, problem $(a_1, a_2, \ldots, a_n : b)$ is problem (CS) where the rows of M are the maximal points of $\{x \in \mathbb{Z}_+^n \mid ax \leq b\}$.

2. Preliminary lemmas

The following lemmas will be needed in the remainder of this article. Let J be a subset of $\{1, 2, \ldots, n\}$. We let $\mathbb{Z}^J$ denote the set $\{x \in \mathbb{Z}^n \mid x_l = 0 \text{ for } l \notin J\}$, and $\mathbb{R}^J$ denote the set $\{x \in \mathbb{R}^n \mid x_l = 0 \text{ for } l \notin J\}$.

Lemma 2.1. *Let P be a polyhedron in $\mathbb{R}_+^n$, and let us assume that P has the decomposition property. Then for any subset J of $\{1, 2, \ldots, n\}$, the polyhedron $P \cap \mathbb{R}^J$ has the decomposition property.*

Proof. Let w be an element of $kP \cap \mathbb{Z}^J$. Then $w \in kP$, and since P has the integral decomposition property, there exist x^i $(i = 1, 2, \ldots, k)$ such that $w = \sum_{i=1}^k x^i$ and $x^i \in P \cap \mathbb{Z}^n$ for each i. But $w_l = 0$ for $l \notin J$ implies that $x_l^i = 0$ for $l \notin J$ and

$i=1,2,\ldots,k$. Therefore w is the sum of k vectors, each of which belongs to $P\cap\mathbb{Z}^J$. Thus $P\cap\mathbb{R}^J$ has the integral decomposition property. □

Lemma 2.2. *Let P' be a polyhedron in $\mathbb{R}^n_+$, and let P be the polyhedron*

$$\{(x_1,x_2,\ldots,x_n,x_{n+1})\,|\,(x_1,x_2,\ldots,x_n+x_{n+1})\in P'\}.$$

If P' has the decomposition property, then so does P.

Proof. Let us assume that P' has the decomposition property, and let w belong to $kP\cap\mathbb{Z}^{n+1}$ for some $k\in\mathbb{Z}_+$. Let z be defined as follows:

$$z_i=\frac{w_i}{k}\quad\text{for } i=1,2,\ldots,n-1,\qquad z_n=\frac{w_n+w_{n+1}}{k}.$$

By definition of P, z belongs to P', and thus kz can be expressed as $\sum_{i=1}^k x^i$, where $x^i\in P'\cap\mathbb{Z}^n$ for each i. If we define the collection of vectors $\{y^i\}_{i=1}^k$ as follows:

$$y^i_j=x^i_j\quad\text{for } j=1,2,\ldots,n-1,$$

$$y^i_n=\min\left\{x^i_n,\ w_n-\sum_{l=1}^{i-1}y^l_n\right\},\qquad y^i_{n+1}=x^i_n-y^i_n,$$

it is easy to verify that $w=\sum_{i=1}^k y^i$. □

For any positive integer b, we define the *master knapsack* polyhedron (denoted by P^b) to be the convex hull of $\{x\in\mathbb{Z}^n_+\,|\,\sum_{i=1}^b ix_i\leq b\}$. The following lemma is a straightforward consequence of Lemmas 2.1 and 2.2.

Lemma 2.3. *Let b be any positive integer. If the master knapsack polyhedron P^b has the integral decomposition property, then so does the polyhedron $P=\operatorname{conv}\{x\in\mathbb{Z}^n_+\,|\,ax\leq b\}$, where $a\in\mathbb{Z}^n_+$.*

Proof. (a) If all the components of a are distinct, the result follows from Lemma 2.1.

(b) If two components of a are equal, we may assume without loss of generality that $a_{n-1}=a_n$. Let P' be the polyhedron $\{x\in\mathbb{Z}^n_+\,|\,a_1x_1+a_2x_2+\cdots+a_{n-1}x_{n-1}\leq b\}$. It is clear that $P=\{(x_1,x_2,\ldots,x_n)\,|\,(x_1,x_2,\ldots,x_{n-1}+x_n)\in P'\}$. The result then follows from (a) and Lemma 2.2. If $a_i=a_j$ for more than one pair $\{i,j\}$ of indices, the result follows by applying Lemma 2.2 repeatedly. □

By the argument given in the introduction, Lemma 2.3 is equivalent to the following statement: the problem $(1,2,\ldots,b:b)$ has the IRU property if and only if for any positive integer n and any vector $a\in\mathbb{Z}^n_+$ whose components satisfy $a_i\leq b$, the problem $(a_1,a_2,\ldots,a_n:b)$ has the IRU property.

