Mathematical Programming 58 (1992) 53-88
North-Holland

The graphical relaxation: A new framework for the Symmetric Traveling Salesman Polytope

Denis Naddef*

ARTEMIS Université Joseph Fourier de Grenoble and ESA Université des Sciences Sociales de Grenoble, 38041 Grenoble, France

Giovanni Rinaldi

Istituto di Analisi dei Sistemi ed Informatica del CNR, 00185 Roma, Italy

Received 19 October 1989
Revised manuscript received 10 April 1991

A present trend in the study of the *Symmetric Traveling Salesman Polytope* (STSP(n)) is to use, as a relaxation of the polytope, the *graphical relaxation* (GTSP(n)) rather than the traditional *monotone relaxation* which seems to have attained its limits. In this paper, we show the very close relationship between STSP(n) and GTSP(n). In particular, we prove that every non-trivial facet of STSP(n) is the intersection of $n+1$ facets of GTSP(n), n of which are defined by the degree inequalities. This fact permits us to define a standard form for the facet-defining inequalities for STSP(n), that we call *tight triangular,* and to devise a proof technique that can be used to show that many known facet-defining inequalities for GTSP(n) define also facets of STSP(n). In addition, we give conditions that permit to obtain facet-defining inequalities by composition of facet-defining inequalities for STSP(n) and general lifting theorems to derive facet-defining inequalities for STSP($n+k$) from inequalities defining facets of STSP(n).

Key words: Symmetric traveling salesman problem, graphical traveling salesman problem, polyhedron, facet, linear inequality, lifting, composition of inequalities.

1. Introduction and notation

Let $K_n = (V_n, E_n)$ be the complete graph on n vertices and let $\mathbb{R}^{E_n}$ represent the set of all real vectors whose components are indexed by the set E_n. We denote by $e = (u, v)$ the element of E_n having u and v as endpoints. For every real vector x in $\mathbb{R}^{E_n}$ we denote by x_e, or by $x(u, v)$, the component of x indexed by $e = (u, v)$. A *Hamiltonian cycle H* of K_n is the edge set of a connected spanning subgraph of K_n for which every node has degree 2. Given a vector $l \in \mathbb{R}^{E_n}$, which assigns the length l_e to every edge $e \in E_n$, the *Symmetric Traveling Salesman Problem* consists of finding a Hamiltonian cycle of K_n with minimum length. This is one of the most extensively studied combinatorial optimization problems; we assume the reader to be familiar

* Partially financed by P.R.C. Mathématique et Informatique.

with the basic concepts related to it. We refer to Lawler et al. (1985) for the necessary background.

Let $\mathcal{H}_n$ be the set of Hamiltonian cycles of K_n. With every $H \in \mathcal{H}_n$ we associate a unique incidence vector χ^H in $\mathbb{R}^{E_n}$ by setting

$$\chi_e^H = \begin{cases} 1 & \text{if } e \in H, \\ 0 & \text{otherwise.} \end{cases}$$

The Symmetric Traveling Salesman Polytope (STSP(n)) is the convex hull of the set of the incidence vectors of all Hamiltonian cycles of K_n, i.e.:

$$\text{STSP}(n) = \text{conv}\{\chi^H \mid H \in \mathcal{H}_n\}.$$

Let $\mathcal{P}$ be a polyhedron in $\mathbb{R}^m$; an inequality $fx \geq f_0$ defined on $\mathbb{R}^m$ is said to be *valid* for $\mathcal{P}$ if it is satisfied by all points of $\mathcal{P}$, it is said to be *supporting* for $\mathcal{P}$ if it is valid and the set $\{x \in \mathbb{R}^m \mid fx = f_0,\ x \in \mathcal{P}\}$ is nonempty, finally it is said to be *facet-defining* for $\mathcal{P}$ if it is supporting and the set $\{x \in \mathbb{R}^m \mid fx = f_0,\ x \in \mathcal{P}\}$ is a facet of $\mathcal{P}$. For the definition of facet, and for all the related basic topics of polyhedral theory we refer to Nemhauser and Wolsey (1988).

By a theorem due to Weyl (1935) it is known that there exists a finite linear system of inequalities facet-defining for STSP(n) whose set of solutions is given by STSP(n). However, it is very unlikely that such a system can be completely described and that it can be given by classes of inequalities for which there exists an *$\mathcal{NP}$-description* (see Pulleyblank (1983)). To date, only a partial description of the linear system of STSP(n) is known. However, this incomplete characterization of the polytope can be efficiently used to solve large instances of the problem to optimality by polyhedral cutting-plane algorithms (see Padberg and Grötschel (1985)). Recently, by means of polyhedral methods it has been possible to solve to optimality very large instances of the problem: Grötschel and Holland (1991) report on the solution of instances up to 1000 nodes and Padberg and Rinaldi (1987, 1991) report on the solution of some "real world" instances with up to 2392 nodes. These results show the validity of the polyhedral approach to the solution of this hard combinatorial problem, and motivate to continue the study of the associated polytope in order to enlarge its partial linear description. Besides this algorithmic issue, since the traveling salesman problem is one of the most investigated combinatorial optimization problems, the study of the linear system of STSP(n) is attracting and challenging by itself: it has interested many researchers and it began far before the successful computational results based on the polyhedral cutting-plane algorithms were obtained.

In Dantzig, Fulkerson and Johnson (1954), a class of valid inequalities, the *subtour elimination* inequalities, is introduced. In Chvátal (1973), another class of valid inequalities, the *comb inequalities*, is given. These inequalities are a generalization of the 2-*matching inequalities* defined in Edmonds (1965) where they are used to give a complete linear characterization of the 2-matching polytope. In Chvátal (1973), an inequality defined by the Petersen graph is shown to be facet-defining for STSP(10). In Maurras (1975), it is shown that the inequality defined by the graph obtained from the Petersen graph by replacing an edge and its endpoints by a clique of size $k \geq 2$ is facet-defining for STSP($n+k-2$). In Grötschel and Padberg (1979a), (1979b), the subtour elimination inequalities are proven to be facet-defining for STSP(n), $n \geq 4$. Moreover, the Chvátal comb inequalities are generalized and the members of the new class, called the *comb inequalities*, are proven to be facet-defining for STSP(n), $n \geq 6$. In Grötschel and Pulleyblank (1986), the *clique-tree inequalities* are defined and proven to be facet-defining for STSP(n), with $n \geq 11$. This new class properly contains the comb inequalities. For a complete description of all these classes of inequalities we refer to Grötschel and Padberg (1985). Very recently Boyd (1988) and Hartman (1988) have independently proved that the *chain inequalities*, defined in Padberg and Hong (1980), are facet-defining for STSP(n), $n \geq 8$. In Naddef and Rinaldi (1992), the *crown* inequalities are defined and shown to define facets of STSP(n), $n \geq 8$. In a sequel paper (Naddef and Rinaldi (1988)) we exploit the results presented here to show that some large families of inequalities define facets of STSP(n). These families generalize the comb, the clique-tree and the chain inequalities. Finally, for the sake of completeness we want to mention a few more inequalities that do not belong to the above families. These inequalities are the *ladder inequality*, introduced in Boyd and Cunningham (1991) and proven to be facet-defining for STSP(8); three inequalities described in Christof, Jünger and Reinelt (1990), that define facets of STSP(8), and two inequalities discovered by Queyranne and Wang (1989), that are facet-defining for STSP(9). At present we do not know how these inequalities can be generalized for higher values of n, even though it is likely that the extensions described in Section 4 of this paper apply to them. To our knowledge to date no other inequalities are known to be facet-defining for STSP(n).

The incidence vectors of all Hamiltonian cycles of K_n satisfy the following system of equations, called the *degree equations*:

$$A_n x = \mathbf{2},$$

where A_n denotes the node-edge incidence matrix of K_n and $\mathbf{2}$ denotes the vector in $\mathbb{R}^{V_n}$ with all the components equal to 2. Consequently, STSP(n) is not full dimensional. With two different techniques it is shown in Grötschel and Padberg (1975, 1979a) and Maurras (1975), respectively, that the dimension of STSP(n) is $|E_n| - n$; we give an alternative proof of this result in Section 2. Let $\mathcal{P}$ be a polyhedron in $\mathbb{R}^m$; we say that two inequalities defined on $\mathbb{R}^m$ are *equivalent* if they define the same face of $\mathcal{P}$. If $\mathcal{P}$ is full dimensional, then two facet-defining inequalities $f^1x \geq f_0^1$ and $f^2x \geq f_0^2$ are equivalent if one can be obtained from the other by multiplication by a positive real number. Since STSP(n) is not full dimensional two facet-defining inequalities $f^1x \geq f_0^1$ and $f^2x \geq f_0^2$ are equivalent if there exist a positive number π and a vector $\lambda \in \mathbb{R}^{V_n}$ such that $f^2 = \pi f^1 + \lambda A_n$ and $f_0^2 = \pi f_0^1 + \lambda \mathbf{2}$. This makes the recognition of two equivalent inequalities more complicated than in the case of a full dimensional polyhedron. For this reason when dealing with a polyhedron $\mathcal{P}$ which is not full dimensional, it is customary to embed it into a larger polyhedron

$\mathcal{P}_R$ such that $\mathcal{P}_R$ is full dimensional, and $\mathcal{P}$ is a face of $\mathcal{P}_R$. The polyhedron $\mathcal{P}_R$ is called a *relaxation* of $\mathcal{P}$. The usual way to proceed is to first describe inequalities that define facets of $\mathcal{P}_R$ and then look for conditions that guarantee that these inequalities, that are valid for $\mathcal{P}$, are also facet-defining for it. A desirable property that not all relaxations have, is that every facet of $\mathcal{P}$ is contained in exactly one of the facets of $\mathcal{P}_R$ which do not contain the entire $\mathcal{P}$. If this property holds, then there is a one to one correspondence between a subset of these facets of $\mathcal{P}_R$ and all facets of $\mathcal{P}$. To date, the most studied relaxation of STSP(n) is the *Monotone Traveling Salesman Polytope* (MTSP(n)) (see Grötschel and Padberg (1985)), that does not have this nice property (see Section 2).

In this paper, we make use of a different relaxation, the *Graphical Traveling Salesman Polyhedron* and show how this polyhedron is strongly related to STSP(n).

Let $G=(V, E)$ be a graph; a *family of edges* of G is a collection F of elements of E. Several copies of the same element of E may appear in the collection. For every element e of E, we call *multiplicity* of e in F the number of times e appears in F. As usual, a *set of edges* of G is a family where every element has multiplicity 1. Let F_1 and F_2 be two families of edges of G and let F_1+F_2 denote the family such that the multiplicity of every element is given by the sum of its multiplicities in F_1 and F_2, respectively. By $F+e$ and $F-e$ we denote the families for which the element e has multiplicity one more and one less than in F, respectively. Finally, $k\{e\}$ denotes the family containing only the element e with multiplicity k.

Let F be a family of edges of $G=(V, E)$. By $G[F]$ we denote the multigraph having node set V and having, for every pair of distinct nodes u and v in V, as many edges with endpoints u and v as the multiplicity of (u, v) in F. For every node v in V the *degree of v in F* is the degree of v in the multigraph $G[F]$, and the *neighbors of v in F* are the neighbors of v in the multigraph $G[F]$. With every family F of edges of G we associate a unique incidence vector $\chi^F \in \mathbb{R}^E$ by setting χ_e^F equal to the multiplicity of e in F for every $e \in E$. If c is a vector in $\mathbb{R}^E$, the *c-length* of F, also denoted by $c(F)$, is defined as $c(F)=c\chi^F$.

A *tour* of a graph $G=(V, E)$ is a family T of edges of G such that:

(i) the degree in T of every $v \in V$ is positive and even;

(ii) $G[T]$ is connected.

