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FACETS OF THE LINEAR ORDERING POLYTOPE

Martin GRÖTSCHEL, Michael JÜNGER and Gerhard REINELT
Universität Augsburg, Institut für Mathematik, Memminger Str. 6, 8900 Augsburg, FR Germany

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Let D_n be the complete digraph on n nodes, and let P^n_{LO} denote the convex hull of all incidence vectors of arc sets of linear orderings of the nodes of D_n (i.e. these are exactly the acyclic tournaments of D_n). We show that various classes of inequalities define facets of P^n_{LO}, e.g. the 3-dicycle inequalities, the simple k-fence inequalities and various Möbius ladder inequalities, and we discuss the use of these inequalities in cutting plane approaches to the triangulation problem of input–output matrices, i.e. the solution of permutation resp. linear ordering problems.

Key words: Facets of Polyhedra, Linear Ordering Problem, Triangulation Problem, Permutation Problem.

1. Introduction and notation

This paper is a continuation of our paper Grötschel, Jünger and Reinelt (1985) on the acyclic subgraph polytope. The polytope associated with linear orderings is a face of the acyclic subgraph polytope. Our main objective is to investigate which of the inequalities shown to define facets of the acyclic subgraph polytope in our former paper also define facets of the linear ordering polytope. We adopt the notations in graph theory and polyhedral theory of that paper.

A *linear ordering* (or *permutation*) of a finite set V with $|V| = n$ is a bijective mapping $\sigma: \{1, 2, \ldots, n\} \to V$. For $u, v \in V$ we say that u is 'better than' or 'before' v if $\sigma^{-1}(u) < \sigma^{-1}(v)$. Among all possible linear orderings of V we want to find a linear ordering which is the best according to some criterion. In many applications a 'value' or a 'cost' can be associated with a linear ordering in the following way. For every two elements $u, v \in V$ a value c_{uv} and a value c_{vu} are given which can be interpreted as the profit we obtain from having u 'before' v resp. v 'before' u in a linear ordering. Then the total value of a linear ordering clearly is given by

$$\sum_{\sigma^{-1}(u) < \sigma^{-1}(v)} c_{uv}.$$

Given a linear ordering of the nodes V of a digraph then the arc set $\{(u, v) \mid \sigma^{-1}(u) < \sigma^{-1}(v)\}$ forms an acyclic tournament on V, and similarly, if (V, T) is an acyclic tournament then this induces a linear ordering of V. Using this graph theoretical interpretation we can state the *linear ordering problem* as follows.

Given a complete digraph $D_n = (V, A_n)$ with arc weights c_{ij} for all $(i,j) \in A_n$, find a spanning acyclic tournament (V, T) in D_n such that

$$c(T) := \sum_{(i,j)\in T} c_{ij}$$

is as large as possible. This problem is NP-complete, cf. Garey and Johnson [1979].

For ease of notation, whenever we shall use the word tournament in the sequel we shall mean the arc set of a spanning tournament.

The linear ordering problem is sometimes also called the *permutation problem* (Young (1979)) or the *triangulation problem* (Korte and Oberhofer (1968), (1969)) and is closely related to the *feedback arc set problem* and the *acyclic subgraph problem*, see Grötschel, Jünger and Reinelt (1985) for a discussion of these relations, and see Lenstra (1973), Marcotorchino and Michaud (1979) and Wessels (1981) for real world applications of the linear ordering problem in triangulation of input-output matrices, scheduling (minimizing average weighted completion time), sports, archeology, social sciences, and psychology.

In subsequent constructions we will frequently have to manipulate acyclic tournaments. The following notation will be convenient: $\langle i_1, i_2, \ldots, i_n\rangle$ denotes the arc set of the acyclic tournament $\{(i_j, i_k) \mid j < k\}$, i.e. $\langle i_1, i_2, \ldots, i_n\rangle$ is the acyclic tournament induced by the linear ordering defined by the mapping $\sigma(j) = i_j$ for $j = 1, \ldots, n$.

2. Dimension, valid inequalities

Let $D_n = (V, A_n)$ be the complete digraph of order n, and set

$$\mathcal{A}_n := \{A \subseteq A_n \mid A \text{ is acyclic}\}, \tag{2.1}$$

$$\mathcal{T}_n := \{T \subseteq A_n \mid T \text{ is an acyclic tournament}\}. \tag{2.2}$$

Clearly, $\mathcal{T}_n \subseteq \mathcal{A}_n$ and for every $A \in \mathcal{A}_n$ there exists a $T \in \mathcal{T}_n$ with $A \subseteq T$. Given weights c_{ij} for every arc $(i,j) \in A_n$, then the acyclic subgraph problem (for D_n) is to solve $\max\{c(A) \mid A \in \mathcal{A}_n\}$ while the linear ordering problem can be stated as $\max\{c(T) \mid T \in \mathcal{T}_n\}$. In the following way polytopes can be associated with the acyclic subgraph problem and the linear ordering problem.

Let $\mathbb{R}^m$, $m := |A_n| = n(n-1)$, denote the real vector space where every component of a vector $x \in \mathbb{R}^m$ is indexed by an arc $(i,j) \in A_n$. For convenience we write x_{ij} instead of $x_{(i,j)}$. For every arc set $A \subseteq A_n$ the incidence vector $x^A \in \mathbb{R}^m$ of A is defined as follows: $x^A_{ij} = 1$, if $(i,j) \in A$, and $x^A_{ij} = 0$, if $(i,j) \notin A$. The *acyclic subgraph polytope* P^n_{AC} on D_n is the convex hull of the incidence vectors of all acyclic arc set in D_n, i.e.

$$P^n_{\mathrm{AC}} = \operatorname{conv}\{x^A \in \mathbb{R}^m \mid A \in \mathcal{A}_n\}. \tag{2.3}$$

(This polytope is denoted $P_{\mathrm{AC}}(D_n)$ in Grötschel, Jünger and Reinelt (1985). We use here the shorter notation (2.3).) Similarly, the *linear ordering polytope* P^n_{LO} on

D_n is the convex hull of the incidence vectors of all acyclic tournaments in D_n, i.e.

$$P_{LO}^n = \text{conv}\{x^T \in \mathbb{R}^m \mid T \in \mathcal{T}_n\}. \tag{2.4}$$

Thus, every vertex of P_{AC}^n resp. P_{LO}^n corresponds to an acyclic arc set resp. acyclic tournament in D_n and vice versa.

In order to be able to apply linear programming techniques to solve the linear ordering problem, we try to find a nonredundant system of equations and inequalities which is as large as possible and satisfies $P_{LO}^n \subseteq \{x \in \mathbb{R}^m \mid Ax \leq b,\ Dx = d\}$.

First we want to determine the dimension of P_{LO}^n and to find a minimal equation system for P_{LO}^n.

(2.5) Theorem. *Let $n \geq 2$, then the system*

$$x_{ij} + x_{ji} = 1 \quad \text{for all } i, j \in V,\ i \neq j, \tag{2.6}$$

is a minimal equation system for P_{LO}^n.