Lemma 2.4. *Let* (CS_1) *be the cutting stock problem* $(a_1,a_2,\ldots,a_n:b)$, *where* $a_1=1$, *and* (CS_2) *be the problem* $(a_2, \ldots, a_n : b)$. *Then* (CS_1) *has the rounding property if and only if* (CS_2) *has the rounding property.*

Proof. Let P^1 and P^2 be the knapsack polyhedra corresponding to (CS_1) and (CS_2) respectively. If P^1 has the integral decomposition property, then by Lemma 2.1, P^2 also has the property. Conversely let us assume that P^2 has the decomposition property, and let $w' \in kP^1 \cap \mathbb{Z}^n$. Then there exist λ_i (for $i = 1, 2, \ldots, p$) and z^i (for $i = 1, 2, \ldots, p$) such that $w' = \sum_{i=1}^{p} \lambda_i z^i$, $\lambda_i \geq 0$ for $i = 1, 2, \ldots, p$, $\sum_{i=1}^{p} \lambda_i = k$ and $z^i \in P^1 \cap \mathbb{Z}^n$. Let us define the $\mathbb{Z}^{n-1}$-vectors w and y^i for $1 \leq i \leq p$ by $w_j = w'_j$ for $j = 2, 3, \ldots, n$ and $y^i_j = z^i_j$ for $j = 2, 3, \ldots, n$. Thus $w = \sum_{i=1}^{p} \lambda_i y^i$ and $w \in kP^2$. Since P^2 has the decomposition property, there exist $x^i \in \mathbb{Z}^{n-1} \cap P^2$ (for $i = 1, 2, \ldots, k$) such that $w = \sum_{i=1}^{k} x^i$.

We define u^i_1 (for $i = 1, 2, \ldots, k$) recursively as follows:

$$u^1_1 = \min\left\{ w'_1, b - \sum_{j=2}^{n} a_j x^1_j \right\}, \qquad u^{i+1}_1 = \min\left\{ w'_1 - \sum_{j=1}^{i} u^j_1, b - \sum_{j=2}^{n} a_j x^{i+1}_j \right\}.$$

It is easy to verify that $w'_1 - \sum_{i=1}^{k-1} u^i_1 \leq b - \sum_{j=2}^{n} a_j x^k_j$, and we conclude that $u^k_1 = w'_1 - \sum_{i=1}^{k-1} u^i_1$. We now define $u^i_j = x^i_j$ for $j = 2, 3, \ldots, n$ and $i = 1, 2, \ldots, k$. Clearly, for $2 \leq j \leq n$, $\sum_{i=1}^{k} u^i_j = \sum_{i=1}^{k} x^i_j = w_j = w'_j$. We have just shown that $w'_1 = \sum_{i=1}^{k} u^i_1$. Therefore $w' = \sum_{i=1}^{k} u^i$, and since $u^i \in P^1 \cap \mathbb{Z}^n$ for $1 \leq i \leq k$, it follows that P^1 has the decomposition property. This completes the proof of the lemma. □

3. Cutting stock problems of the form $(a_1, a_2 : b)$

We claim that a cutting stock problem of the form $(a_1, a_2 : b)$ has the IRU property. This fact is a simple consequence of the following theorem.

Theorem 3.1. *Let P be a lower comprehensive polyhedron in $\mathbb{R}^2$ with integral extreme points, and let M be the matrix whose rows are the maximal integral points of P. Then the covering problem*

$$\begin{aligned} \min \quad & 1 \cdot y \\ s.t. \quad & yM \geq w, \\ & y \geq 0, \ y \text{ integral}, \end{aligned}$$

has the IRU *property.*

Proof. It suffices to show that P has the integral decomposition property. Let $w \in kP$. Then there exist real numbers λ_1 and λ_2 $(0 \leq \lambda_1, \lambda_2 \leq 1)$ and vectors y^1 and y^2 belonging to $P \cap \mathbb{Z}^2$ such that $\lambda_1 + \lambda_2 = 1$ and $w/k = \lambda_1 y^1 + \lambda_2 y^2$. Therefore $w = \mu_1 y^1 + \mu_2 y^2$, where $\mu_1 = k\lambda_1$, $\mu_2 = k\lambda_2$ and thus $\mu_1 + \mu_2 = k$. If μ_1 and μ_2 are integers, w is clearly the sum of k vectors belonging to $P \cap \mathbb{Z}^2$, namely μ_1 copies of y^1 and