Observe that a Hamiltonian cycle in G is a tour where every node has degree 2, and that a tour is not in general a Hamiltonian cycle. The set of all tours of K_n is denoted by $\mathcal{T}_n^*$. While $\mathcal{H}_n$ is a finite set (since it contains $\frac{1}{2}(n-1)!$ elements), $\mathcal{T}_n^*$ contains an infinite number of elements, since if $T \in \mathcal{T}_n^*$, then $T+k\{e\} \in \mathcal{T}_n^*$ for every $e \in E_n$ and for every positive even number k. In the following, we will be mainly interested in tours which are minimal under the operation of removing edges; therefore we define the set of *minimal* tours by

$$\mathcal{T}_n=\{T \in \mathcal{T}_n^* \mid \nexists T' \in \mathcal{T}_n^*, \chi^{T'} < \chi^T\}.$$

By definition, the components of the incidence vector of a minimal tour have value 0, 1 or 2.

Now, we define two subsets of $\mathcal{H}_n$ and $\mathcal{T}_n$, respectively, that are associated with every inequality in $\mathbb{R}^{E_n}$ and that are often used in this paper. For every inequality $fx \geq f_0$ in $\mathbb{R}^{E_n}$, we call *extremal* the Hamiltonian cycles and the minimal tours whose incidence vectors satisfy the inequality with equality. The set of extremal Hamiltonian cycles and extremal tours are denoted by $\mathcal{H}_f^=$ and $\mathcal{T}_f^=$, respectively, and formally defined by

$$\mathcal{H}_f^= = \{H \in \mathcal{H}_n \mid f\chi^H = f_0\},$$

$$\mathcal{T}_f^= = \{T \in \mathcal{T}_n \mid f\chi^T = f_0\}.$$

The *Graphical Traveling Salesman Polyhedron* (GTSP(n)) is the convex hull of the set of the incidence vectors of the elements of $\mathcal{T}_n^*$, i.e.:

$$\text{GTSP}(n) = \text{conv}\{\chi^T \mid T \in \mathcal{T}_n^*\}.$$

Since $\mathcal{H}_n \subset \mathcal{T}_n^*$ it follows that STSP(n) $\subset$ GTSP(n), and so GTSP(n) is a relaxation of STSP(n).

For every graph G, the set $\mathcal{T}_G^*$ of all tours of G represents the solution set of the *Graphical Traveling Salesman Problem* defined in Cornuéjols, Fonlupt and Naddef (1985) and in Fleischmann (1988). In the first paper the polyhedral structure of the polyhedron GTSP(G) = conv$\{\chi^T \mid T \in \mathcal{T}_G^*\}$ (the convex hull of the incidence vectors of all tours of G) is investigated. It is shown that GTSP(G) is full dimensional if G is connected and that, under connectivity conditions that are always satisfied by K_n, the *cocycle inequalities*

$$x(\delta(U)) \geq 2, \quad \text{for } \emptyset \neq U \subset V,$$

where $\delta(U)$ is the cocycle $\{(u, v) \in E \mid u \in U,\ v \in V - U\}$ of the set U, are facet-defining for GTSP(G). Moreover, a family of inequalities, the *path inequalities*, is defined and proven to be facet-defining for GTSP(G). In Naddef and Rinaldi (1991) a composition of inequalities is defined. This composition yields new facet-defining inequalities from facet-defining inequalities for GTSP(G). In the same paper the new family of *path-tree inequalities* is defined and its members are shown to be facet-defining for GTSP(G).

In Section 2, we investigate the relationship between the polyhedra GTSP(n) and STSP(n), and we show that every nontrivial facet of STSP(n) is contained in exactly $n+1$ facets of GTSP(n), n of which are defined by the *degree inequalities*, i.e., by the cocyle inequalities of the subsets of V with only one element. This permits to define a special form for the inequalities facet-defining for STSP(n) that is particularly suitable for proving most of the results of the other sections and to devise a proof technique that is used to show when an inequality defines a facet of STSP(n). We call such a form, *tight triangular*. The transformation of a facet-defining inequality into the tight triangular form allows one to easily check whether or not two given inequalities define the same facet of STSP(n). In Section 3, we extend the results on the composition of facet-defining inequalities for GTSP(n), given in Naddef and Rinaldi (1991), to STSP(n). In Section 4, we give some general lifting theorems that are used to derive facet-defining inequalities for STSP(n^*), with $n^* > n$, from inequalities defining facets of STSP(n).

In the sequel paper Naddef and Rinaldi (1988) we show, using these results, that the path and the path-tree inequalities define facets of STSP(n). These inequalities generalize the comb inequalities and the clique-tree inequalities, respectively. Moreover, we give some examples of application of the results of the Sections 3 and 4 and we prove, e.g., that some new inequalities that generalize the chain inequalities are facet-defining for STSP(n).

2. The polyhedra GTSP and STSP

In this section, we show how the polyhedron GTSP(n) and the polytope STSP(n) are closely related. We start by introducing two concepts concerning the inequalities in $\mathbb{R}^{E_n}$ that are extensively used in the following: the definition of the edge set $\Delta_f(u)$ associated with an inequality $fx \geqslant f_0$ and with every node u in V_n, and the definition of *tight triangular inequality.*

For any inequality $fx \geqslant f_0$ and for each node $u \in V_n$ the edge set $\Delta_f(u) \subseteq E_n$ is defined as follows:

$$\Delta_f(u) = \{(v, w) \in E_n \mid u \neq v, u \neq w, f(v, w) = f(u, v) + f(u, w)\}.$$

The set $\Delta_f(u)$ is a key concept in our treatment: most of the results we give in the following are expressed in terms of properties of $\Delta_f(u)$ for every node u in V_n.

Definition 2.1. An inequality $fx \geqslant f_0$ defined on $\mathbb{R}^{E_n}$ is said to be *tight triangular*[1] or *in tight triangular form* if the following conditions are satisfied:

(a) The coefficients f_e satisfy the triangular inequality, i.e.: $f(u, v) \leqslant f(u, w) + f(w, v)$ for every triple u, v, w of distinct nodes in V_n.

(b) $\Delta_f(u) \neq \emptyset$ for all u in V_n.

We abbreviate *tight triangular* by TT and *tight triangular form* by TT *form.*

Almost every inequality facet-defining for GTSP(n) is tight triangular as stated in the following proposition. As customary we use the notation $\delta(u)$ instead of $\delta(\{u\})$ to denote the cocycle of the singleton $\{u\}$. We call $x_e \geqslant 0$, for $e \in E_n$, the *trivial inequalities.* The *trivial facets* are those defined by the trivial inequalities. We call *nontrivial* every inequality and every facet which is not trivial.

Proposition 2.2. *An inequality* $cx \geqslant c_0$ *facet-defining for* GTSP(n) *falls in one of the following three categories*:

(i) *trivial inequalities* $x_e \geqslant 0$ *for all* $e \in E_n$;

(ii) *degree inequalities* $x(\delta(v)) \geqslant 2$ *for all* $v \in V_n$;

(iii) *tight triangular inequalities.*

[1] In a previous version of this paper and in many talks on this subject we used the term *strongly triangular.*

Proof. Let $cx \geq c_0$ be a facet-defining inequality for GTSP(n). First, suppose that the inequality does not satisfy the condition (a) of Definition 2.1. It follows that there exists a triple u, v, w of distinct nodes in V_n with $c(u, v) > c(u, w) + c(w, v)$. But then, for every tour T in $\mathcal{T}_c^=$ containing the edge (u, v), the tour $T' = T - (u, v) + (u, w) + (v, w)$ has c-length strictly less than c_0, contradicting the assumption that the inequality is valid. Consequently, no tours of the set $\mathcal{T}_c^=$ contain the edge $e = (u, v)$, and so the inequality is the trivial inequality $x_e \geq 0$.

Now, suppose that the inequality satisfies (a) but not (b) of Definition 2.1. It follows that there exists a node u in V_n such that for every pair v, w of distinct nodes in $V_n - \{u\}$, $c(u, v) + c(u, w) > c(v, w)$, which further implies $c(u, v) > 0$ for all v in $V_n - \{u\}$. Let T be a tour of $\mathcal{T}_n^*$ where u has degree higher than 2. If T contains $2\{(u, v)\}$, let $T' = T - 2\{(u, v)\}$. Otherwise there is a pair of distinct neighbors y, z of u in T such that the edge family $T' = T - (u, y) - (u, z) + (y, z)$ is a tour. In both cases T' is a tour of $\mathcal{T}_n^*$ with $c(T') < c(T)$. Consequently in none of the tours satisfying $cx \geq c_0$ with equality the node u has degree higher than 2, and so the inequality is the degree inequality $x(\delta(u)) \geq 2$. □

It is straightforward to see that if an inequality $cx \geq c_0$ defined on $\mathbb{R}^{E_n}$ satisfies the condition (a) of Definition 2.1, then $c_e \geq 0$ for all $e \in E_n$. Hence Proposition 2.2 implies that all facet-defining inequalities for GTSP(n) have nonnegative coefficients.

From Proposition 2.2 it follows that for any nontrivial facet of STSP(n) there is an inequality defining that facet which is tight triangular.

From the proof of Proposition 2.2 we get the following observation.

Proposition 2.3. *Let $cx \geq c_0$ be a* TT *inequality facet-defining for* GTSP(n). *Then*

(a) *for every edge $e \in E_n$ there exists a tour $T \in \mathcal{T}_c^=$ such that $e \in T$;*

(b) *for every node $v \in V_n$ there exists a tour $T' \in \mathcal{T}_c^=$ such that v has degree at least* 4 *in T'.* □

Since the definition of TT inequality applies only to inequalities defined on complete graphs, Propositions 2.2 and 2.3 do not hold in general for the polyhedron GTSP(G) associated with a general graphy G.

Let T be a tour in $\mathcal{T}_c^=$ with $t > n$ edges. To prove some of the following results, we often need to derive from T a new tour T' in $\mathcal{T}_c^=$ having a smaller number of edges. For this purpose, we show how, for TT inequalities, this can be achieved by a sequence of elementary operations.

Definition 2.4. For every ordered triple $\langle u, v, w\rangle$ of distinct nodes in V_n, we call *shortcut* on $\langle u, v, w\rangle$ the vector $s_{uvw} \in \mathbb{R}^{E_n}$ defined by

$$s_{uvw}(e) = \begin{cases} 1 & \text{if } e = (v, w), \\ -1 & \text{if } e \in \{(u, v), (u, w)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.5. *Let $cx \geq c_0$ be a tight triangular inequality which is supporting for* GTSP(n), *and let $T \in \mathcal{T}_c^=$ be a tour having $t > n$ edges and containing the edge e. For* *every node $u \in V_n$ with degree $k \geqslant 4$ in T, there exists a shortcut s_{uvw} such that the edge family having incidence vector $\chi^T + s_{uvw}$ is a tour with $t-1$ edges belonging to $\mathcal{T}_c^=$ and containing the edge e. Moreover, the edge (v, w) belongs to $\Delta_c(u)$.*

Proof. Let u be a node in V_n with degree $k \geqslant 4$ in T. Since every tour of $\mathcal{T}_c^=$ contains at most twice the same edge, only three cases are possible (see Figure 1).

Case (a). The families $2\{(u, v)\}$ and $2\{(u, w)\}$, with $w \neq v$, are contained in T.

Case (b). The node u is adjacent in T to 3 distinct nodes v, w, and z and $2\{(u, v)\}$ is contained in T. Without loss of generality we assume here that $(u, w) \neq e$.

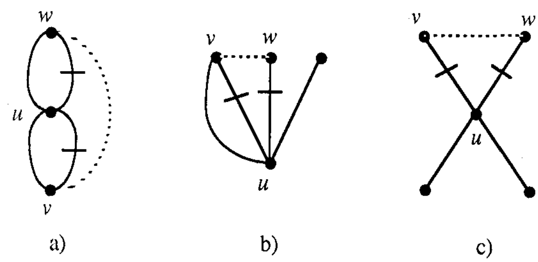

Fig. 1. Shortcut reductions of a tour.