Proof. We have to prove (a) that every incidence vector of an acyclic tournament satisfies the equation system (2.6), (b) that the matrix defined by (2.6) has full rank, and (c) that every other equation $d^Tx = d_0$ with $P_{LO}^n \subseteq \{x \mid d^Tx = d_0\}$ can be written as a linear combination of the equations (2.6).

By definition, if i, j are two different nodes of a tournament T, then T contains arc (i, j) or arc (j, i) but not both, thus every incidence vector of a (acyclic) tournament satisfies the equations (2.6). This proves (a). The proof of (b) is even more obvious.

To prove (c) we assume that $P_{LO}^n \subseteq \{x \mid d^Tx = d_0\}$ where d is a nonzero vector in $\mathbb{R}^m$. We first show that d satisfies $d_{ij} = d_{ji}$ for all $i \neq j$. Let $i \neq j$ be any two nodes in V, then the incidence vectors of the acyclic tournaments $T_i := \langle i, j, \alpha \rangle$ and $T_j := \langle j, i, \alpha \rangle$, where α is a linear ordering of $V \backslash \{i, j\}$, satisfy $d^Tx^{T_i} = d^Tx^{T_j} = d_0$. Hence $0 = d^Tx^{T_i} - d^Tx^{T_j} = d_{ij} - d_{ji}$ which implies

$$d_{ij} = d_{ji} \quad \text{for all } i \neq j.$$

Let $a_{ij}^T x = 1$ denote the equation $x_{ij} + x_{ji} = 1$ for $i < j$, then we obtain from the relation above that $d = \sum_{i<j} d_{ij} a_{ij}$ (implying $d_0 = \sum_{i<j} d_{ij}$) holds, and we are done. □

(2.7) Corollary. *For $n \geq 2$,*

$$\dim P_{LO}^n = \binom{n}{2}. \quad \square$$

The proof of Theorem (2.5) shows in addition that the equation system (2.6) is also a minimal equation system for the *tournament polytope*, i.e. the convex hull of the incidence vectors of the tournaments contained in D_n. (The tournament polytope is obviously the polytope associated with the bases of a partition matroid on A_n, and a complete and nonredundant linear description of this polytope is given by the equation system (2.6) and the nonnegativity constraints.)

Acyclic tournaments are by definition exactly the acyclic arc sets in A_n which for every two nodes $i \neq j$ contain one of the arcs (i, j) or (j, i) but not both. Thus, the incidence vectors of acyclic tournaments are exactly those vertices of P^n_{AC} satisfying the equations (2.6). Since the inequalities $x_{ij} + x_{ji} \leq 1$ for all $i \neq j$ are valid inequalities for P^n_{AC} we can conclude

(2.8) Remark. For $n \geq 2$, the linear ordering polytope P^n_{LO} is a $\binom{n}{2}$-dimensional face of the acyclic subgraph polytope P^n_{AC}. □

Remark (2.8) has an important consequence. Namely, every inequality valid with respect to P^n_{AC} is also valid with respect to P^n_{LO}, and moreover, every complete system for P^n_{AC} induces a complete system for P^n_{LO}. It is, however, not true—as we shall see—that every inequality defining a facet of P^n_{AC} also defines a facet of P^n_{LO}.

We now describe the classes of inequalities valid for P^n_{AC} which were introduced in Grötschel, Jünger and Reinelt (1985). All these inequalities define facets of P^n_{AC}. In the next section we shall investigate which of these inequalities define facets of P^n_{LO}.

By definition, an acyclic arc set contains no dicycle. This implies that the intersection of the arc set of every dicycle C with every acyclic arc set contains at most $|C|-1$ arcs. This immediately implies that the inequalities

$$x(C) := \sum_{(i,j) \in C} x_{ij} \leq |C| - 1, \quad C \text{ a dicycle in } A_n \tag{2.9}$$

are valid with respect to P^n_{AC} and P^n_{LO}. If C is a k-dicycle we call $x(C) \leq k-1$ a *k-dicycle inequality.*

For every integer $k \geq 3$ a digraph $D = (V, A)$ of order $2k$ is called a *simple k-fence* if V consists of two disjoint node sets $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_k\}$ such that

$$A = \bigcup_{i=1}^{k} (\{(u_i, w_i)\} \cup \{(w_i, v) \mid v \in U \setminus \{u_i\}\}).$$

The nodes in U are called the *upper nodes*, those in W the *lower nodes*. The arcs (u_i, w_i) going 'down' are called *pales*, the arcs (w_i, u_j), $i \neq j$, going 'up' are called *pickets*, see Fig. 2.1 for simple 3-fence.

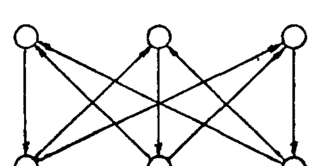

Fig. 2.1.

A simple k-fence is a particular orientation of the complete bipartite graph $K_{k,k}$. IF $A \subseteq A_n$ is the arc set of a simple k-fence, then

$$x(A) \leq k^2 - k + 1 = |A| - k + 1 \tag{2.10}$$

is called a *simple k-fence inequality.* All simple k-fence inequalities are valid with respect to P^n_{AC} and thus with respect to P^n_{LO}.

Let $C_1, C_2, \ldots, C_k$ be a sequence of different dicycles in the complete digraph D_n such that the following holds:

(2.11) $k \geq 3$ and k is odd.

(2.12) C_i and C_{i+1} $(i = 1, \ldots, k-1)$ have a directed path P_i in common, C_1 and C_k have a dipath P_k in common.

(2.13) Given any dicycle C_j, $j \in \{1, \ldots, k\}$ set

$$J = \{1, \ldots, k\} \cap (\{j-2, j-4, j-6, \ldots\} \cup \{j+1, j+3, j+5, \ldots\}).$$

Then every set $(\bigcup_{i=1}^{k} C_i) \setminus \{e_i \mid i \in J\}$ contains exactly one dicycle (namely C_j), where e_i, $i \in J$, is any arc contained n the dipath P_i.

(2.14) The cardinality of every smallest feedback arc set in $\bigcup_{i=1}^{k} C_k$ is $(k+1)/2$ (or equivalently the largest acyclic arc set has cardinality $|\bigcup_{i=1}^{k} C_i| - (k+1)/2$.)

Then we call the arc set $M = \bigcup_{i=1}^{k} C_i$ a *Möbius ladder.* For convenience we say that the dicycles C_i, C_{i+1}, $i = 1, \ldots, k-1$ and C_1, C_k are *adjacent* (with respect to M.)

Assumption (2.14) implies immediately that for any Möbius ladder M contained in D_n the equality

$$x(M) \leq |M| - \frac{k+1}{2} \tag{2.15}$$

is valid with respect to P^n_{AC} and thus also with respect to P^n_{LO}.