μ_2 copies of y^2. If μ_1 and μ_2 are not integers, w can be expressed as $\lfloor\mu_1\rfloor y^1+\lfloor\mu_2\rfloor y^2+y^3$, where y^3 belongs to $P\cap\mathbb{Z}^2$ and $\lfloor\mu_i\rfloor$ denotes the greatest integer smaller than or equal to μ_i. In that case also w is the sum of k vectors belonging to $P\cap\mathbb{Z}^2$. We have thus shown that P has the integral decomposition property. □

The following corollary was first pointed out to us by G. Nemhauser.

Corollary 3.2. *Cutting stock problems of the form $(a_1, a_2\colon b)$ have the* IRU *property.*

4. Cutting stock problems of the form $(a_1, a_2, \ldots, a_n : b)$, where $a_1|a_2|\cdots|a_n$

Cutting stock problems such that $a_1|a_2|\cdots|a_n$ (i.e., such that a_i divides a_{i+1} for $i=1,2,\ldots,n-1$) are said to have the property of successive divisibility. In this section we show that the IRU property holds for cutting stock problems which have the property of successive divisibility. We also describe a set of inequalities which define the convex hull of the set $\{x\in\mathbb{Z}^n_+\,|\,ax\leqslant b\}$ when $a_i\,|\,a_{i+1}$ for $i=1,2,\ldots,n-1$. By Lemma 2.4, we may take a_1 to be 1; we shall thus assume that $a_1=1$ in what follows, unless otherwise indicated. P denotes the convex hull of the set $\{x\in\mathbb{Z}^n_+\,|\,ax\leqslant b\}$.

Lemma 4.1. *Let $x\in rP$ where r is a positive real number. Then*

$$x_j\leqslant r\left\lfloor\frac{b}{a_j}\right\rfloor-\sum_{l=j+1}^{n} q^j_l x_l \text{ for } j=1,2,\ldots,n, \tag{4.2}$$

where

$$q^j_l=\frac{a_l}{a_j}\quad\text{for } l>j\quad\left(\sum_{l=j+1}^{n} q^j_l x_l=0 \text{ when } j=n\right).$$

Proof. The lemma follows from the definition of rP and from the fact that P is the convex hull of integral points which satisfy 4.2 with $r=1$. □

In order to prove that P has the integral decomposition property whenever $a_i\,|\,a_{i+1}$ for $i=1,2,\ldots,n-1$, we shall use the following 'greedy' procedure: let x be any vector belonging to $\mathbb{R}^n_+$. Then x' is defined by

$$x'_n=\min\left\{x_n,\left\lfloor\frac{b}{a_n}\right\rfloor\right\},$$

$$x'_j=\min\left\{x_j,\left\lfloor\frac{b-\sum_{l=j+1}^{n} a_l x'_l}{a_j}\right\rfloor\right\}\quad\text{for } j=n-1,\ n-2,\ldots,1. \tag{4.3}$$

Lemma 4.4 and Theorem 4.5 imply that this simple procedure (also known as the First-Fit Decreasing algorithm) will yield an integral decompositon of w, for any $w \in kP \cap \mathbb{Z}^n$, and hence an optimal solution of (CS).

Lemma 4.4. *Let $x \in \mathbb{R}^n_+$ be such that $x_j \in \mathbb{Z}$ for $j \geq 2$, and suppose that $ax = rb$ for some real number $r > 1$. Furthermore let us assume that x satisfies property* 4.2 *for $j = 1, 2, \ldots, n$, and x' is the vector defined by* 4.3. *Then the following hold:*

(i) $0 \leq x' \leq x$ *and* $x' \in \mathbb{Z}^n_+$;

(ii) $ax' = b$;

(iii) $x - x'$ *satisfies property* 4.2 *for $j = 1, 2, \ldots, n$, where $r-1$ replaces r in each inequality.*

Proof. (i) The fact that $x'_j \leq x_j$ for every j follows directly from the definition of x', and a simple induction argument shows that $b - \sum_{l=j+1}^n a_l x'_l$ is always nonnegative.