Case (c). The node u is adjacent to 4 distinct nodes. Let $(u, v) \neq e$ and $(u, y) \neq e$ be any pair of distinct edges in T incident with u. If the edge family T' having incidence vector $\chi^{T'} = \chi^T + s_{uvy}$ is connected, we set $w = y$. Otherwise, let $(u, z) \neq e$ be an edge in T with $z \neq y$ and $z \neq v$. Now, the family T' having incidence vector $\chi^{T'} = \chi^T + s_{uvz}$ is necessarily connected and we set $w = z$.

In all cases, the edge family T' having incidence vector $\chi^{T'} = \chi^T + s_{uvw}$ is connected, and so it is a tour and contains e. Since the triangular inequality holds and $cx \geqslant c_0$ is supporting, it follows that $c_0 \leqslant c(T') \leqslant c(T) = c_0$; hence $T' \in \mathcal{T}_c^=$ and $(v, w) \in \Delta_c(u)$. □

Observe that, by repeatedly applying Lemma 2.5, it is possible to obtain a Hamiltonian cycle $H \in \mathcal{H}_c^=$ containing the edge e from a tour $T \in \mathcal{T}_c^=$ containing an edge e. We say that the tour T has been *reduced* to the cycle H or that the cycle H has been obtained from T *by shortcut reductions*. The following corollary is an immediate consequence of this observation.

Corollary 2.6. *If $cx \geqslant c_0$ is tight triangular and facet-defining for* GTSP(n), *then for every edge e in E_n there exists a Hamiltonian cycle of $\mathcal{H}_c^=$ that contains e.* □

The following lemma permits to show how the polyhedral structures of STSP(n) and GTSP(n) are related.

Lemma 2.7. *Let $cx \geq c_0$ be a facet-defining inequality for* STSP(n). *An inequality $fx \geq f_0$ equivalent to $cx \geq c_0$ is tight triangular if and only if $f = \lambda A_n + \pi c$ and $f_0 = \lambda 2 + \pi c_0$, where $\pi > 0$ and $\lambda \in \mathbb{R}^{V_n}$ satisfy*

$$\lambda_u = \tfrac{1}{2}\pi \max\{c(v, w) - c(u, v) - c(u, w) \mid u, v, w \in V_n, u \neq v \neq w\}. \tag{2.1}$$

Proof. It is easy to verify that if λ and π satisfy (2.1) the inequality $fx \geq f_0$ is tight triangular. Suppose now that $fx \geq f_0$ is tight triangular. By imposing the condition (a) of Definition 2.1 we have

$$\lambda_u \geq \tfrac{1}{2}\pi(c(v, w) - c(u, v) - c(u, w)) \quad \text{for all } u \neq v \neq w \in V_n. \tag{2.2}$$

By condition (b) of Definition 2.1, for every $u \in V_n$ there is a triple of distinct nodes $u, v, w \in V_n$ for which (2.2) holds with equality. □

Many known inequalities defining facets of STSP(n) can be written in the following form:

$$\sum_{i=1}^{t} \alpha_i x(\gamma(S_i)) \leq \sum_{i=1}^{t} \alpha_i |S_i| - r(\mathscr{S}), \tag{2.3}$$

or equivalently

$$\sum_{i=1}^{t} \alpha_i x(\delta(S_i)) \geq 2r(\mathscr{S}), \tag{2.3'}$$

where $\mathscr{S} = \{S_1, S_2, \ldots, S_t\}$ is a collection of subsets of V_n, $\gamma(S) \subseteq E_n$ denotes the set of edges with both the endpoints contained in S, and $r(\cdot)$ denotes a suitable rank function defined for $\mathscr{S}$.

The *support graph* of an inequality $cx \geq c_0$ defined on $\mathbb{R}^{E_n}$ is the weighted graph $G_c = (V_n, E_n, c)$, where the weight of each edge $e \in E_n$ is given by c_e.

In the Figures 2 and 3 we show the support graph of two facet-defining inequalities for STSP(n). We represent the support graph of an inequality in the form (2.3) by a collection of subsets of V_n; next to each subset S_i we write its associated coefficient α_i. Here and in the following the support graph of a TT inequality is represented

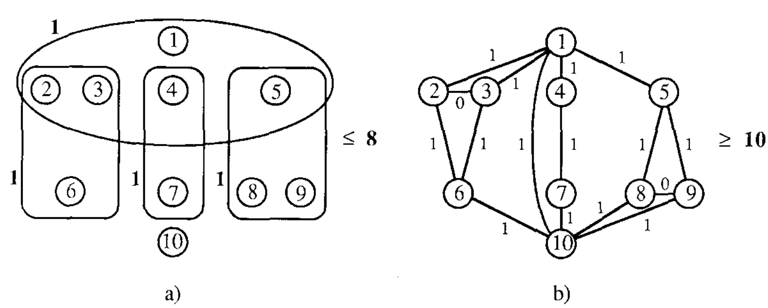

Fig. 2. Equivalent comb inequalities for STSP(10).

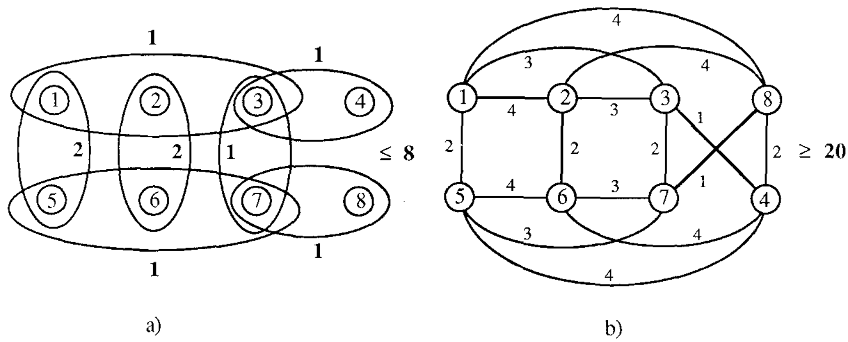

Fig. 3. Equivalent ladder inequalities for STSP(8).

as a weighted graph where some edges are missing to make the picture more readable. The coefficients of a missing edge is the length of the shortest path connecting its endpoints in the graph. Figure 2(a) shows a comb inequality for STSP(10) in the form (2.3) and Figure 2(b) shows its equivalent TT inequality. Figure 3(a) shows a ladder inequality for STSP(8)in the form (2.3) and Figure 3(b) shows its equivalent TT inequality.

Note that (2.1) implies that if $cx \geqslant c_0$ is a facet-defining TT inequality for STSP(n), then every TT inequality $fx \geqslant f_0$ equivalent to it is obtained by multiplication by a positive real number. This observation suggests to associate with every nontrivial facet of STSP(n) an inequality which is tight triangular and that defines that facet. Like in the case of full dimensional polyhedra this inequality is unique to within multiplication by a positive real number. Therefore, from now on we consider only inequalities facet-defining for STSP(n) which are tight triangular. Observe that given an inequality facet-defining for STSP(n) which is not tight triangular, by Lemma 2.7 the coefficients of a TT inequality equivalent to it can be computed in $O(n^3)$ time. In this way we have a polynomial algorithm to check whether or not two inequalities define the same facet of STSP(n).

Using the definition of tight triangular inequality we give now an alternative proof that the dimension of STSP(n) is $|E_n| - n$. The proof is based on the following lemma.

Lemma 2.8. *If $cx \geqslant c_0$ is a TT inequality satisfied with equality by the incidence vectors of all Hamiltonian cycles of K_n, then c is the null vector.*

Proof. The c-length of all Hamiltonian cycles of K_n is c_0. Consequently for all $w \in V_n$ there exist two reals a_w and b_w such that for all distinct u and $v \in V_n$ (see Berenguer (1979))

$$c(u, v) = a_u + b_v. \tag{2.4}$$

Since $c(u, v) = c(v, u)$, $a_u - b_u = a_v - b_v$. It follows that, for some real t,

$$a_w - b_w = t \quad \text{for all } w \in V_n. \tag{2.5}$$

Since the inequality is tight triangular, for all $w \in V_n$ there exist u and $v \in V_n$, with $u \neq v \neq w$, such that $c(u, v) = c(u, w) + c(v, w)$, and so (2.4) and (2.5) imply that $a_w = \frac{1}{2}t$ and $b_w = -\frac{1}{2}t$ for all $w \in V_n$, and the lemma follows. □

Theorem 2.9. *The dimension of* STSP(n) is $|E_n| - n$.

Proof. Lemma 2.8 implies that every equation satisfied by the incidence vectors of all Hamiltonian cycles of K_n is a linear combination of the degree equations $A_n x = \mathbf{2}$. Since A_n has full row rank the theorem follows. □

Observe that the TT form for an inequality facet-defining for STSP(n) has been introduced here mainly for theoretical reasons to emphasize and to exploit the relationship between the two polyhedra. However, the form (2.3) may sometimes be more suitable to represent a facet-defining inequality for STSP(n) in a polyhedral based cutting-plane algorithm: the efficiency of these algorithms usually increases by using inequalities with a large number of zero coefficients (see Padberg and Rinaldi (1991)). While a *simple* inequality (see Section 4.2) has no zero coefficients, there always exists an equivalent inequality which is not tight triangular, has at least n zero coefficients, and it can be obtained by the reduction procedure described in Grötschel and Pulleyblank (1986). On the other hand this is not always the case: one can construct examples of inequalities in TT form obtained by *zero-lifting* (see Section 4.2) of simple inequalities, having more zero coefficients than their equivalent inequalities in the form (2.3).

A *basis* of an inequality $cx \geqslant c_0$ defining a facet of GTSP(n) is a set $\mathscr{B}_c$ of $|E_n|$ tours in $\mathscr{T}_c^=$ whose incidence vectors are linearly independent. We say that a tour T is *almost Hamiltonian in* u if $u \in V_n$ has degree 4 in T and every other node in V_n has degree 2 in T.

Definition 2.10. A basis $\mathscr{C}_c$ of an inequality $cx \geqslant c_0$ *defining a facet of* GTSP(n) is called *canonical*, if it contains $|E_n| - n$ Hamiltonian cycles and n almost Hamiltonian tours (i.e., if for every $u \in V_n$, there exists a tour $T_u \in \mathscr{C}_c$ almost Hamiltonian in u such that its incidence vector satisfies the equation $x(\delta(u)) = 4$, and such that for every tour $T \in \mathscr{C}_c - T_u$ the vector χ^T satisfies the equation $x(\delta(u)) = 2$).

The following theorem is a consequence of Definition 2.10.

Theorem 2.11. *A nontrivial* TT *inequality* $cx \geqslant c_0$ *which is facet-defining for* STSP(n) *defines a facet of* GTSP(n).