The inequalities (2.10) can be generalized as follows. If $D = (V, A)$ is a digraph, (i, k) an arc of D and j a node not in D, then the digraph $D' = (V', A')$ with $V' := V \cup \{j\}$, $A' := (A \setminus \{(i, k)\}) \cup \{(i, j), (j, k)\}$ is called the digraph obtained by subdividing arc (i, k).

A digraph $D = (V, A)$ is called *k-fence* if it can be obtained from a simple k-fence by repeated subdivision of arcs.

For an arc set A in $D_n = (V, A)$ let $V(A) \subseteq V$ denote the set of nodes in D_n occuring as endnodes of arcs in A. Then the following inequalities are valid with respect to P^n_{AC} and P^n_{LO}:

$$x(A) \leq |A| - k + 1 \quad \text{for all } k\text{-fences } (V(A), A). \tag{2.16}$$

Clearly, the inequalities (2.10) are special cases of (2.16). A main result of Grötschel, Jünger and Reinelt (1985) is that all inequalities (2.9), (2.15), (2.16) define facets of P^n_{AC}.

Let $G_n = [V, E]$ denote the skeleton of P^n_{LO}, i.e. G_n is a graph whose nodes are the vertices of P^n_{LO}, and two nodes are adjacent in G_n if and only if they are adjacent (as vertices) on P^n_{LO}. It is well-known in algebra that all $n!$ permutations can be obtained by starting with any permutation, applying a transposition and continuing this procedure further $(n!-2)$ times. One can easily show that the incidence vectors of two permutations (linear orderings) obtained by a transposition from each other are adjacent on P^n_{LO}, cf. Young (1978). Thus we can conclude that G_n is hamiltonian. Moreover, Young (1978) showed the deeper result that G_n has diameter two, thus in principle, it is possible to reach any vertex of P^n_{LO} from any other vertex in at most two steps walking along edges of P^n_{LO}.

3. Facets of P^n_{LO}

We shall now determine which of the inequalities (2.9), (2.15), (2.16) define facets of the linear ordering polytope. We start by proving a useful lemma.

(3.1) Trivial-Lifting Lemma. *Suppose $a^Tx \leq a_0$ defines a facet of P^n_{LO}, $n \geq 2$. Setting $\bar{a}_{ij} := a_{ij}$ for all $(i,j) \in A_n$ and $\bar{a}_{i,n+1} := \bar{a}_{n+1,i} := 0$ for $i = 1, \dots, n$ then $\bar{a}^Tx \leq a_0$ defines a facet of P^{n+1}_{LO}.*

Proof. First note that a set of vectors in P^n_{LO} is affinely independent if and only if it is linearly independent since the affine hull of P^n_{LO} does not contain the zero vector. Moreover, it is obvious that $\bar{a}^Tx \leq a_0$ is valid with respect to P^{n+1}_{LO} and is not satisfied by all vectors $x \in P^{n+1}_{\mathrm{LO}}$ with equality. It remains to prove that P^{n+1}_{LO} contains $d_1 = \dim P^{n+1}_{\mathrm{LO}} = \binom{n+1}{2}$ linearly independent vectors satisfying $\bar{a}^Tx \leq a_0$ with equality.

Since $a^Tx \leq a_0$ defines a facet of P^n_{LO} there are $d := \dim P^n_{\mathrm{LO}} = \binom{n}{2}$ linearly independent incidence vectors of acyclic tournaments $T_1, T_2, \dots, T_d$ in D_n satisfying this inequality with equality. Considering the incidence vector x^{T_i}, $i = 1, \dots, d$ as the i-th row of a $(d, n(n-1))$-matrix M' then the linear independence of the vectors implies the existence of a nonsingular (d, d)-submatrix M of M'. Let $B \subseteq A_n$ denote the set of d arcs of D_n corresponding to the columns of M.

We construct $d_1 = d + n$ acyclic tournaments T'_j, $j = 1, \dots, d$, and S_i, $i = 1, \dots, n$ of D_{n+1} as follows. If T_j, $j \in \{1, \dots, d\}$ is given by the linear ordering $\langle i_1, i_2, \dots, i_n \rangle$ then set $T'_j := \langle n+1, i_1, \dots, i_n \rangle$. Moreover, choose any acyclic tournament, say S, in D_n whose incidence vector satisfies $a^Tx \leq a_0$ with equality. Assume $S = \langle j_1, j_2, \dots, j_n \rangle$ then set $S_i := \langle j_1, j_2, \dots, j_i, n+1, j_{i+1}, \dots, j_n \rangle$, $i = 1, \dots, n$. The incidence vectors of the acyclic tournaments T'_j, S_i in D_{n+1} satisfy $\bar{a}^Tx \leq a_0$ with equality by construction.

Now consider the following (d_1, d_1)-matrix N. The first d rows of N are formed by the incidence vectors of the tournaments $T'_1, T'_2, \dots, T'_d$, and the last n rows $d+1, d+2, \dots, d_1$ are formed by the incidence vectors of the tournaments $S_1, S_2, \dots, S_n$. the first d columns of N correspond to the arc set $B \subseteq A_n$ defined above, and the last n columns to the arcs $(i, n+1)$, $i = 1, \dots, n$.

Clearly, the principal (d, d)-submatrix of N is the nonsingular matrix M. The (d, n)-submatrix N_1 consisting of the first d rows and the last n columns of N is a zero matrix by construction. The (n, n)-submatrix N_2 consisting of the last n rows and last n columns of N has the following form

$$N_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 0 \\ 1 & & \cdots & 1 \end{pmatrix}$$

and thus is nonsingular. M and N_2 nonsingular and N_1 a zero matrix implies that N is nonsingular.

Hence, we have shown that the incidence vectors of the d_1 acyclic tournaments T'_j, S_i are linearly independent which implies that $\bar{a}^T x \leq a_0$ defines a facet of P_{LO}^{n+1}. □

Lemma (3.1) implies the interesting fact that whenever we know that an inequality $\sum a_{ij} x_{ij} \leq a_0$ defines a facet of P_{LO}^n then the same inequality also defines a facet for all linear ordering polytopes P_{LO}^k, $k \geq n$, in other words, a linear ordering polytope 'inherits' all facets of linear ordering polytopes of lower dimension.

The trivial inequalities, i.e. the hypercube constraints $0 \leq x_{ij} \leq 1$, define facets of P_{LO}^n. However, the classes of facets given by the upper resp. lower bounds are identical.

(3.2) Proposition. *Let $n \geq 2$, then the following holds.*

(a) For all $(i, j) \in A_n$, $x_{ij} \geq 0$ defines a facet of P_{LO}^n. No two of these facets are equivalent with respect to P_{LO}^n.

(b) For all $(i, j) \in A_n$, $x_{ij} \leq 1$ defines a facet of P_{LO}^n. No two of these facets are equivalent with respect to P_{LO}^n.

(c) Two inequalities $x_{ij} \geq 0$ and $x_{pq} \leq 1$ are equivalent with respect to P_{LO}^n if and only if $p = j$, $q = i$.