(ii) Let R_j be the remainder of the division of b by a_j; namely $R_j = b - \lfloor b/a_j \rfloor a_j$. For $j \geq 2$, we let $a^{j-1} = (a_1, a_2, \ldots, a_{j-1})$ and $x^{j-1} = (x_1, x_2, \ldots, x_{j-1})$. It is easily checked that

$$a^{j-1}x^{j-1} \geq rR_j \quad \text{for every } j \geq 2, \tag{4.5}$$

and that

$$\sum_{l=j}^n a_l x'_l = b - R_j \quad \text{whenever } x'_j = \left\lfloor \frac{b - \sum_{l=j+1}^n a_l x'_l}{a_j} \right\rfloor. \tag{4.6}$$

Let k be the smallest index such that $k > 1$ and

$$x_k > \left\lfloor \frac{b - \sum_{l=k+1}^n a_l x'_l}{a_k} \right\rfloor.$$

If k does not exist, $x'_j = x_j$ for $j \geq 2$, and thus

$$x'_1 = \min\left\{x_1, b - \sum_{l=2}^n a_l x_l\right\} = b - \sum_{l=2}^n a_l x_l \quad (\text{since } ax = rb > b)$$

$$= b - \sum_{l=2}^n a_l x'_l \qquad (\text{since } x'_l = x_l \text{ for } l \geq 2).$$

Hence $ax' = b$ in this case.

If k does exist, however, 4.5 implies that $a^{k-1}x^{k-1} \geq rR_k$ and 4.6 implies that $\sum_{l=k}^n a_l x'_l = b - R_k$. From the definition of k we know that $x'_j = x_j$ for $j = 2, 3, \ldots, k-1$. It follows by an easy calculation that $x'_1 = R_k - \sum_{l=2}^{k-1} a_l x_l$, and hence that $ax' = b$. Thus (ii) holds in this case also.

(iii) Let $w = x - x'$. We must show that

$$w_j \leq (r-1)\left\lfloor \frac{b}{a_j} \right\rfloor - \sum_{l=j+1}^n q^j_l w_l \quad \text{for } j = 1, 2, \ldots, n.$$

We only sketch the proof since the computations are tedious. Let us consider first the case where $w_j = 0$. If $w_l = 0$ for every index l greater than j, there is nothing to prove. If $w_l > 0$ for some l greater than j, we let k denote the smallest index such that $k > j$ and $w_k > 0$. By definition of k, $w_l = 0$ for $j \leq l < k$; also, by definition of k, $w_k > 0$, which implies that $x'_k < x_k$. Thus

$$x'_k = \left\lfloor \frac{b - \sum_{l=k+1}^{n} a_l x'_l}{a_k} \right\rfloor$$

and finally $\sum_{l=k}^{n} a_l x'_l = b - R_k$ by 4.6. From this we are able to conclude that $w_j = 0 \leq (r-1)\lfloor b/a_j \rfloor - \sum_{l=j+1}^{n} q^j_l w_l$.

We now turn to the case where $w_j > 0$. In this case $x'_j < x_j$ and thus

$$x'_j = \left\lfloor \frac{b - \sum_{l=j+1}^{n} a_l x'_l}{a_j} \right\rfloor.$$

which implies that

$$\sum_{l=j}^{n} a_l x'_l = b - R_j = a_j \left\lfloor \frac{b}{a_j} \right\rfloor$$

by 4.6. Using 4.2 and the above equality, it is possible to show that

$$w_j \leq (r-1) \left\lfloor \frac{b}{a_j} \right\rfloor - \frac{1}{a_j} \left\{ \sum_{l=j+1}^{n} a_l w_l \right\}.$$

This completes the proof of (iii) and the proof of the lemma. □

The previous lemmas imply that polyhedron P has the decomposition property; actually Lemma 4.4 implies the following theorem, which is stronger than the statement that P has the integral decomposition property.