Proof. For every $u \in V_n$, construct a tour $T_u \in \mathscr{T}_c^=$ in the following way: Let $e = (v, w)$ be any edge in $\Delta_c(u)$ and $H_e \in \mathscr{H}_c^=$ be a cycle containing e; this cycle exists otherwise the inequality would be equivalent to the trivial inequality $x_e \geqslant 0$. Then define $T_u = H_e - e + (u, v) + (u, w)$. Since $cx \geqslant c_0$ is facet-defining, there is a set $\mathscr{B}$ of $|E_n| - n$

Hamiltonian cycles of $\mathscr{H}_c^=$ whose incidence vectors are linearly independent. Suppose that the vectors $\{\chi^H \mid H \in \mathscr{B}\} \cup \{\chi^{T_u} \mid u \in V_n\}$ are not linearly independent. Then there exist $\lambda \in \mathbb{R}^{\mathscr{B}}$ and $\mu \in \mathbb{R}^{V_n}$ such that

$$\sum_{H \in \mathscr{B}} \lambda_H \chi^H + \sum_{u \in V_n} \mu_u \chi^{T_u} = 0. \tag{2.6}$$

Since the vectors $\{\chi^H \mid H \in \mathscr{B}\}$ are linearly independent, at least one component of μ is nonzero. Let v and w be any two distinct nodes of V_n. By multiplying all vectors in (2.6) by the incidence vectors of $\delta(v)$ and $\delta(w)$, we get

$$2 \sum_{H \in \mathscr{B}} \lambda_H + 2 \sum_{u \in V_n - \{v,w\}} \mu_u + 4\mu_v + 2\mu_w = 0$$

and

$$2 \sum_{H \in \mathscr{B}} \lambda_H + 2 \sum_{u \in V_n - \{v,w\}} \mu_u + 2\mu_v + 4\mu_w = 0,$$

respectively. From the last two equations it follows that $\mu_v = \mu_w$, and so we can assume, without loss of generality, that $\mu_u = 1$ for all u in V_n. This implies that $\sum_{H \in \mathscr{B}} \lambda_H = -(n+1)$. Let us now multiply all vectors in (2.6) by c. Since all these vectors satisfy the inequality $cx \geqslant c_0$ to equality, the resulting equation is

$$\left(\sum_{H \in \mathscr{B}} \lambda_H\right) c_0 + nc_0 = -(n+1)c_0 + nc_0 = 0,$$

which implies $c_0 = 0$ and contradicts the assumption that the inequality $cx \geqslant c_0$ is facet-defining. Consequently $\mathscr{B} \cup \{\chi^{T_u} \mid u \in V_n\}$ is a canonical basis of $cx \geqslant c_0$. □

To see why Theorem 2.11 does not apply to trivial inequalities in TT form, take a trivial inequality $x_e \geqslant 0$, with $e = (u, v)$. The inequality is facet-defining for STSP(n) if $n \geqslant 5$, and so we assume that we are in this case. By Lemma 2.7, we can find a TT inequality $fx \geqslant f_0$ equivalent to it with respect to STSP(n). It is easy to see that $f_0 = 2(n-2)$, and that for all $e \in E_n$ (see Figure 4)

$$f_e = \begin{cases} 1 & \text{if } e \in \delta(\{u, v\}), \\ 2 & \text{otherwise.} \end{cases}$$

The inequality $fx \geqslant f_0$ is supporting for GTSP(n), but is not facet-defining. Since $\Delta_f(w) = \{(u, v)\}$ for all $w \in V_n - \{u, v\}$, and since $(u, v) \notin H$ for all $H \in \mathscr{H}_f^=$, it follows

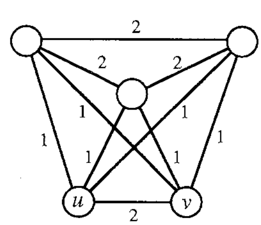

Fig. 4. A trivial inequality in TT form.

that in all tours of $\mathcal{T}_f^=$ every node of the set $V_n - \{u, v\}$ has degree 2. Hence $fx \geq f_0$ defines a face of GTSP(n) of at most dimension $\frac{1}{2}n(n-1)-n+2$.

The inequalities $x_e \leq 1$ for $e \in E_n$ define facets of STSP(n), for $n \geq 4$. Some authors (see, e.g., Grötschel and Padberg (1985)) also call these inequalities trivial. In our setting the inequality $x_e \leq 1$ is nontrivial and its TT form is the cocyle inequality $x(\delta(\{u, v\})) \geq 2$, where $e = (u, v)$, which is facet-defining for GTSP(n), as said in Section 1.

Theorem 2.12. *Every nontrivial facet of* STSP(n) is contained in exactly $n+1$ *facets of* GTSP(n), *n of which are defined by the degree inequalities.*

Proof. Let F be a nontrivial facet of STSP(n). F belongs to the n facets of GTSP(n) defined by the degree inequalities. Every other facet of GTSP(n) containing F is defined by a TT inequality. By Lemma 2.7, there is only one such a facet and the theorem follows. □

This theorem shows that the polyhedral structure of STSP(n) is very closely related to that of GTSP(n). This strong relationship between the two polyhedra motivates our choice of GTSP(n) as a relaxation of STSP(n).

It is very simple to see that Theorem 2.12 does not hold when GTSP(n) is replaced by the MTSP(n). Take, e.g., a comb inequality in the form (2.3), where the sets S_i, $i = 1, \ldots, t$, are the *handle* and the *teeth* of the comb. The inequality is facet-defining for MTSP(n), since it satisfies the conditions of Theorem 9 in Grötschel and Padberg (1985, p. 272). If we substitute any of the sets S_i of the inequality by its complement $\bar{S}_i$, we obtain an equivalent inequality for STSP(n) that still satisfies the conditions of the above mentioned Theorem 9, and so it defines a different facet of MTSP(n).

A natural question to ask at this point is whether or not any TT inequality facet-defining for GTSP(n) defines also a facet of STSP(n). Corollary 2.6 is not sufficient to guarantee a positive answer to this question; on the other hand we do not know any example of a TT inequality facet-defining for GTSP(n) which does not define a facet of STSP(n). Definition 2.10 implies that a TT inequality facet-defining for GTSP(n) defines a facet of STSP(n) if and only if it has a canonical basis. The basic observation that permits us to derive conditions that guarantee that an inequality has a canonical basis is contained in the following remark.

Remark 2.13. Let $cx \geq c_0$ be a TT inequality facet-defining for GTSP(n) and $\mathcal{B}_c$ be one of its bases. Let $\{T_u \mid u \in V_n\}$ be a set of n almost Hamiltonian tours of $\mathcal{T}_c^=$, where by T_u we denote a tour almost Hamiltonian in u. If every tour T of $\mathcal{B}_c$ can be reduced to a cycle of $\mathcal{H}_c^=$ by using only shortcuts obtained by a linear combination of the incidence vectors of elements of $\mathcal{A}_c = \mathcal{H}_c^= \cup \{T_u \mid u \in V_n\}$, then $\mathcal{A}_c$ contains a canonical basis of $cx \geq c_0$. To put it differently, the incidence vector of T can be expressed as a linear combination of the incidence vectors of elements of $\mathcal{A}_c$. In general, it is sufficient that every shortcut be obtained as a linear combination of the incidence vectors of elements of $\mathscr{A}_c$ with coefficients in $\mathbb{R}$. However, in order to obtain simple sufficient conditions based on some properties of $\mathscr{H}_c^=$, we restrict the coefficients of the linear combination to the set $\{-1, 0, 1\}$.

Let $e=(u, v)$ and $f=(w, y)$ be two distinct edges in E_n and let z be a node in V_n. We say that e and f are *c-adjacent* if there exists a Hamiltonian cycle $H \in \mathscr{H}_c^=$ containing both e and f. We say that e and f are *c-adjacent in z* if:

(i) e and f belong to $\Delta_c(z)$;

(ii) there exists a tour $T_z \in \mathscr{T}_c^=$ almost Hamiltonian in z that contains (z, u), (z, v), (z, w) and (z, y);

(iii) $T_z-(z, u)-(z, v)+e$ is a Hamiltonian cycle (and so $T_z-(z, w)-(z, y)+f$ is also a Hamiltonian cycle).

A set of edges $J \subseteq E_n$ is said to be *c-connected* if for every pair of distinct edges f_1 and $f_2 \in J$ there exists a sequence of k edges $e_1, \ldots, e_k$ in J, with $e_1 \equiv f_1$ and $e_k \equiv f_2$, such that e_i is c-adjacent to e_{i+1}, for $i=1, \ldots, k-1$. A set of edges $J \subseteq E_n$ is said to be *c-connected in z* if for every pair of distinct edges f_1 and $f_2 \in J$ there exists a sequence of k edges $e_1, \ldots, e_k$ in E_n (not necessarily belonging to J), with $e_1 \equiv f_1$ and $e_k \equiv f_2$, such that e_i is c-adjacent in z to e_{i+1}, for $i=1, \ldots, k-1$. Observe that the notion of c-connectedness in z is "weaker" than that of c-connectedness, in the sense that, contrarily to what happens for the usual concept of connectivity, in this case every subset of a set c-connected in z is c-connected in z as well.

Lemma 2.14. *Let $cx \geq c_0$ be a* TT *inequality facet-defining for* GTSP(n). *If $\Delta_c(u)$ is c-connected in u for every $u \in V_n$ then $cx \geq c_0$ has a canonical basis, and so it is facet-defining for* STSP(n).

Proof. Let u be a node in V_n and $\{T_v \mid v \in V_n\}$ be any set of n almost Hamiltonian tours of $\mathscr{T}_c^=$. This set always exists by Corollary 2.6, since $cx \geq c_0$ is tight triangular, and can be constructed as in the proof of Theorem 2.11. By Lemma 2.5 there exists a shortcut $s_{u\bar{y}\bar{z}}$, with $(\bar{y}, \bar{z}) \in \Delta_c(u)$ that can be used to reduce T_u to a cycle H in $\mathscr{H}_c^=$. Consequently we have

$$s_{u\bar{y}\bar{z}} = \chi^H - \chi^{T_u},$$

and so $s_{u\bar{y}\bar{z}}$ is a linear combination with coefficients in $\{-1, 1\}$ of the incidence vectors of elements in $\mathscr{A}_c = \mathscr{H}_c^= \cup \{T_u \mid u \in V_n\}$. If $|\Delta_c(u)|=1$ we are done. Otherwise let $e=(v, w)$ and $f=(y, z)$ be two distinct edges in $\Delta_c(u)$ c-adjacent in u. The fact that the two edges may or may not have a common endpoint does not change the proof. Let us assume that the shortcut s_{uyz} is a linear combination with coefficients in $\{-1, 1\}$ of the incidence vectors of elements of $\mathscr{A}_c$. Since e and f are c-adjacent in u there exists a tour $T'_u \in \mathscr{T}_c^=$ almost Hamiltonian in u containing (u, v), (u, w), (u, y) and (u, z), and the two Hamiltonian cycles

$$H_1 = T'_u - (u, y) - (u, z) + (y, z),$$

$$H_2 = T'_u - (u, v) - (u, w) + (v, w)$$

belong to $\mathcal{H}_c^=$ by the triangular inequality. The shortcut s_{uvw} can be expressed as follows:

$$s_{uvw} = \chi^{H_2} - \chi^{H_1} + s_{uyz},$$

and so it is a linear combination with coefficients in $\{-1, 1\}$ of the incidence vectors of tours in $\mathcal{A}_c$. By the assumption that $\Delta_c(u)$ is c-connected in u, it follows that for all $u \in V_n$ and for all $(v, w) \in \Delta_c(u)$ the shortcuts s_{uvw} can be expressed as a linear combination of the incidence vectors of elements in $\mathcal{A}_c$, and by Remark 2.13 the lemma follows. □

The conditions of Lemma 2.14 are too strong, since the choice of the basis $\mathcal{B}_c$ of the inequality $cx \geq c_0$ and the choice of the shortcuts in the process of reducing a tour in $\mathcal{B}_c$ to a cycle in $\mathcal{H}_c^=$ can be completely arbitrary. If we consider a particular basis of the inequality and we restrict ourselves to a suitably chosen subset of shortcuts for the reductions of the tours of the basis, we can weaken the conditions of Lemma 2.14 by requiring that only a subset of $\Delta_c(u)$ be c-connected in u. The weak version of Lemma 2.14 is the following lemma, that is the one we use in the following sections to prove that some families of inequalities define facets of STSP(n).

Lemma 2.15. *Let $cx \geq c_0$ be a* TT *inequality facet-defining for* GTSP(n). *If there exists a basis $\mathcal{B}_c$ of $cx \geq c_0$ and if for every $u \in V_n$ there exists a nonempty set of edges $J_u \subseteq \Delta_c(u)$ c-connected in u, such that every tour $T \in \mathcal{B}_c$ can be reduced to an element of $\mathcal{H}_c^=$ by using only shortcuts in the set $\{s_{uvw} \mid (v, w) \in J_u, u \in V_n\}$, then $cx \geq c_0$ has a canonical basis, and so it is facet-defining for* STSP(n).