Proof. It is trivial to see that $x_{ij} \geq 0$ defines a facet of P_{LO}^2. Thus lemma (3.1) implies that the nonnegativity constraints give facets of P_{LO}^n for all $n \geq 2$. The nonequivalence of two different nonnegativity constraints is obvious. This proves (a).

Since P_{LO}^n is contained in the affine space defined by the equation system (2.6), we get that $x_{ij} = 0$ holds if and only if $x_{ji} = 1$ holds. From this and (a), (c) and (b) follow immediately. □

(3.3) Theorem. *Let $n \geq 3$, then for every 3-dicycle $\{(i, j), (j, k), (k, i)\}$ contained in D_n the 3-dicycle inequality*

$$x_{ij} + x_{jk} + x_{ki} \leq 2$$

defines a facet of P_{LO}^n.

Proof. This is trivial to show for P_{LO}^3 and follows for P_{LO}^n, $n \geq 3$, by Lemma (3.1). □

It was shown in Grötschel, Jünger and Reinelt (1985) that all k-dicycle inequalities, $2 \leq k \leq n$, define facets of the acyclic subgraph polytope P^n_{AC}. We shall see now that the 3-dicycle inequalities are the only ones of this class that also define facets of P^n_{LO}. We first prove a lemma.

(3.4) Lemma. *Suppose $a^T x \leq a_0$, $a \geq 0$, defines a facet of P^n_{LO}, $n \geq 3$, and let (V, A) be the subgraph of $D_n = (V, A_n)$ induced by a, i.e. $A = \{(i, j) \in A_n \mid a_{ij} \neq 0\}$. Suppose (V, A) contains a node j which is contained in exactly two arcs (i, j), $(j, k) \in A$ with $i \neq k$. Then every vector $x \in P^n_{\mathrm{LO}}$ satisfying $a^T x \leq a_0$ with equality also satisfies the* 3-dicycle inequality $x_{ij} + x_{jk} + x_{ki} \leq 2$ *with equality.*

Proof. Let $T(a)$ denote the set of acyclic tournaments whose incidence vectors satisfy $a^T x \leq a_0$ with equality. We first prove that $a_{ij} = a_{jk}$.

Suppose $a_{ij} < a_{jk}$. First observe that each tournament $T \in T(a)$ contains at least one of the arcs (i, j) and (j, k), for otherwise the tournament T' resulting from T by moving j to the last position yields $a^T x^{T'} > a_0$.

Since $a^T x \leq a_0$ defines a facet there exists $T = \langle v_1, \ldots, v_n \rangle \in T(a)$ with $(i, j) \in T$ but $(j, k) \notin T$. And since the outdegree of j is one we may even assume that $v_n = j$. Setting $S = \langle v_n, v_1, \ldots, v_{n-1} \rangle$ we obtain $a^T x^S - a^T x^T = a_{jk} - a_{ij} > 0$; but this contradicts the assumption that $a^T x \leq a_0$ is valid. Similarly, we can prove that a_{ij} is not larger than a_{jk}, and hence $a_{ij} = a_{jk}$, say $\alpha := a_{ij} = a_{jk} > 0$.

Let $b^T x = \alpha$ denote the equation $\alpha(x_{ik} + x_{ki}) = \alpha$ which is satisfied by all $x \in P^n_{\mathrm{LO}}$. Then the inequality $a^T x + b^T x \leq a_0 + \alpha$ is equivalent to $a^T x \leq a_0$ with respect to P^n_{LO}. Now denote the 3-dicycle inequality $\alpha(x_{ij} + x_{jk} + x_{ki}) \leq 2\alpha$ by $c^T x \leq 2\alpha$, then we obtain that $a^T x \leq a_0$ is equivalent to the inequality $c^T x + d^T x \leq 2\alpha + a_0 - \alpha$, where d is a vector arising from a by setting

$$\begin{aligned} d_{pq} &:= a_{pq} \quad \text{for all } (p, q) \in A_n \setminus \{(i, j), (j, k), (i, k)\} \\ d_{ij} &:= d_{jk} := 0, \; d_{ik} := a_{ik} + \alpha. \end{aligned} \qquad 3.5)$$

It is easy to deduce from the validity of $a^T x \leq a_0$ that $d^T x \leq a_0 - \alpha$ is valid for P^n_{LO}. Hence we obtain that $a^T x \leq a_0$ is equivalent to an inequality which is the sum of two valid inequalities one of which is the 3-dicycle inequality $c^T x \leq 2\alpha$ and the other $d^T x \leq a_0 - \alpha$. This implies that for every vector $x \in P^n_{\mathrm{LO}}$, $a^T x \leq a_0$ is satisfied with equality if and only if both $c^T x \leq 2\alpha$ and $d^T x \leq a_0 - \alpha$ are satisfied with equality. this proves the lemma. □

Lemma (3.4) together with theorem (3.3) implies the following:

(3.6) Corollary. *If $a^T x \leq a_0$, $a \geq 0$, is a valid inequality with respect to P^n_{LO}, $n \geq 3$, such that the subdigraph (V, A) of D_n induced by a contains a node j contained in exactly two arcs (i, j), $(j, k) \in A$, $i \neq k$, then $a^T x \leq a_0$ is either equivalent to the 3-dicycle inequality $x_{ij} + x_{jk} + x_{ki} \leq 2$ or does not define a facet of P^n_{LO}.* □

(3.7) Corollary. *No k-dicycle inequality* (2.9), $k \geq 4$, *defines a facet of* P^n_{LO}.

Proof. Observe that the subdigraph induced by a k-dicycle inequality, $k \geq 4$, contains nodes j as required in (3.6) and that no k-dicycle inequality is equivalent to a 3-dicycle inequality. $\square$

Moreover, lemma (3.4) shows that facet lifting by subdivision of arcs does not work with respect to P^n_{LO}, namely:

(3.8) Corollary. *Suppose* $a^T x \leq a_0$, $a \geq 0$, *defines a facet of* P^n_{LO}, $n \geq 3$, *and let* (i, j) *be an arc of* A_n *with* $a_{ij} > 0$. *Set* $\bar{a}_{pq} := a_{pq}$ *for all* $(p, q) \in A_n \setminus \{(i, j)\}$, $\bar{a}_{ij} := 0$, $\bar{a}_{i,n+1} := \bar{a}_{n+1,j} := a_{ij}$, *and* $\bar{a}_{p,n+1} := \bar{a}_{n+1,p} := 0$ *for all* $p \neq i, j, n+1$, *then* $\bar{a}^T x \leq a_0 + a_{ij}$ *is valid for* P^{n+1}_{LO} *but does not define a facet of* P^{n+1}_{LO}. $\square$

The procedure with which $\bar{a}^T x \leq a_0 + a_{ij}$ is obtained from $a^T \leq a_0$ is called subdivision of an arc. As shown in Grötschel, Jünger and Reinelt (1985), for the acyclic subgraph polytope P^n_{AC} this subdivision method always produces facet defining inequalities from (nontrivial) facet defining ones. By Corollary (3.8) this is not true for the linear ordering polytope P^n_{LO}.