Theorem 4.7. *Let x be a vector satisfying the hypotheses of Lemma 4.4 for some positive real number r. Let $k = \lceil r \rceil$. Then there exist vectors $x^1, x^2, \ldots, x^k$ such that $x = \sum_{i=1}^{k} x^i$, $x^i \in \mathbb{Z}^n_+$ and $ax^i = b$ for $i = 1, 2, \ldots, k-1$ and $x^k \geq 0$ and $ax^k = (r-k+1)b$.*

Proof. The proof is by induction on k. For $k = 1$, there is nothing to prove, since we may take $x^k = x$. When $k = 2$, we have $1 < r \leq 2$, and by Lemma 4.4, there exists an x' such that $0 \leq x' \leq x$, $x' \in \mathbb{Z}^n_+$ and $ax' = b$. It is clear that $x^1 = x'$ and $x^2 = x - x'$ verify the conclusion of the theorem, since $x = x^1 + x^2$, $x^1 \in \mathbb{Z}^n_+$, $ax^1 = b$ and $ax^2 = ax - ax^1 = rb - b = (r-k+1)b$. Thus the theorem is true for $k = 2$. Let us now assume that $k > 2$ and that the theorem is true for all $k' < k$. By Lemma 4.4 again, there exists a vector $x' \in \mathbb{Z}^n_+$ such that $0 \leq x' \leq x$, $ax' = b$ and $x - x'$ satisfies inequalities 4.2 for $j = 1, 2, \ldots, n$, where $r-1$ replaces r in each inequality. But then $k' = \lceil r-1 \rceil = k-1 < k$, and we may apply the induction hypothesis to conclude that there exist $k-1$ vectors $x^2, x^3, \ldots, x^k$ such that $x - x' = \sum_{i=2}^{k} x^i$, $x^i \in \mathbb{Z}^n_+$ and $ax^i = b$ for $i = 2$,

$3, \ldots, k-1$ and $ax^k = ((r-1)-(k-1)+1)b = (r-k+1)b$. Let $x^1 = x'$. Then $x = x' + (x - x') = \sum_{i=1}^{k} x^i$, where $x^i \in \mathbb{Z}_+^n$ and $ax^i = b$ for $i = 1, 2, \ldots, k-1$ and $ax^k = (r-k+1)b$. This completes the proof of the theorem □

The fact that P has the integral decomposition property is a straightforward consequence of Theorem 4.7.

Corollary 4.8. *The polyhedron P has the integral decomposition property and* (CS) *has the* IRU *property, provided successive divisibility holds.*

Proof. Let $x \in kP \cap \mathbb{Z}_+^n$. Then there exist real scalars λ_i and n-vectors y^i such that $\lambda_i \geq 0$ for $i = 1, 2, \ldots, p$, $\sum_{i=1}^{p} \lambda_i = 1$, $y^i \in P \cap \mathbb{Z}_+^n$ for $i = 1, 2, \ldots, p$ and $x = k \sum_{i=1}^{p} \lambda_i y^i$. Let us define z^i as follows:

$$z_j^i = y_j^i \quad \text{for } j = 2, 3, \ldots, n, \qquad z_1^i = y_1^i + (b - ay^i).$$

Let us also define $\tilde{x}$ as $k \sum_{i=1}^{p} \lambda_i z^i$. By Lemma 4.1, Property 4.2 (with r replaced by k) holds for $\tilde{x}$; therefore $\tilde{x}$ satisfies the hypotheses of Lemma 4.4, and by Theorem 4.7, there exist vectors $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^k$ such that $\tilde{x}^i \in \mathbb{Z}_+^n$ for $i \leq k-1$, $a\tilde{x}^i = b$ for every i and $\tilde{x} = \sum_{i=1}^{k} \tilde{x}^i$.

We define x^i recursively for $i = 1, 2, \ldots, k$ as follows:

$$x_j^i = \tilde{x}_j^i \quad \text{for } j = 2, 3, \ldots, n, \qquad x_1^i = \min\left\{\tilde{x}_1^i,\ x_1 - \sum_{l=1}^{i-1} x_1^l\right\}.$$

It is a simple matter to show that x_1^i is an integer for $i = 1, 2, \ldots, k$, and that $x_1 = \sum_{l=1}^{k} x_1^l$. We have thus shown that x is the sum of k vectors, each of which belongs to $P \cap \mathbb{Z}^n$. Since this is true for all $k \geq 1$ and all $x \in kP \cap \mathbb{Z}_+^n$, we have proved that the decomposition property holds for P, and hence that the IRU property holds for (CS). □

Lemmas 4.1 and 4.4 have enabled us to prove that P has the integral decomposition property; but they can also be used to describe P by means of linear inequalities.