Proof. For every $u \in V_n$ let (v, w) be any edge in J_u. By Corollary 2.6 there exists $H \in \mathcal{H}_c^=$ containing the edge (v, w). Let T_u be the almost Hamiltonian tour defined as $T_u = H - (v, w) + (u, v) + (u, w)$. By Remark 2.13 and a process analogous to that of the proof of Lemma 2.14 it follows that the set $\mathcal{A}_c = \mathcal{H}_c^= \cup \{T_u \mid u \in V_n\}$ contains a canonical basis for $cx \geq c_0$ and the lemma follows. □

3. Composition of facet-defining inequalities

Due to the complex polyhedral structure of the polyhedra GTSP(n) and STSP(n) the description of families of inequalities facet-defining for them may become a very complicated task. To simplify this description we use the following approach. We assume that some "elementary" facet-defining inequalities are known and we use them as "building blocks" of more complex facet-defining inequalities. To do so we describe some operations that permit to obtain new facet-defining inequalities from the building blocks. We give two kinds of operations: the node lifting, that we describe in Section 4 and the composition of inequalities that we describe in this section. An application of this approach is given in Naddef and Rinaldi (1988) where we prove that *path inequalities* are facet-defining for STSP(n) and then we use them as building blocks to which we apply the operations described in these two sections.

In Naddef and Rinaldi (1991) it is described a composition operation that permits to obtain new facet-defining inequalities from pairs of facet-defining inequalities for GTSP(n). This operation is called *s-sum* and is based on the s-sum operation of the support graphs of the two inequalities. We describe here this operation for $s=2$ (2-sum) and then we show how it produces inequalities that are, under additional conditions, facet-defining for STSP(n).

We say that two weighted graphs $G^1=(V^1, E^1, c^1)$ and $G^2=(V^2, E^2, c^2)$ are *isomorphic* if there exists a one-to-one correspondence ρ between their node sets that preserves the weight function, i.e., for every edge $(u, v)\in E^1$, the edge $(\rho(u), \rho(v))$ belongs to E^2 and $c^1(u, v)=c^2(\rho(u), \rho(v))$.

Definition 3.1. Let $c^1x^1\geqslant c_0^1$ and $c^2x^2\geqslant c_0^2$ be two TT inequalities valid for GTSP(n_1) and GTSP(n_2), respectively, and let $e_1=(u_1, v_1)\in E_{n_1}$ and $e_2=(u_2, v_2)\in E_{n_2}$ be two edges such that $c^1(u_1, v_1)=c^2(u_2, v_2)=\varepsilon>0$. Denote by V^1 the set $V_{n_1}-\{u_1, v_1\}$ and by V^2 the set $V_{n_2}-\{u_2, v_2\}$. Then a 2-sum of the two inequalities, obtained by *identifying* u_1 with u_2 and v_1 with v_2, is the inequality $cx\geqslant c_0^1+c_0^2-2\varepsilon$ defined on $\mathbb{R}^{E_n}$, with $n=n_1+n_2-2$, whose support graph $G_c=(V_n, E_n, c)$ is defined as follows:

(i) $V_n=V^1+V^2+\{u, v\}$;

(ii) the subgraph of G_c induced by $V^1+\{u, v\}$ is isomorphic to G_{c^1} and u and v correspond to u_1 and v_1, respectively, in the isomorphism;

(iii) the subgraph of G_c induced by $V^2+\{u, v\}$ is isomorphic to G_{c^2} and u and v correspond to u_2 and v_2, respectively, in the isomorphism;

(iv) the coefficients of the edges with one endpoint in V_1 and the other in V_2, that we call the crossing edges of the 2-sum, are computed in the following way: an ordering of the *crossing edges* $e_1, e_2, \ldots, e_k$ is given. For every $i\in\{1, \ldots, k\}$, let T^i be a minimum c-length tour among all tours in $G^i=(V_n, E^i)$, where $E^i=E_n-\{e_{i+1}, \ldots, e_k\}$, that contain the edge e_i. Then c_{e_i} is the value for which $c(T^i)=c_0^1+c_0^2-2\varepsilon$.

The inequalities $c^1x^1\geqslant c_0^1$ and $c^2x^2\geqslant c_0^2$ are called the *component* inequalities of the 2-sum.

The condition that $c^1(u_1, v_1)=c^2(u_2, v_2)$ given in Definition 3.1 is not restrictive, since it can always be satisfied by multiplying one of the two inequalities by a suitable positive number.

The procedure described at the point (iv) of Definition 3.1 is the usual sequential lifting procedure (see, e.g., Padberg (1973)). The ordering of the crossing edges is called *lifting sequence.* In general the values of the coefficients of the crossing edges depend on the lifting sequence, therefore for a given pair of inequalities there may be many 2-sum inequalities obtained by different lifting sequences.

We call a 2-sum inequality *h-liftable* if the coefficients of its crossing edges can be computed by a lifting sequence such that, for every crossing edge e_i, the minimum c-length tour in G^i containing e_i can be reduced to a Hamiltonian cycle in G^i having c-length $c_0^1 + c_0^2 - 2\varepsilon$ and containing e_i.

For the sake of simplicity, from now on every time that the correspondence between nodes, edges, and tours of each of the graph G_{c_1} and G_{c_2} and their corresponding isomorphic subgraphs of G_c is evident, we omit to mention it explicitly.

Let $cx \geqslant c_0$ be an inequality supporting for GTSP(n); a node $v \in V_n$ is said to be *k-critical* for the inequality if the c-length of a minimum c-length tour of $K_n - \{v\}$ is $c_0 - k$.

Remark 3.2. The value k in the above definition cannot exceed the value $\bar{k} = 2\min\{c_e \mid e \in \delta(v)\}$, because otherwise there is a tour of K_n with c-length less than c_0. In this paper we consider only two types of critical nodes, 0-critical and $\bar{k}$-critical, where $\bar{k}$ is defined as before. Observe that to show that v is $\bar{k}$-critical it is sufficient to exhibit any tour of $K_n - \{v\}$ of c-length $c_0 - \bar{k}$.

Remark 3.3. For the remaining of this section we assume that $c^1x^1 \geqslant c_0^1$ and $c^2x^2 \geqslant c_0^2$ are TT inequalities supporting for GTSP(n_1) and GTSP(n_2), respectively, and that the nodes $u_1 \neq v_1 \in V_{n_1}$ and $u_2 \neq v_2 \in V_{n_2}$ are such that $c^1(u_1, v_1) = c^2(u_2, v_2) = \varepsilon$. We assume that the inequality $cx \geqslant c_0$ defined on $\mathbb{R}^{E_n}$, with $n = n_1 + n_2 - 2$, is the 2-sum of the two inequalities obtained by identifying u_1 with u_2 and v_1 with v_2 and we call v and u, respectively, the nodes resulting from these identifications. Finally, we denote by V^1 and V^2 the node sets $V_{n_1} - \{u_1, v_1\}$ and $V_{n_2} - \{u_2, v_2\}$, respectively.

The following theorem proven in Naddef and Rinaldi (1991) gives conditions for a 2-sum inequality to be facet-defining for GTSP(n).

Theorem 3.4. *Under the assumptions of Remark* 3.3, *let* $c^1x^1 \geqslant c_0^1$ *and* $c^2x^2 \geqslant c_0^2$ *be facet-defining for* GTSP(n_1) *and* GTSP(n_2), *respectively. The* 2-*sum inequality* $cx \geqslant c_0$ *is facet-defining for* GTSP(n) *if* v_1 *is* 2ε-*critical for* $c^1x^1 \geqslant c_0^1$ *and at least one of the two nodes* u_2 *and* v_2 *is* 2ε-*critical for* $c^2x^2 \geqslant c_0^2$. □

The condition given in Naddef and Rinaldi (1991) for Theorem 3.4 is weaker than the one given here. In fact it is required that there exist $T_1 \in \mathcal{T}_{c_1}^{=}$ with $2\{(u_1, v_1)\} \subset T_1$, and $T_2 \in \mathcal{T}_{c_2}^{=}$ with $2\{(u_2, v_2)\} \subset T_2$. This condition is implied by the 2ε-criticality of v_1 and u_2 (or v_2). In this paper we prefer to give the theorem in the current form, because 2ε-criticality is necessary in the following Theorem 3.5.

The next theorem is the corresponding of Theorem 3.4 for STSP(n).

Theorem 3.5. *Under the assumptions of Remark* 3.3, *let* $c^1x^1 \geqslant c_0^1$ *and* $c^2x^2 \geqslant c_0^2$ *be nontrivial and facet-defining for* STSP(n_1) *and* STSP(n_2), *respectively. The* 2-*sum inequality* $cx \geqslant c_0$ *is facet-defining for* STSP(n) *if it is h-liftable and*:

(a) v_1 *is* 2ε-*critical for* $c^1x^1 \geqslant c_0^1$,

(b) $\delta(u_2)$ *is* c^2-*connected,*

and either (*case* (A))

(c') u_2 *is* 2ε-*critical for* $c^2x^2 \geqslant c_0^2$,

(d') $\delta(v_1)$ *is* c^1-*connected,*

or (*case* (B))

(c'') v_2 *is* 2ε-*critical for* $c^2x^2 \geqslant c_0^2$,

(d'') $\delta(u_1)$ *is* c^1-*connected,*

(e'') *there exists a Hamiltonian cycle* $H_1 \in \mathcal{H}_{c_1}^{=}$ *containing the edge* (u_1, v_1) *and any edge* $e_1 \in \Delta_{c_1}(v_1)$,

(f'') *there exists a Hamiltonian cycle* $H_2 \in \mathcal{H}_{c_2}^{=}$ *containing the edge* (u_2, v_2) *and any edge* $e_2 \in \Delta_{c_2}(v_2)$.

Proof. To prove the theorem we first construct a basis $\mathcal{B}_c$ of $cx \geqslant c_0$. Then we define for every $w \in V_n$ a set of edges $J_w \subseteq \Delta_c(w)$ such that every tour in $\mathcal{B}_c$, where the node w has degree higher than 2, can be reduced to a tour where w has degree 2, by using only shortcuts in the set $\{s_{wyz} \mid (y, z) \in J_w\}$. Then by Lemma 2.15 it is sufficient to show that J_w is c-connected in w for all $w \in V_n$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two canonical bases of $c^1x^1 \geqslant c_0^1$ and $c^2x^2 \geqslant c_0^2$, respectively. $\mathcal{C}_1$ and $\mathcal{C}_2$ always exist since the two inequalities are nontrivial and facet-defining for STSP(n_1) and STSP(n_2), respectively. For $i = 1, 2$, we call T_{u_i} and T_{v_i} the tours in $\mathcal{C}_i$ which are almost Hamiltonian in u_i and v_i, respectively. Let Γ_1 be a Hamiltonian cycle of $K_{n_1} - \{v_1\}$ of c^1-length $c_0^1 - 2\varepsilon$ and let Γ_2 be a Hamiltonian cycle of c^2-length $c_0^2 - 2\varepsilon$ of the graph $K_{n_2} - \{u_2\}$ if we are in case (A), and of the graph $K_{n_2} - \{v_2\}$ if we are in case (B). By Lemma 2.5 the cycles Γ_1 and Γ_2 always exist. If we are in case (A) we assume without loss of generality that

$$T_{u_1} = \Gamma_1 \cup 2\{(u_1, v_1)\},$$

$$T_{v_2} = \Gamma_2 \cup 2\{(u_2, v_2)\}.$$

If we are in case (B) we assume without loss of generality that

$$T_{u_1} = \Gamma_1 \cup 2\{(u_1, v_1)\},$$

$$T_{v_1} = H_1 - e_1 + (w', v_1) + (w'', v_1),$$

$$T_{u_2} = \Gamma_2 \cup 2\{(u_2, v_2)\},$$

$$T_{v_2} = H_2 - e_2 + (z', v_2) + (z'', v_2),$$

where $(w', w'') = e_1$ and $(z', z'') = e_2$.