(3.9) Theorem. *Let* $n \geq 6$ *and let* $A \subseteq A_n$ *be the arc set of a simple k-fence,* $k \geq 3$. *Then the simple k-fence inequality*

$$x(A) \leq k^2 - k + 1$$

defines a facet of P^n_{LO}.

Proof. Suppose (V, A) is the given simple k-fence in D_n. If we can show that $x(A) \leq k^2 - k + 1$ defines a facet of P^{2k}_{LO}, i.e. of the linear ordering polytope on the complete graph with node set V, then the trivial-lifting lemma (3.1) implies that the simple k-fence inequality also defines a facet of P^n_{LO}.

Thus we may assume that $n = 2k$, and for notational convenience we assume that $V = \{1, 2, \ldots, n\}$, $U = \{1, \ldots, k\}$, $W = \{k+1, \ldots, n\}$ where U resp. W are the set of upper resp. lower nodes of the simple k-fence (V, A). Moreover, we denote the minimal equation system (2.6) for P^n_{LO} by $Hx = 1$ where H is a $(\binom{n}{2}, 2\binom{n}{2})$-matrix of full rank.

Let F be the face of P^n_{LO} defined by the k-fence inequality, i.e. $F = \{x \in P^n_{\mathrm{LO}} \mid x(A) = k^2 - k + 1\}$. To prove the theorem, we assume that there is an inequality $b^T x \leq b_0$ valid with respect to P^n_{LO} such that $F \subseteq G := \{x \in P^n_{\mathrm{LO}} \mid b^T x = b_0\}$. If we can show that there are a number $\mu \geq 0$ and a vector $\lambda \in \mathbb{R}^{\binom{n}{2}}$ such that $b^T = \mu a^T + \lambda^T H$, where $a^T x := x(A)$, then we are done.

By adding to b an appropriate multiple of the row of H corresponding to $x_{ij} + x_{ji} = 1$ we can make sure that either $b_{ij} = a_{ij}$ or $b_{ji} = a_{ji}$ for all $(i, j) \in A_n$. Thus, since (V, A) is an orientation of the complete bipartite graph $K_{k,k}$ and therefore contains no antiparallel arcs and since U and W are stable node sets, we may assume that our initial vector b satisfies

$$b_{ij} = a_{ij} \quad \text{for all } (i,j) \in A, \tag{1}$$

$$b_{ij} = a_{ij} = 0 \quad \text{for all arcs } (i,j) \in A_n \text{ with } 1 \leq i < j \leq k \text{ or } k+1 \leq i < j \leq n. \tag{2}$$

It is known, cf. Grötschel, Jünger and Reinelt (1985, Prop. (2.11)), that an acyclic arc set $B \subseteq A$ satisfies $a^T x^B = k^2 - k + 1$ if it either contains one pale and all pickets, or two pales, say $(i, k+i)$ and $(j, k+j)$ and all pickets except for one of the two pickets $(k+i, j)$, $(k+j, i)$.

An acyclic tournament containing the pale $(k, 2k)$ and all pickets is $T = \langle \pi, k, 2k, \sigma \rangle$ where π is any linear ordering of $k+1, \ldots, 2k-1$ and σ is any ordering of $1, \ldots, k-1$. Hence $a^T x^T = k^2 - k + 1$ and therefore, by our assumption $F \subseteq G$, $b^T x^T = b_0$. Now suppose $T_1 = \langle \pi, k, 2k, 1, \ldots, k-2, k-1 \rangle$ and $T_2 = \langle \pi, k, 2k, 1, \ldots, k-1, k-2 \rangle$ then $0 = b^T x^{T_1} - b^T x^{T_2} = b_{k-2,k-1} - b_{k-1,k-2}$. By (2) we have $b_{k-2,k-1} = 0$ and hence $b_{k-1,k-2} = 0$. With the same argument we obtain

$$b_{ji} = a_{ji} = 0 \quad \text{for all } 1 \leq i < j \leq k. \tag{3}$$

Similarly we can consider the acyclic tournaments

$$T_3 = \langle k+1, k+2, \ldots, 2k-1, k, 2k, \sigma \rangle,$$

$$T_4 = \langle k+2, k+1, k+3, \ldots, 2k-1, k, 2k, \sigma \rangle$$

whose incidence vectors x^{T_3} and x^{T_4} are in F and therefore also in G. Again we get $0 = b^T x^{T_3} - b^T x^{T_4} = b_{k+1,k+2} - b_{k+2,k-1}$ which, since $b_{k+1,k+2} = 0$ by (2), implies $b_{k+2,k+1} = 0$. Repeating this argument yields

$$b_{ji} = a_{ji} = 0 \quad \text{for all } k+1 \leq i < j \leq 2k. \tag{4}$$

Now consider the following three acyclic tournaments:

$$S_1 = \langle k+1, k+2, \ldots, 2k-2, k-1, 2k-1, k, 2k, 1, 2, \ldots, k-2 \rangle,$$

$$S_2 = \langle k+1, k+2, \ldots, 2k-2, 2k-1, k, 2k, k-1, 1, 2, \ldots, k-2 \rangle,$$

$$S_3 = \langle k+1, k+2, \ldots, 2k-2, 2k, k-1, 2k-1, k, 1, 2, \ldots, k-2 \rangle.$$

By construction S_1 contains two pales and all but one picket, S_2 and S_3 contain one pale and all pickets. Thus the incidence vectors of S_1, S_2 and S_3 are in F and therefore in G. By taking differences we get

$$0 = b^T x^{S_2} - b^T x^{S_3} = b_{2k-1,2k} + b_{2k-1,k-1} + b_{k,2k} + b_{k,k-1} - b_{2k,2k-1} - b_{k-1,2k-1} - b_{2k,k} - b_{k-1,k}.$$

Six of the eight values b_{ij} are known from (1), ..., (4), and hence we obtain $b_{2k-1,k-1} = b_{2k,k}$. Let us set $\alpha := b_{2k,k}$. Repeating this argument we obtain that each arc in A_n antiparallel to a pale has b-value α, i.e.

$$b_{k+i,i} = \alpha, \quad i = 1, \ldots, k. \tag{5}$$

We now show that each arc in A_n antiparallel to a picket has b-value α. Namely, observing (1), ..., (5), consider

$$0 = b^T x^{S_1} - b^T x^{S_2} = b_{k-1,2k-1} + b_{k-1,k} + b_{k-1,2k} - b_{2k-1,k-1} - b_{k,k-1} - b_{2,k,k-1}$$
$$= 1 + 0 + b_{k-1,2k} - \alpha - 0 - 1,$$

then we have $b_{k-1,2k} = \alpha$; which by analogous arguments implies

$$b_{ij} = \alpha \quad \text{for } 1 \leqslant i \leqslant k, \ k+1 \leqslant j \leqslant n, \ j \neq k+i. \tag{6}$$

Let $\lambda \in \mathbb{R}^{\binom{n}{2}}$ be the vector defined by $\lambda_{ij} = 0$ if $1 \leqslant i < j \leqslant k$ or $k+1 \leqslant i < j \leqslant n$, $\lambda_{ij} = \alpha$ else, then (1), ..., (6) imply

$$b^T = (1-\alpha)a^T + \lambda^T H. \tag{7}$$

Now consider the acyclic tournament S_2 (which has one pale) and the reverse tournament $\bar{S}_2$. Since $x^{\bar{S}_2} \in P^n_{\mathrm{LO}}$ we get

$$b^T x^{S_2} = b_0 = k(k-1) + 1 + (k-1)\alpha \geqslant b^T x^{\bar{S}_2} = (k(k-1)+1)\alpha + k - 1.$$

This implies $\alpha \leqslant 1$, and therefore, (7) is the desired representation of b. This completes the proof. □

The following observations that follow from Corollary (3.6) are immediate.