Theorem 4.9. The polyhedron P is precisely the set of nonnegative vectors y which satisfy

$$y_j \leq \left\lfloor \frac{b}{a_j} \right\rfloor - \sum_{l=j+1}^{n} q_l^j y_l \quad \text{for } j = 1, \ldots, n, \tag{4.10}$$

where $q_l^j = a_l / a_j$ for $l > j$.

Proof. Let Q be the set of nonnegative vectors y for which 4.10 holds. It follows from Lemma 4.1, by taking $r = 1$, that P is a subset of Q. Conversely let y be an extreme point of Q. It is well known (see for instance Gale (1960)) that y is the solution of a subsystem of 4.10 consisting of n linearly independent equalities. Since all the coefficients of system 4.10 are integers, the components of y are rationals. Let k be the least common multiple of the denominators of the components of y. Then $ky \in \mathbb{Z}_+^n$ and satisfies property 4.2 for $j = 1, 2, \ldots, n$, where r has been replaced by k. Let us define $\tilde{y}$ as follows:

$$\tilde{y}_j = ky_j \quad \text{for } j = 2, 3, \ldots, n, \qquad \tilde{y}_1 = ky_1 + k(b - ay).$$

It is clear that $\tilde{y} \in \mathbb{Z}_+^n$. Furthermore, since $\tilde{y}_j = ky_j$ for $j = 2, 3, \ldots, n$ and the inequalities for $j = 2, 3, \ldots, n$ contain components of index $j = 2, 3, \ldots, n$ only, $\tilde{y}$ satisfies the inequalities 4.2 for $j = 2, 3, \ldots, n$, where r has been replaced by k. Finally, the inequality 4.2 for $j = 1$ and $r = k$ reduces to $x_1 \leqslant kb - \sum_{l=2}^n a_l x_l$, which is clearly satisfied by $\tilde{y}$. We have thus shown that $\tilde{y}$ satisfies all the hypotheses of Lemma 4.4, and hence all the hypotheses of Theorem 4.7. We conclude that there exist vectors $x^1, x^2, \ldots, x^k$ belonging to $P \cap \mathbb{Z}^n$ such that $\tilde{y} = \sum_{i=1}^k x^i$. Therefore $\tilde{y} \in kP$ and since P is lower comprehensive, $y \leqslant \tilde{y}/k$ implies that $y \in P$. We have thus shown that every extreme point of Q belongs to P. Since P is a convex set and Q is the convex hull of its extreme points, it follows that Q is contained within P. Thus $P = Q$, and the theorem is proved. □

5. Cutting stock problems of the form $(a_1, a_2, \ldots, a_p, a_{p+1}, \ldots, a_n : b)$, where $a_i \mid b$ for every i and $a_i \mid a_{i+1}$ for $i \neq p, n$

Let P be the convex hull of $\{x \in \mathbb{Z}_+^n \mid ax \leqslant b\}$, where a_i divides b for every i and a_i divides a_{i+1} for $i \neq p, n$; that is, $a_1 | a_2 | \cdots | a_p$ and $a_{p+1} | a_{p+2} | \cdots | a_n$. Theorem 4.7 implies the following theorem:

Theorem 5.1. *P has the decomposition property.*

Proof. Since a_i divides b for every i, the polyhedron P is equal to the polyhedron $\{x \in \mathbb{R}_+^n \mid ax \leqslant b\}$ and its extreme points are 0 and the vectors $(b/a_i)e^i$, where e^i is the ith unit vector. Let w be a vector belonging to $kP \cap \mathbb{Z}_+^n$. Then there exist nonnegative scalars λ_i (for $i = 1, 2, \ldots, n$) such that $\sum_{i=1}^n \lambda_i \leqslant k$ and

$$w = \sum_{i=1}^n \lambda_i \left(\frac{b}{a_i}\right) e^i = \sum_{i=1}^p \lambda_i \left(\frac{b}{a_i}\right) e^i + \sum_{i=p+1}^n \lambda_i \left(\frac{b}{a_i}\right) e^i.$$