Let $\mathcal{B}_1 \subset \mathcal{T}_c^{=}$ be the set of tours obtained by adding the edges of Γ_2 to each tour in $\mathcal{C}_1$ and $\mathcal{B}_2 \subset \mathcal{T}_c^{=}$ be the set obtained by adding the edges in Γ_1 to each tour in

$\mathscr{C}_2$. Finally, since $cx \geq c_0$ is h-liftable there is a sequence of the crossing edges $\{e_1, e_2, \ldots, e_k\}$ and a sequence of Hamiltonian cycles $\{H_1, H_2, \ldots, H_k\}$ belonging to $\mathscr{H}_c^=$, such that the cycle H_i, with $1 \leq i \leq k$, contains only edges in $E_{n_1} \cup E_{n_2} \cup \{e_j \mid 1 \leq j \leq i\}$. Let $\mathscr{B}_3 \subset \mathscr{H}_c^=$ be the set

$$\mathscr{B}_3 = \{H_i \mid 1 \leq i \leq k\}.$$

The set $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3$ contains $\frac{1}{2}n(n-1)$ elements (the tour $\Gamma_1 \cup \Gamma_2 \cup 2\{(u, v)\}$ is contained in both $\mathscr{B}_1$ and $\mathscr{B}_2$) and (see Naddef and Rinaldi (1991)) is a basis for $cx \geq c_0$, and so this inequality is facet-defining for GTSP(n). Since $\mathscr{B}_3$ is a set of Hamiltonian cycles, to prove that $cx \geq c_0$ is facet-defining for STSP(n) it is sufficient to apply Lemma 2.15 to the set $\mathscr{B}_1 \cup \mathscr{B}_2$. Every node $w \in V_n - \{u, v\}$ has degree 2 in all tours in $\mathscr{B}_1 \cup \mathscr{B}_2$ except one, where w has degree 4. Consequently, J_w has cardinality 1, and so it is c-connected in w, for $w \in V_n - \{u, v\}$.

Case (A). The sets J_u and J_v are given by:

$$J_u = \{(y', y'') \mid y' \in V^2 \cup \{v\}, y'' \in \{t_1, z_1\}\},$$

$$J_v = \{(y', y'') \mid y' \in V^1 \cup \{u\}, y'' \in \{t_2, z_2\}\},$$

where t_1 and z_1 are the neighbors of u in Γ_1 and t_2 and z_2 are the neighbors of v in Γ_2. Take now two nodes s_1 and s_2 in $V^1 \cup \{u\}$ such that the edges (v, s_1) and (v, s_2) are c_1-adjacent, and so they both belong to a cycle $H \in \mathscr{H}_{c_1}^=$. Since the tour $T_v = H \cup \Gamma_2$, almost Hamiltonian in v, contains (v, s_1), (v, s_2), (v, t_2), and (v, z_2) and since by triangular inequality all edges (s_1, t_2), (s_1, z_2), (s_2, t_2), and (s_2, z_2) belong to $\Delta_c(v)$, it follows that these four edges are pair wise c-adjacent in v. The edge set $\delta(v)$ is c_1-connected, and so J_v is c-connected in v. Analogously it can be proven that J_u is c-connected in u and this completes the proof for the case (A).

Case (B). In this case the node v has degree 2 in all tours of $\mathscr{B}_1 \cup \mathscr{B}_2$, except for $T_{v_1} + \Gamma_2$ and $T_{v_2} + \Gamma_1$. Consequently, the sets J_u and J_v are given by $J_u = J_u^1 \cup J_u^2$ and $J_v = \{e_1, e_2\}$, where

$$J_u^1 = \{(y', y'') \mid y' \in V^1 \cup \{v\}, y'' \in \{t_2, z_2\}\}$$

and

$$J_u^2 = \{(y', y'') \mid y' \in V^2 \cup \{v\}, y'' \in \{t_1, z_1\}\},$$

and where t_1, z_1, t_2, and z_2 are defined as in case (A). Using the same argument as in case (A), it follows that J_u^1 and J_u^2 are c-connected in u, and since their intersection is nonempty it follows that J_u is c-connected in u. Finally, consider the Hamiltonian cycles H_1 and H_2 of the conditions (e″) and (f″). The Hamiltonian cycle $H_1 \cup H_2 - (u_1, v_1) - (u_2, v_2)$ belongs to $\mathscr{H}_c^=$ and contains e_1 and e_2. Consequently, J_v is c-connected in v. □

As already shown in Naddef and Rinaldi (1991) the 2-sum operation can be applied to two inequalities which are themselves 2-sum inequalities. The process can go on indefinitely and produce the *tree-inequalities*. To show that a tree-inequality is facet-defining for GTSP(n) or STSP(n) the Theorems 3.4 and 3.5 can be inductively applied. To do so it is essential that some properties of the component inequalities that are required by these theorems are inherited by the resulting 2-sum inequality. The following lemmata give conditions for these properties to be hereditary.

Lemma 3.6. *Under the assumptions of Remark* 3.3, *let $w \in V_{n_1}$ be k-critical for $c^1x^1 \geq c_0^1$, and $c_e^1 = \frac{1}{2}k$ for some edge $e \in \delta(w)$ in K_{n_1}. The corresponding node $w \in V_n$ is k-critical for $cx \geq c_0$ if $cx \geq c_0$ is supporting for* GTSP(n) *and any of the following conditions holds*:

(a) $w \neq v_1$ *and u_2 is 2ε-critical for $c^2x^2 \geq c_0^2$*,

(b) $w \neq u_1$ *and v_2 is 2ε-critical for $c^2x^2 \geq c_0^2$*.

Proof. Let T^1 be a minimal c^1-length tour of $K_{n_1} - \{w\}$ of c^1-length $c_0^1 - k$. Let T^2 denote a minimal c^2-length tour of $K_{n_2} - \{u_2\}$ if (a) holds and of $K_{n_2} - \{v_2\}$ if (b) holds. The c^2-length of T^2 is $c_0^2 - 2\varepsilon$. The tour of $K_n - \{w\}$ whose edges correspond to the members of $T^1 + T^2$ has c-length $c_0^1 + c_0^2 - 2\varepsilon - k$. Since the edge corresponding to e belongs to $\delta(w)$ in K_n, the lemma follows by Remark 3.2. □

Lemma 3.7. *Under the assumptions of Remark* 3.3, *if there exists a Hamiltonian cycle $H \in \mathcal{H}_{c_1}^=$ containing two nonadjacent edges e and $f \in E_{n_1}$ and at least one of the two nodes u_2 and v_2 is 2ε-critical for $c^2x^2 \geq c_0^2$, then there exists a Hamiltonian cycle $H^* \in \mathcal{H}_c^=$ containing the edges in E_n corresponding to e and f, respectively.*

Proof. Without loss of generality, let v_2 be 2ε-critical. Then by Lemma 2.5 there exists a Hamiltonian cycle H^2 of $K_{n_2} - \{v_2\}$ of c^2-length $c_0^2 - 2\varepsilon$. The tour of K_n whose edges correspond to the elements of $H + H^2$ is almost Hamiltonian in u_2 and belongs to $\mathcal{T}_c^=$. By Lemma 2.5 this tour can be reduced to a Hamiltonian cycle in $\mathcal{H}_c^=$ that contains both e and f since they cannot be both incident with the node u_2. □

Lemma 3.8. *Under the assumptions of Remark* 3.3, *let $\delta(w)$ in K_{n_1} be c^1-connected for every node $w \in V_{n_1}$ and let $\delta(w)$ in K_{n_2} be c^2-connected for every node $w \in V_{n_2}$. Then $\delta(w)$ in K_n is c-connected for every node $w \in V_n$ if the following conditions hold*:

(i) *$cx \geq c_0$ is h-liftable*;

(ii) *at least one of the two nodes u_1 and v_1 is 2ε-critical for $c^1x^1 \geq c_0^1$*;

(iii) *at least one of the two nodes u_2 and v_2 is 2ε-critical for $c^2x^2 \geq c_0^2$*.

Proof. For $i = 1$, 2 and for every $w \in V_{n_i}$ let $\delta^i(w) \subset E_n$ denote the set of edges corresponding to $\delta(w)$ in K_{n_i}. Let w be a node of $V_{n_1} - \{u_1, v_1\}$ and let (y, w) and (z, w) be two edges of $\delta^1(w)$ whose corresponding edges of K_{n_1} are c^1-adjacent. It follows that there exists a cycle $H \in \mathcal{H}_{c^1}^=$ containing (y, w) and (z, w). Without loss of generality, let v_2 be 2ε-critical for $c^2x^2 \geq c_0^2$. Then, there exists a Hamiltonian cycle H^2 of $K_{n_2} - \{u_2\}$ having length $c_0^2 - 2\varepsilon$. The tour $H + H^2$ of K_n is almost Hamiltonian, has length c_0, and can be reduced to a cycle containing both (y, w)

and (z, w). Since $\delta(w)$ in K_{n_1} is c^1-connected, it follows that $\delta^1(w)$ is c-connected. The edge set $\delta(w)$ of K_n is the union of $\delta^1(w)$ and of a subset of the crossing edges. Consequently, since the inequality $cx \geq c_0$ is h-liftable, it is easy to show that also $\delta(w)$ is c-connected. Similarly, it can be shown that $\delta(w)$ is c-connected for all w in $V_{n_2} - \{u_2, v_2\}$.

Consider now the case $w = v$ and let H, (y, w), and (z, w) be defined as for the previous case. If v_2 is 2ε-critical for $c^2x^2 \geq c_0^2$, then as before one can construct a Hamiltonian cycle in K_n containing both (y, w) and (z, w), and show that the two edges are c-adjacent. Otherwise, u_2 is 2ε-critical, and so there exists a Hamiltonian cycle H^2 of $K_{n_2} - \{u_2\}$ with length $c_0^2 - 2\varepsilon$. Let w' and w'' be the two neighbors of v_2 in H^2. The edge (y, u) is c-adjacent to the edge (u, w') because the cycle $H + H^2 - (z, u) - (u, w'') + (z, w'')$ in $\mathcal{H}_c^=$ contains both of them. The edges (z, u) and (u, w') are c-adjacent because they belong to the cycle $H + H^2 - (y, u) - (u, w'') + (y, w'')$ in $\mathcal{H}_c^=$. Following the same argument for two c^2-connected edges of $\delta^2(w)$ and observing that $\delta(u) = \delta^1(u) \cup \delta^2(u)$ and $\delta^1(u) \cap \delta^2(u) = \{(u, v)\}$, it is easy to show that $\delta(w)$ is c-connected. The case $w = u$ is completely analogous to the previous one. □

4. Node lifting

Figure 5(a) shows the support graph of a 2-matching inequality defined on $\mathbb{R}^{E_6}$. The graphs of the Figures 5(b) and 5(c) are the support graphs of comb inequalities defined on $\mathbb{R}^{E_7}$. The coefficients of the three inequalities coincide for all edges with both the endpoints in the set $\{1, 2, \ldots, 6\}$. Therefore each of the two inequalities of the Figures 5(b) and 5(c) can be considered as an "extension" of the inequality of Figure 5(a).

In Grötschel and Padberg (1979b) the operations that leads to extended inequalities is called "lifting" and some conditions are given under which the lifting applied to a facet-defining inequality produces a facet-defining inequality in a higher dimensional space. Then the comb inequalities are proven to be facet-defining in a two-step process. First it is shown that the 2-matching inequalities are facet-defining

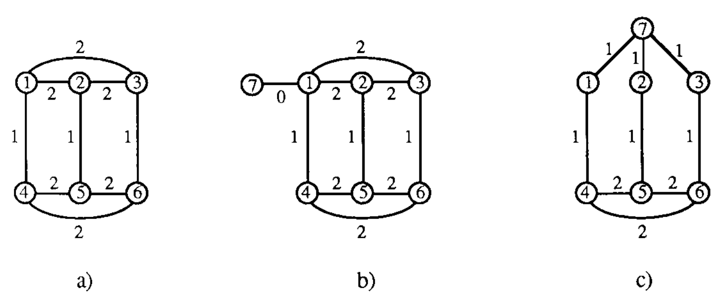

Fig. 5.

for STSP(n), $n \geq 6$. Then it is shown that all comb inequalities are obtained by applying the lifting to the 2-matching inequalities and that the aforementioned conditions for this lifting are satisfied.

In this section we describe many kinds of extensions for facet-defining inequalities for STSP(n) in TT form. We call the extension *node lifting* and we give conditions that guarantee that the extended inequality is facet-defining. These conditions are satisfied not only by the 2-matching inequalities, but by most of the facet-defining inequalities known to date.