(3.10) Remark. (a) The simple k-fence inequalities are the only ones in the class of k-fence inequalities (2.16) that define facets of P^n_{LO}. (b) No two different simple k-fence inequalities are equivalent with respect to P^n_{LO}.

Corollary (3.6) also implies that a Möbius ladder having a node of indegree and outdegree equal to one cannot induce a facet of P^n_{LO}. Let us therefore call a Möbius ladder, cf. (2.11), .., (2.15), *simple* if the digraph $(V(M), M)$ does not contain a node with indegree and outdegree equal to one. We now show

(3.11) Theorem. *Let M be the arc set of a simple Möbius ladder in D_n consisting of $k \geqslant 3$ dicycles $C_1, C_2, \ldots, C_k$ of length four such that each pair of adjacent dicycles C_i, C_{i+1}, $i = 1, \ldots, k-1$, C_1, C_k intersects in exactly one arc, say (a_i, b_i), $i = 1, \ldots, k$, and such that the arcs (a_i, b_i), $i = 1, \ldots, k$ form a matching in D_n, cf. Fig.* 3.1. *Then the simple Möbius Ladder inequality*

$$x(M) \leqslant |M| - \frac{k+1}{2} = 3k - \frac{k+1}{2}$$

defines a facet of P^n_{LO}.

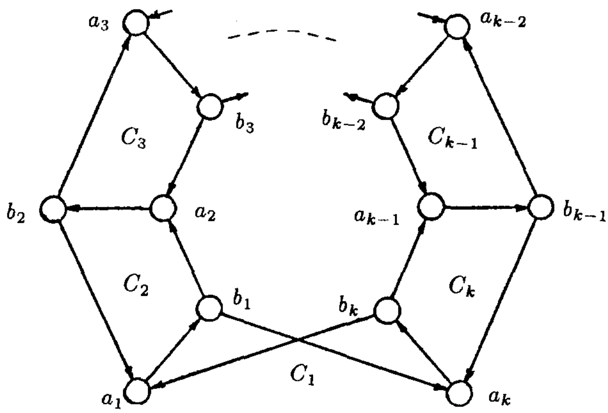

Fig. 3.1.

Proof. We shall proceed in a similar way as in the proof of theorem (3.9). In view of the trivial lifting lemma (3.1) it is sufficient to show that $x(M) \leq |M| - (k+1)/2$ defines a facet of P_{LO}^{2k}. So we assume $n = 2k$.

As before we denote the minimal equation system (2.6) for P_{LO}^n by $Hx = 1$. Denoting $c^T x = x(M)$, $c_0 = 3k - (k+1)/2$ and assuming that there is a valid inequality $d^T x \leq d_0$ for P_{LO}^n with $\{x \in P_{\mathrm{LO}}^n \mid c^T x = c_0\} \subseteq \{x \in P_{\mathrm{LO}}^n \mid d^T x = d_0\}$ it is sufficient for the proof of the theorem to show that there are a number $\mu \geq 0$ and a vector $\lambda \in \mathbb{R}^{\binom{n}{2}}$ such that $d^T = \mu c^T + \lambda^T H$.

For notational convenience, we set $\{1, 3, 5, \ldots, 2k-1\} = \{a_1, a_2, \ldots, a_k\}$, and $\{2, 4, 6, \ldots, 2k\} = \{b_1, b_2, \ldots, b_k\}$, i.e. the node set of D_n is $V = \{1, 2, \ldots, 2k\}$. This implies that the intersection of the dicycles C_i, C_{i+1}, $i = 1, \ldots, k-1$, resp. C_1, C_k is the arc $(2i-1, 2i)$, $i = 1, \ldots, k$.

It is clear that if for any pair of different nodes $i, j \in V$ we can present a partial ordering T on V whose incidence vector x^T satisfies $c^T x^T = c_0$ and which does not imply $i < j$ or $j < i$, we have shown the existence of two linear orderings on V in one of which i is directly before j and in the other j directly before i. More precisely, there are linear orderings

$$T_1 = \langle \alpha, i, j, \beta \rangle, \qquad T_2 = \langle \alpha, j, i, \beta \rangle,$$

where $\alpha \cup \beta$ consists of all nodes in $V \setminus \{i, j\}$, satisfying $c^T x^{T_i} = c_0$ and therefore $d^T x^{T_i} = d_0$, $k = 1, 2$. In such a case we have

$$0 = d_0 - d_0 = d^T x^{T_1} - d^T x^{T_2} = d_{ij} - d_{ji}$$

and therefore $d_{ij} = d_{ji}$.

With the same argument as in the proof of (3.9) we can assume that

$$\begin{aligned} d_{ij} &= c_{ij} && \text{for all } (i, j) \in M, \\ d_{ij} &= c_{ij} = 0 && \text{for all } (i, j) \in A_n \setminus \{(r, s) \mid (r, s) \in M \text{ or } (s, r) \in M\} \text{ with } i < j. \end{aligned} \tag{3.12}$$

First we show that $d_{ij}=0$ for all $(i,j)\in A\backslash\{(r,s)\,|\,(r,s)\in M$ or $(s,r)\in M\}$ (i.e. the arcs which are neither in M nor antiparallel to any arc in M). Clearly, by symmetry it is sufficient to do this for the arcs $(i,1)$, $i\in I=\{3,5,6,7,\ldots,2k-3,2k-2,2k-1\}$ and for $(j,2)$, $j\in J=\{4,5,6,7,\ldots,2k-3,2k-2,2k\}$. By the discussion above we only have to show for every pair $\{1,i\}$ resp. $\{2,j\}$ the existence of partial orderings on V which neither imply $i<1$ nor $1<i$ resp. neither $j<2$ nor $2<j$ and whose incidence vectors satisfy $c^Tx=c_0$ (and therefore $d^Tx=d_0$).

The partial ordering defined by the Möbius ladder M minus the arcs $(1,2)$, $(3,4)$, $(7,8),\ldots,(2k-7,2k-6)$, $(2k-3,2k-2)$ does this for all pairs $\{1,i\}$, $i\in I\backslash\{2k-2,2k-1\}$, and for all $\{2,j\}$, $j\in J\backslash\{2k-3,2k\}$. For $i\in\{2k-2,2k-1\}$ and $j\in\{2k-3,2k\}$ we can take the partial ordering defined by M minus the arcs $(1,2)$, $(5,6)$, $(9,10),\ldots,(2k-9,2k-8)$, $(2k-5,2k-4)$, $(2k-1,2k)$. This proves the first claim.