The components of $\sum_{i=1}^p \lambda_i (b/a_i) e^i$ whose indices are $p+1, p+2, \ldots, n$ are all equal to zero, while the components of $\sum_{i=p+1}^n \lambda_i (b/a_i) e^i$ whose indices are $1, 2, \ldots, p$ are equal to 0. Hence if we let $w^1 = (w_1, w_2, \ldots, w_p)$ and $w^2 = (w_{p+1}, w_{p+2}, \ldots, w_n)$ we have (for appropriately dimensioned unit vectors):

$$w^1 = \sum_{i=1}^p \lambda_i \left(\frac{b}{a_i}\right) e^i \quad \text{and} \quad w^2 = \sum_{i=p+1}^n \lambda_i \left(\frac{b}{a_i}\right) e^{i-p}.$$

Let $r_1 = \sum_{i=1}^{p} \lambda_i$ and $k_1 = [\sum_{i=1}^{p} \lambda_i]$; w^1 clearly satisfies the hypotheses of Theorem 4.7 with r replaced by r_1. Thus there exist vectors $x^1, x^2, \ldots, x^{k_1}$ such that $w^1 = \sum_{i=1}^{k_1} x^i$, $x^i \in \mathbb{Z}_+^p$ and $a^1 x^i = b$ for $i = 1, 2, \ldots, k_1 - 1$ and $x^{k_1} \in \mathbb{Z}_+^p$ and $a^1 x^{k_1} = (r_1 - k_1 + 1)b$, where $a^1 = (a_1, a_2, \ldots, a_p)$. Similarly, if we let r_2 denote $\sum_{i=p+1}^{n} \lambda_i$ and k_2 denote $[\sum_{i=p+1}^{n} \lambda_i]$, w^2 satisfies the hypotheses of Theorem 4.7 and we may conclude that there exist vectors $y^1, y^2, \ldots, y^{k_2}$ such that $w^2 = \sum_{i=1}^{k_2} y^i$, $y^i \in \mathbb{Z}_+^{n-p}$ and $a^2 y^i = b$ for $i = 1, 2, \ldots, k_2 - 1$ and $y^{k_2} \in \mathbb{Z}_+^{n-p}$ and $a^2 y^{k_2} = (r_2 - k_2 + 1)b$, where $a^2 = (a_{p+1}, a_{p+2}, \ldots, a_n)$.

If $k_1 + k_2 \leq k$, we can clearly write w as $\sum_{i=1}^{k_1} (x^i, 0) + \sum_{i=1}^{k_2} (0, y^i)$, where $(x^i, 0)$ (for $i = 1, 2, \ldots, k_1$) and $(0, y^i)$ (for $i = 1, 2, \ldots, k_2$) are elements of $\mathbb{Z}_+^n$. Thus w is the sum of at most k vectors, each of which belongs to $P \cap \mathbb{Z}_+^n$. On the other hand, if $k_1 + k_2 > k$, we have $k_1 + k_2 = k + 1$ and $(r_1 - k_1 + 1) + (r_2 - k_2 + 1) \leq 1$. Thus we can write w as $\sum_{i=1}^{k_1 - 1} (x^i, 0) + \sum_{i=1}^{k_2 - 1} (0, y^i) + (x^{k_1}, y^{k_2})$. Since $a(x^{k_1}, y^{k_2}) = a^1 x^{k_1} + a^2 y^{k_2} = (r_1 - k_1 + 1)b + (r_2 - k_2 + 1)b \leq b$, w is the sum of k vectors, each of which belongs to $P \cap \mathbb{Z}_+^n$. This completes the proof of the theorem. □

It follows from Theorem 5.1 that the cutting stock problem

$$\begin{aligned} \min \quad & 1 \cdot y \\ \text{s.t.} \quad & yM \geq w, \\ & y \geq 0, \ y \text{ integral}, \end{aligned}$$

where the rows of M are the maximal integral points of P, has the IRU property.

6. Conclusion

We have shown that the rounding property holds for several classes of cutting stock problems. It is also possible to show that the rounding property holds for small values of b (a proof may be found in Marcotte (1982) for $b \leq 8$). Finally, it can be shown that when the right-hand side of the knapsack inequality is large compared to the coefficients of the left-hand side, the optimal value of (CS) is less than or equal to $[r_w^*] + 1$, where r_w^* is the optimal value of the linear relaxation of (CS) (Marcotte (1982)). The relationship between (CS) and its linear programming relaxation thus seems worthy of further investigation.

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