Let $K_n = (V_n, E_n)$ and $K_{n^*} = (V_{n^*}, E_{n^*})$ be two complete graphs with n nodes and m edges and $n^* > n$ nodes and m^* edges, respectively. We denote by $V_n = \{u_1, u_2, \ldots, u_n\}$ the nodes of K_n and by $V_{n^*} = V_n \cup \{u_{n+1}, \ldots, u_{n^*}\}$ the nodes of K_{n^*}. We say that an inequality $c^*x^* \geq c_0^*$ defined on $\mathbb{R}^{E_{n^*}}$ is obtained by *node lifting* of the inequality $cx \geq c_0$ defined on $\mathbb{R}^{E_n}$ if

$$c^*(u_i, u_j) = c(u_i, u_j) \quad \text{for all } 1 \leq i < j \leq n.$$

An inequality $c^*x^* \geq c_0^*$ obtained by node lifting of the inequality $cx \geq c_0$ is completely defined by c_0^* and the coefficients of the edge sets $\delta(v)$, $v \in \{u_{n+1}, \ldots, u_{n^*}\}$.

A special case of node lifting occurs when $c^*(v, u) = 0$ for some $u \in V_n$ and all $v \in \{u_{n+1}, \ldots, u_{n^*}\}$. We say in this case that the inequality $c^*x^* \geq c_0^*$ is obtained from $cx \geq c_0$ by *zero-lifting of node u*. For a TT inequality obtained by zero-lifting the following property holds as a direct consequence of Definition 2.1.

Proposition 4.1. *For a* TT *inequality $c^*x^* \geq c_0$ obtained from $cx \geq c_0$ by zero-lifting of node $u \in V_n$ the following holds*:

(i) $c^*(v, w) = c(u, w)$ *for all* $w \in V_n - \{u\}$ *and all* $v \in \{u_{n+1}, \ldots, u_{n^*}\}$;

(ii) $\Delta_{c^*}(v) = \Delta_c(u) \cup \bigcup \{\delta(v') - (v', v) \mid v' \in \{u, u_{n+1}, \ldots, u_{n^*}\}, v' \neq v\}$ *for all* $v \in \{u_{n+1}, \ldots, u_{n^*}\}$, *where* $\delta(v')$ *is a subset of* E_{n^*}. □

The zero-lifting of a node u can be seen as the operation of replacing u with a clique of $q > 1$ nodes in the support graph of an inequality $cx \geq c_0$ defined on $\mathbb{R}^{E_n}$. The coefficient of the inequality $c^*x^* \geq c_0$ associated with the resulting graph are defined as follows. The value of the coefficients of all edges of the clique is set to zero and for every node v of the clique and every node $w \in V_n - \{u\}$ the coefficient $c^*(v, w)$ is set equal to $c(u, w)$. For this reason the zero-lifting is also called *clique-lifting* in Padberg and Rinaldi (1990), where a general separation procedure for inequalities obtained by clique-lifting is described.

4.1. 1-Node lifting

When $n^* = n + 1$ we have a special case of node lifting that we call 1-*node lifting.* The next theorem gives conditions that are satisfied by c_0^* and the coefficients of the edge set $\delta(u_{n+1})$ when $c^*x^* \geq c_0^*$ and $cx \geq c_0$ are facet-defining for GTSP($n+1$) and GTSP(n), respectively.

Theorem 4.2. *Let $c^*x^* \geqslant c_0^*$ be an inequality facet-defining for* GTSP$(n+1)$ *which is obtained by 1-node lifting of a* TT *inequality $cx \geqslant c_0$ facet-defining for* GTSP(n); *then the following conditions hold*:

(i) *$c^*x^* \geqslant c_0^*$ is tight triangular*;

(ii) $c_0^* = c_0$;

(iii) *for all $e \in \Delta_{c^*}(u_{n+1})$ there exist $e' \neq e$, $e' \in \Delta_{c^*}(u_{n+1})$ and $H \in \mathscr{H}_c^=$ such that e and e' belong to H*;

(iv) *every connected component of the graph $(V_n, \Delta_{c^*}(u_{n+1}))$ contains at least one odd cycle.*

Proof. If $c^*x \geqslant c_0^*$ is not tight triangular, by Proposition 2.2 it is either a trivial or a degree inequality. Since these inequalities cannot be obtained by 1-node lifting of TT inequalities (i) follows.

The coefficients of the inequality $c^*x^* \geqslant c_0^*$ satisfy the triangular inequality; this implies that $c_0^* \geqslant c_0$. Since the inequality $c^*x^* \geqslant c_0^*$ is tight triangular there exists an edge $(v, w) \in \Delta_{c^*}(u_{n+1})$ and since the $cx \geqslant c_0$ is facet-defining for GTSP(n), by Proposition 2.3 there exists a tour $T \in \mathscr{T}_c^=$ containing (v, w). For the tour T^* of K_{n+1} obtained from T by removing (v, w) and adding (v, u_{n+1}) and (u_{n+1}, w) we have $c^*\chi^{T^*} = c_0 \geqslant c_0^*$ and (ii) follows.

Let e be any edge in $\Delta_{c^*}(u_{n+1})$. Since $c^*x^* \geqslant c_0^*$ is nontrivial there exists by Corollary 2.6 a Hamiltonian cycle $H \in \mathscr{H}_{c^*}^=$ containing e. Let v and w be the neighbors of u_{n+1} in H. The Hamiltonian cycle $H-(u_{n+1}, v)-(u_{n+1}, w)+(v, w)$ belongs to $\mathscr{H}_c^=$, contains e and (v, w). The edge (v, w) belongs to $\Delta_{c^*}(u_{n+1})$ and (iii) follows.

The graph $(V_n, \Delta_{c^*}(u_{n+1}))$ spans K_n. In fact for any node $u \in V_n$ there exists, as before, $H \in \mathscr{H}_{c^*}^=$ containing (u, u_{n+1}). Let w be the second neighbor of u_{n+1} in H. Then the cycle $H' = H-(u_{n+1}, v)-(u_{n+1}, w)+(u, w)$ belongs to $\mathscr{H}_c^=$ and the edge (u, w) belongs to $\Delta_{c^*}(u_{n+1})$. We consider two cases.

Case (a). The inequality $c^*x^* \geqslant c_0^*$ is obtained by zero-lifting of some node $u \in V_n$. By Proposition 4.1 the graph $(V_n, \Delta_{c^*}(u_{n+1}))$ is connected and contains at least one odd cycle and (iv) follows.

Case (b). $c^*(u_{n+1}, v) > 0$ for all $v \in V_n$. Assume that (iv) is not verified. It follows that there exists a connected component (U, F) of the graph $(V_n, \Delta_{c^*}(u_{n+1}))$, with $|U| \geqslant 2$, that does not contain odd cycles. Let ε be defined as follows:

$$\varepsilon = \tfrac{1}{3}\min\{\varepsilon', \varepsilon''\},$$

where

$$\varepsilon' = \min\{c^*(u_{n+1}, v) + c^*(u_{n+1}, w) - c^*(v, w),\ (v, w) \in E_n - \Delta_{c^*}(u_{n+1})\},$$

$$\varepsilon'' = \min\{c^*(u_{n+1}, v),\ v \in V_n\}.$$

Note that ε' is nonzero by definition of $\Delta_{c^*}(u_{n+1})$ and that ε'' is nonzero by the assumption that $c^*(u_{n+1}, v) > 0$ for all $v \in V_n$; hence ε is strictly positive.

Let $\bar{u}$ be a node of the component (U, F) and define two new inequalities $c^+x^* \geq c_0^*$ and $c^-x^* \geq c_0^*$ in the following way:

- $c^+(u_{n+1}, \bar{u}) = c^*(u_{n+1}, \bar{u}) + \varepsilon$;
- $c^+(u_{n+1}, v) = c^*(u_{n+1}, v) + \varepsilon$ for all $v \neq \bar{u}$ belonging to U and connected to $\bar{u}$ in (U, F) by a path with an even number of edges;
- $c^+(u_{n+1}, v) = c^*(u_{n+1}, v) - \varepsilon$ for all $v \neq \bar{u}$ belonging to U and connected to $\bar{u}$ in (U, F) by a path with an odd number of edges;
- $c_e^+ = c_e^*$, for every other edge e in E_{n+1};
- for the coefficients of the inequality $c^-x^* \geq c_0^*$:

$$c_e^- = \begin{cases} c_e^* - \varepsilon & \text{if } c_e^+ = c_e^* + \varepsilon, \\ c_e^* + \varepsilon & \text{if } c_e^+ = c_e^* - \varepsilon, \\ c_e^* & \text{if } c_e^+ = c_e^*. \end{cases}$$

These two inequalities are well defined since the component (U, F) does not contain odd cycles and all the coefficients are nonnegative by definition of ε.

By the definition of ε, ε', c^+ and c^- it follows that

$$\Delta_{c^+}(u_{n+1}) = \Delta_{c^-}(u_{n+1}) = \Delta_{c^*}(u_{n+1}),$$

and that

$$\begin{aligned} c^+(v, w) &\leq c^+(u_{n+1}, v) + c^+(u_{n+1}, w) \quad \text{for all } (v, w) \in E_n, \\ c^-(v, w) &\leq c^-(u_{n+1}, v) + c^-(u_{n+1}, w) \quad \text{for all } (v, w) \in E_n. \end{aligned} \tag{4.1}$$

Suppose that $c^+x^* \geq c_0^*$ is not valid, then since the coefficients of $c^+x^* \geq c_0^*$ are nonnegative, there exists a minimum c^+-length tour $T \in \mathcal{T}_{n+1}$ such that $c^+\chi^T < c_0^*$. If the degree of u_{n+1} in T is more than 2 we can reduce T by shortcuts to a tour $\bar{T}$ where u_{n+1} has degree 2. Observe that $c^+(\bar{T}) = c^+(T)$ by (4.1) and since T is a minimum c^+-length tour. Therefore we assume without loss of generality that u_{n+1} has only two (not necessarily distinct) neighbors v and w in T.

If v and w coincide let T' be the tour of K_n defined by $T' = T - 2\{(u_{n+1}, v)\}$. Since $c^+(u_{n+1}, v) > 0$ we have that $c^+\chi^{T'} < c_0^*$. If v and w are distinct let T' be the tour $T' = T - (u_{n+1}, v) - (u_{n+1}, w) + (v, w)$. By (4.1) T' has c^+-length $c^+\chi^{T'} \leq c^+\chi^T < c_0^*$. This leads to a contradiction since T' is a tour of K_n and the components of c^+ and c coincide in E_n. The same argument shows that $c^-x^* \geq c_0^*$ is valid. Let T be any tour of $\mathcal{T}_{c^*}^=$. If $c^+\chi^T > c_0^*$ then $c^-\chi^T < c_0^*$; hence $c^+\chi^T = c_0^*$ and $\mathcal{T}_{c^*}^= \subseteq \mathcal{T}_{c^+}^=$. Since there is no real π such that $c^* = \pi c^+$, the assumption that $c^*x^* \geq c_0^*$ is facet-defining for GTSP$(n+1)$ is contradicted and the theorem follows. □

By condition (ii) of the previous theorem it follows that the inequality $c^*x^* \geq c_0^*$ is completely defined by coefficients of the edges in $\delta(u_{n+1})$. Moreover, the node u_{n+1} is 0-critical for $c^*x^* \geq c_0^*$.