Now let

$$\alpha=\langle 6,10,14,\ldots,2k-8,2k-4,2k\rangle,$$

$$\beta=\langle 7,11,15,\ldots,2k-11,2k-7,2k-3\rangle,$$

$$\gamma=\langle 8,12,16,\ldots,2k-10,2k-6,2k-2\rangle,$$

$$\delta=\langle 5,9,13,\ldots,2k-9,2k-5,2k-1\rangle$$

Assume $d_{21}=\xi$ and consider the following linear orderings on V:

$$T_3=\{\alpha,\beta,1,2,3,4,\gamma,\delta\rangle,\qquad T_4=\langle\alpha,\beta,2,3,4,1,\gamma,\delta\rangle,$$

$$T_5=\langle\alpha,\beta,3,4,1,2,\gamma,\delta\rangle,\qquad T_6=\langle\alpha,\beta,4,1,2,3,\gamma,\delta\rangle.$$

In view of axiom (2.13) it is easy to verify that $c^Tx^{T_i}=c_0$ and therefore $d^Tx^{T_i}=d_0$ for $i\in\{3,4,5,6\}$. Now we have

$$\begin{aligned}0=d_0-d_0&=d^Tx^{T_3}-d^Tx^{T_4}\\&=d_{12}+d_{13}+d_{14}-d_{21}-d_{31}-d_{41}\\&=1+0+d_{14}-d_{21}-0-1\end{aligned}$$

and therefore $d_{14}=d_{21}=\xi$. By taking the other appropriate differences we get

$$d_{21}=d_{14}=d_{43}=d_{32}=\xi$$

and in the obvious way $d_{ij}=\xi$ for all arcs (i,j) such that $(j,i)\in M$

Defining $\lambda\in\mathbb{R}^{\binom{n}{2}}$ by $\lambda_{ij}=\xi$ if $(i,j)\in M$ or $(j,i)\in M$ and $\lambda_{ij}=0$ otherwise, we obtain

$$d^T=(1-\xi)c^T+\lambda^TH.$$

Now consider the acyclic tournament T_3 and the reverse tournament $\bar{T}_3$. By construction we have $d^Tx^{T_3}=d_0$. Using the results about d derived above we get $d^Tx^{T_3}=3k-(k+1)/2+\xi(k+1)/2$ and $d^Tx^{\bar{T}_3}=\xi(3k-(k+1)/2)+(k+1)/2$. Since $x^{\bar{T}_3}\in P^n_{\mathrm{LO}}$

we have by assumption $d^T x^{T_3} \leq d_0$. This implies $\xi \leq 1$, and thus for $\mu = 1 - \xi$

$$d^T = \mu c^T + \lambda^T H$$

is the desired representation. □

Using the same method of proof as above we can show the following more general result.

(3.13) Theorem. *Let M be the arc set of a simple Möbius Ladder in D_n consisting of $k \geq 3$ dicycles $C_1, \ldots, C_k$ having the following (additional) properties:*

(3.13.1) *The length of C_i is three or four, $i = 1, \ldots, k$.*

(3.13.2) *Two adjacent dicycles have exactly one arc in common.*

(3.13.3) *If two nonadjacent dicycles C_i and C_j, $i < j$, have a common node, say v, then v either belongs to all dicycles $C_i, C_{i+1}, \ldots, C_j$ or to all dicycles $C_j, C_{j+1}, \ldots, C_k, C_1, C_2, \ldots, C_i$.*

Then the Möbius Ladder inequality

$$x(M) \leq |M| - \frac{k+1}{2}$$

defines a facet of P_{LO}^n.

The proof of Theorem (3.13) is not more complicated than that of Theorem (3.11). However, quite a number of notational inconveniences arise which make it technical ugly. We therefore give a sketch of the proof only.

Sketch of the proof. We start as in the proof of (3.13)assuming the existence of a valid inequality $d^T x \leq d_0$ defining a face of P_{LO}^n which contains the face defined by the Möbius ladder inequality. We can make assumption (3.12) about the coefficients of d.

First we show that for any two nodes p, q which are on a 4-dicycle C_i of M and are not adjacent on C_i, $d_{pq} = d_{qp} = 0$ holds. This is done by exhibiting a partial ordering where neither p before q nor q before p and extending these partial orderings to linear orderings appropriately.

Then we show that for any two nodes p, q such that (p, q) is an arc of M, i.e. $d_{pq} = 1$, we have $d_{qp} = \xi$. Here we use the same technique as in the proof of (3.11) to show that the arcs antiparallel to arcs of the Möbius ladder have d-value ξ.

Finally we show that, for any two nodes p, q of M which are not on a common cycle of M, $d_{pq} = d_{qp} = 0$ holds. For this we construct a partial ordering in which neither p before q nor q before p holds and extend this partial ordering appropriately to linear orderings. This is the most complicated construction since a number of cases depending on the 'relative location' of p and q in M have to be considered to show that such a partial ordering indeed exists. Of course all linear orderings constructed above must have the property that their incidence vectors satisfy the Möbius ladder inequality with equality. □

It is clear that all Möbius ladders satisfying the assumptions of Theorem (3.11) also satisfy the assumptions of Theorem (3.13), thus the latter theorem is more general. Figures 3.2 and 3.3 show two simple Möbius ladders which satisfy (3.13.1), ..., (3.13.3). The inequalities induced by the Möbius ladders shown in Figs 3.2 and 3.3 define facets of P^n_{LO}, n large enough.

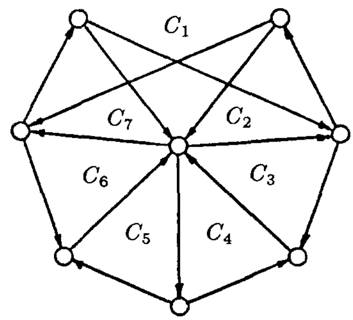

Fig. 3.2.

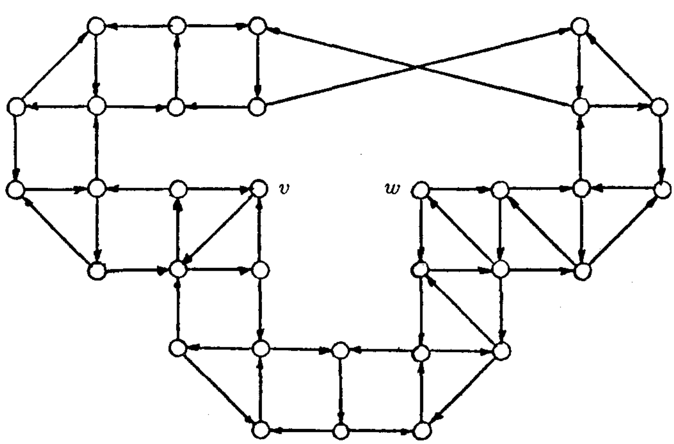

Fig. 3.3.

In Fig. 3.3 we have labeled two nodes v and w. If we identify these two nodes then the resulting graphs is still a Möbius ladder satisfying (3.13.1), (3.13.2), but not (3.13.3). We believe that our assumption (3.13.3) is not a necessary one for the result of (3.13), it only makes the technical details of the proof much easier. We in fact conjecture more generally that all simple Möbius ladders induce facets of P^n_{LO}. However, since we do not have a 'nice' constructive characterization of Möbius ladders we do not see how one can prove this.

It is easy to see that within each of the classes of facet defining inequalities of P^n_{LO} described in (3.2)(a), (3.3), (3.9), and (3.13) no two different inequalities are equivalent; moreover, no two inequalities from different classes are equivalent with one exception, namely, every simple 3-fence is a simple Möbius ladder described in (3.10) consisting of three 4-dicycles. Thus we can conclude:

(3.14) Theorem. *Let $n \geqslant 2$, and let $D_n = (V, A_n)$ be the complete digraph on n nodes. Then the following system of equations*

$$x_{ij} + x_{ji} = 1, \quad i, j \in V, \ i \neq j, \tag{3.15}$$

is a minimal equation system for the linear ordering polytope P^n_{LO}. The following four classes of inequalities define facets of P^n_{LO}

$$x_{ij} \geqslant 0 \qquad 1 \leqslant i, j \leqslant n, \tag{3.16}$$

$$x_{ij} + x_{jk} + x_{ki} \leqslant 2, \qquad 1 \leqslant i < j < k \leqslant n, \tag{3.17}$$

$$x(A) \leqslant k^2 - k + 1 \qquad \text{for all simple } k\text{-fences } A \subseteq A_n, \ k \geqslant 4, \tag{3.18}$$

$$x(M) \leqslant 3k - \frac{k+1}{2} \qquad \text{for all simple Möbius ladders } M \subseteq A_n \text{ as defined in (3.13), } k \geqslant 3. \tag{3.19}$$

The system of equations and inequalities (3.15), ..., (3.19) *is a partial nonredundant linear characterization of P^n_{LO}.* □

4. Final remarks

The partial description of P^n_{LO} given in (3.14) can be used in a linear programming-cutting plane procedure to solve linear ordering problems. We have implemented such a method and combined it with branch-and-bound-techniques. This code seems to be quite successful. We were for instance able to triangulate all input–output tables available to us. The largest dimension of such a table was $n = 60$ (see Grötschel, Jünger and Reinelt (1984b) for an economic analysis of these results). We do not know of any other code that can handle such sizes.

The description of P^n_{LO} given in (3.14) is not complete. We know some further facet defining inequalities different from those in (3.14). Nevertheless, even the partial description given by the equations (3.15), the nonnegativity constraints (3.16) and the 3-dicycle inequalities (3.17) often suffices to prove optimality in an LP-cutting plane approach. In most of our triangulation problems the optimum solution to the LP given by the constraints (3.15), ..., (3.17) was integral. Since the integral points contained in the polyhedron defined by (3.15), ..., (3.17) are exactly the incidence vectors of acyclic tournaments, an optimum linear ordering was found. A description of our code and the computational experience with it can be found in Grötschel, Jünger and Reinelt (1984a).

We can show that the linear ordering polytope is completely described by the system (3.15), ..., (3.17) for $n = 2, 3, 4, 5$. For $n = 6$ the simple Möbius ladders on

6 nodes (these are the ones shown in Fig. 4.1) have to be added, but we do not know whether these are all facets of P^6_{LO}. Actually, Bowman (1972) 'proved' that a complete description of P^n_{LO} is given by the system (3.15), (3.16) and (3.17). Of course, each of the inequalities (3.18) or (3.19) provides a counterexample to this, for $n \geq 6$, see also Young (1978).

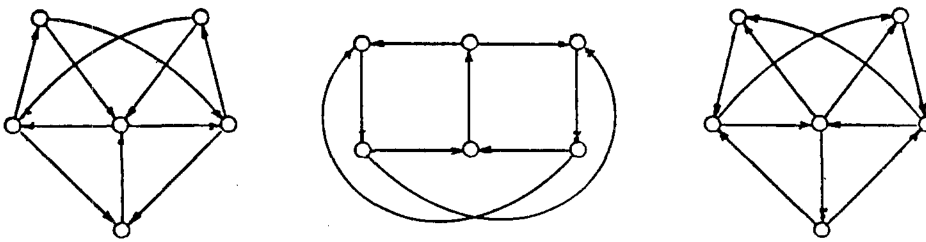

Fig. 4.1.

Because of the simple structure of the equation system (3.15) it is quite easy to eliminate one half of the variables (without losing too much insight into the structure of the inequalities etc.) simply by replacing each variable x_{ij}, $i > j$, by $1 - x_{ji}$. This way we obtain a projection $\bar{P}^n_{\mathrm{LO}}$ of P^n_{LO} contained in the space $\mathbb{R}^{\binom{n}{2}}$. $\bar{P}^n_{\mathrm{LO}}$ is a full-dimensional polytope, and each of its vertices corresponds to an acyclic tournament in D_n and vice versa. Our cutting plane procedure for the linear ordering problem, of course, uses this projection and optimizes over $\bar{P}^n_{\mathrm{LO}}$ instead of P^n_{LO}, since this is more space economical.

To give an example of such a projected polytope we have made a drawing of the polytope $\bar{P}^3_{\mathrm{LO}} \subseteq \mathbb{R}^3$, Fig. 4.2, This polytope has 6 vertices and 8 facets. Two facets are given by the 3-dicycle inequalities, all other facets are trivial.

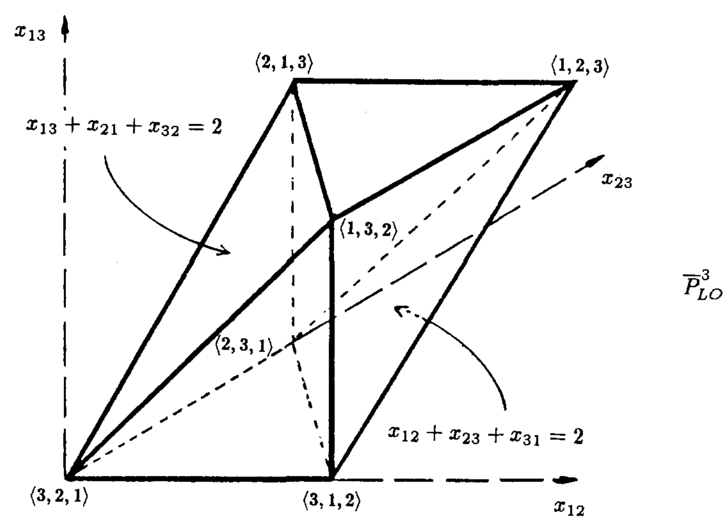

Fig. 4.2.

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