Theorem 4.3. *Let $cx \geq c_0$ be a* TT *inequality facet-defining for* GTSP(n); *an inequality $c^*x^* \geq c_0$ which is obtained by 1-node lifting of $cx \geq c_0$ is facet-defining for* GTSP$(n+1)$

if it is tight triangular and there exist an edge set $F \subseteq \Delta_{c^*}(u_{n+1})$ *and a basis* $\mathcal{B}_c$ *of* $cx \geq c_0$ *such that*:

(i) $F \cap T \neq \emptyset$ *for all* $T \in \mathcal{B}_c$;

(ii) *for all* $e \in F$ *there exist* $e' \neq e$, $e' \in \Delta_{c^*}(u_{n+1})$ *and* $H \in \mathcal{H}_c^=$ *such that* e *and* e' *belong to* H;

(iii) *every connected component of the graph* (V_n, F) *contains at least one odd cycle.*

Proof. The validity of $c^*x^* \geq c_0$ immediately follows from the fact that it is tight triangular. Let T be any tour in $\mathcal{B}_c$; since there exists an edge (u, v) in the intersection $T \cap F$ the tour $T-(u, v)+(u, u_{n+1})+(v, u_{n+1})$ belongs to $\mathcal{T}_{c^*}^=$, and so $\mathcal{T}_{c^*}^=$ is non-empty. Let $f^*x^* \geq f_0$ be any inequality facet-defining for GTSP$(n+1)$ such that $\mathcal{T}_{c^*}^= \subseteq \mathcal{T}_{f^*}^=$. Take any edge $e=(u, v) \in F$. By (ii) there exist $e'=(y, z) \neq e$, $e' \in \Delta_{c^*}(u_{n+1})$ and $H \in \mathcal{H}_c^=$ such that e and e' belong to H. The tours $T^1 = H-(y, z)+(u_{n+1}, y)+(u_{n+1}, z)$ and $T^2 = T^1-(u, v)+(u_{n+1}, u)+(u_{n+1}, v)$ belong to $\mathcal{T}_{c^*}^=$, and so they belong to $\mathcal{T}_{f^*}^=$. Therefore, $f^*(T^1)=f^*(T^2)$ and $f^*(u, v) = f^*(u_{n+1}, u)+f^*(u_{n+1}, v)$, and so

$$F \subseteq \Delta_{f^*}(u_{n+1}). \tag{4.2}$$

Let f be the vector in $\mathbb{R}^{E_n}$ defined by

$$f_e = f_e^* \quad \text{for all } e \in E_n.$$

Take any tour $T \in \mathcal{B}_c$. By (i) T contains an edge $(y, z) \in F$, and so the tour $T^* = T-(y, z)+(u_{n+1}, y)+(u_{n+1}, z)$ belongs to $\mathcal{T}_{c^*}^=$; hence it belongs to $\mathcal{T}_{f^*}^=$. By (4.2) it follows that $T \in \mathcal{T}_f^=$ and this implies that $\mathcal{T}_c^= \subseteq \mathcal{T}_f^=$. Since $cx \geq c_0$ defines a facet of GTSP(n) it follows that $fx \geq f_0$ defines the same facet and

$$f = \pi c, \tag{4.3}$$

for some real $\pi > 0$. Consider now a subset F' of F with $|F'| = n$ and whose corresponding columns of A_n are linearly independent. By (iii) this set exists. In fact it is well known (see, e.g., Grötschel and Pulleyblank (1986)) that the columns of A_n corresponding to a set J of n edges of K_n are linearly independent if and only if every connected component of the graph (V_n, J) has at least one odd cycle (consequently, since $|J| = n$, every connected component of (V_n, J) has exactly one odd cycle and no even cycles). By (4.2) and (4.3) we have

$$f^*(u_{n+1}, y)+f^*(u_{n+1}, z) = f^*(y, z) = \pi c^*(y, z) \quad \text{for } (y, z) \in F'. \tag{4.4}$$

The coefficient matrix of the system of equations (4.4) is the transpose of the submatrix of A_n corresponding to the elements of F', and so it is nonsingular and (4.4) has a unique solution. The vector c^* satisfies the system

$$c^*(u_{n+1}, y)+c^*(u_{n+1}, z) = c^*(y, z) \quad \text{for all } (y, z) \in F'. \tag{4.3'}$$

The systems (4.4) and (4.3′) have the same coefficient matrix and their right hand sides differs by the factor π. It follows that

$$f_e^* = \pi c_e^* \quad \text{for all } e \in \delta(u_{n+1}). \tag{4.5}$$

By (4.3) and (4.5) we have that $f^* = \pi c^*$, with $\pi > 0$, and the theorem follows. □

Observe that in Theorem 4.3 all the conditions except (i) are necessary by Theorem 4.2. We were not able to prove that also (i) is necessary, but we conjecture so.

We give now an equivalent of Theorem 4.3 for the STSP(n) polytope.

Theorem 4.4. *Let* $cx \geqslant c_0$ *be a* TT *inequality that is facet-defining for* STSP(n); *an inequality* $c^*x^* \geqslant c_0$ *which is obtained by* 1*-node lifting of* $cx \geqslant c_0$ *is facet-defining for* STSP($n+1$) *if it is tight triangular and there exist an edge set* $F \subseteq \Delta_{c^*}(u_{n+1})$ *and a canonical basis* $\mathscr{C}_c$ *of* $cx \geqslant c_0$ *such that*:

(i) $F \cap H \neq \emptyset$ *for all* $H \in \mathscr{C}_c$;

(ii) *for all* $e \in F$ *there exist* $e' \neq e$, $e' \in \Delta_{c^*}(u_{n+1})$ *and* $H \in \mathscr{H}_c^=$ *such that* e *and* e' *belong to* H;

(iii) *every connected component of the graph* (V_n, F) *contains at least one odd cycle*;

(iv) F *is* c^**-connected in* u_{n+1}.

Proof. By Theorem 4.3 the inequality $c^*x^* \geqslant c_0$ is facet-defining for GTSP($n+1$). To prove that it is also facet-defining for STSP($n+1$) we first construct a set of tours $\mathscr{B}^* \subseteq \mathscr{T}_{c^*}^=$ containing a basis of $c^*x^* \geqslant c_0$. Then we define for every $w \in V_{n+1}$ a set of edges $J_w \subseteq \Delta_c(w)$ such that every tour in $\mathscr{B}^*$ where the node w has degree greater that 2 can be reduced to a tour where w has degree 2, by using only shortcuts in the set $\{s_{wyz} \mid (y, z) \in J_w\}$. Then by Lemma 2.15 it is sufficient to show that J_w is c^*-connected in w for all $w \in V_{n+1}$. For every $e = (u, v) \in F$ by (ii) there is $e' = (y, z) \neq e \in \Delta_{c^*}(u_{n+1})$ *and* $H_e \in \mathscr{H}_c^=$ such that e and e' belong to H_e. Define the two tours T_e^1 and T_e^2 associated with e as follows:

$$T_e^1 = H_e - (y, z) + (u_{n+1}, y) + (u_{n+1}, z),$$

$$T_e^2 = T_e^1 - (u, v) + (u_{n+1}, u) + (u_{n+1}, v).$$

For every tour $T \in \mathscr{C}_c$ define the tour T' as $T' = T - (y, z) + (u_{n+1}, y) + (u_{n+1}, z)$, where (y, z) is an edge in T belonging to F, that by (i) always exists. Then the set $\mathscr{B}^*$ is defined as

$$\mathscr{B}^* = \{T_e^1, T_e^2 \mid e \in F\} \cup \{T' \mid T \in \mathscr{C}_c\}$$

and contains a basis of $c^*x^* \geqslant c_0$, because it contains all tours used in the proof of Theorem 4.3 to show that $c^*x^* \geqslant c_0$ is facet-defining for GTSP($n+1$). It is easy to see that every node $w \in V_n$ has degree 4 in exactly one tour in $\mathscr{B}^*$ and has degree 2 in all the others. Consequently, J_w has cardinality 1, and so it is c^*-connected in w for all $w \in V_n$. Finally since $J_{u_{n+1}} \subseteq F$, by (iv) the theorem follows. □

4.2. Zero-lifting

By using Proposition 4.1 we derive now a specialization of Theorems 4.3 and 4.4 for the inequalities obtained by zero-lifting.

Theorem 4.5. *Let $c^*x^* \geq c_0$ be a* TT *inequality defined on $\mathbb{R}^{E_{n^*}}$ and obtained by zero-lifting of node $u \in V_n$ from a* TT *inequality $cx \geq c_0$ that is facet-defining for* GTSP(n). *Then $c^*x^* \geq c_0$ is facet-defining for* GTSP(n^*).

Proof. We first assume that $n^* = n+1$. Let F be defined by $F = \delta(u) \cup \{(v, w)\}$, with $(v, w) \in \Delta_c(u)$. Since every tour of K_n contains at least two edges in $\delta(u)$ the conditions (i) and (ii) of Theorem 4.3 are satisfied. The condition (iii) holds because the graph (V_n, F) contains only one connected component with only one cycle on the three nodes u, v and w. The proof is then completed by induction on n^*. □

The result of Theorem 4.5 is given already in Cornuéjols, Fonlupt and Naddef (1985) and has been given here as an application of Theorem 4.3.

Lemma 4.6. *Let $cx \geq c_0$ be a nontrivial* TT *inequality facet-defining for* STSP(n). *If $c(u, v) = 0$ for some $u \neq v \in V_n$ the edge sets $\delta(u)$ and $\delta(v)$ are c-connected.*

Proof. Every edge (w, u) with $w \neq v$ is c-adjacent to (u, v). In fact let $H \in \mathcal{H}_c^=$ be a Hamiltonian cycle containing (w, u), which exists by Corollary 2.6. If (u, v) does not belong to H, let $w' \neq v$ be a neighbor of u and let z and z' be the neighbors of v in H. The Hamiltonian cycle $H' = H - (z, v) - (z', v) + (z, z') - (u, w') + (u, v) + (v, w')$ contains both (w, u) and (u, v) and belongs to $\mathcal{H}_c^=$. Analogously it can be proven that $\delta(v)$ is c-connected. □

Theorem 4.7. *Let $c^*x^* \geq c_0$ be a* TT *inequality defined on $\mathbb{R}^{E_{n^*}}$ and obtained by zero-lifting of node $u \in V_n$ from a nontrivial* TT *inequality that is facet-defining for* STSP(n). *If the edge set $\delta(u) \subseteq E_n$ is c-connected then $c^*x^* \geq c_0$ is facet-defining for* STSP(n^*).

Proof. We first assume that $n^* = n+1$. Let F be defined by $F = \delta(u) \cup \{e\}$, with $e = (v, w) \in \Delta_c(u)$. By the same argument used in the proof of Theorem 4.5 the conditions (i), (ii) and (iii) of Theorem 4.4 are satisfied. Observe that if two edges f and g in $\delta(u)$ are c-adjacent then they are c^*-adjacent in u_{n+1}. By Corollary 2.6 there exists a Hamiltonian cycle in $\mathcal{H}_c^=$ containing the edge e. Let f be an edge of this cycle belonging to $\delta(u)$. Consequently, e and f are c^*-adjacent in u_{n+1} and the condition (iv) of Theorem 4.4 is satisfied. By Lemma 4.6, $\delta(u)$ in K_{n+1} is c^*-connected. Consequently, the proof can be completed by induction on n^*. □

The following lemma gives conditions that guarantee the c-connectivity property in an inequality obtained by general 1-node lifting.

Lemma 4.8. *Let $cx \geqslant c_0$ be a* TT *inequality facet-defining for* STSP(n), $c^*x^* \geqslant c_0$ *be an inequality obtained by* 1*-node lifting of* $cx \geqslant c_0$ *and* F *be a subset of* $\Delta_{c^*}(u_{n+1})$ *such that* $F \cap H$ *is nonempty for all* $H \in \mathscr{H}_c^=$ *and the conditions* (ii), (iii) *and* (iv) *of Theorem* 4.4 *are satisfied. Then the following two conditions hold*:

(a) *The edge set* $\delta(u_{n+1}) \subseteq E_{n+1}$ *is* c^**-connected if the graph* (V_n, F) *is connected.*

(b) *For every* $v \in V_n$ *the edge set* $\delta(v) \subseteq E_{n+1}$ *is* c^**-connected if the edge set* $\delta(v) \subseteq E_n$ *is c-connected.*

Proof. Let $e = (v, w)$ be an edge in F and $H \in \mathscr{H}_c^=$ be a Hamiltonian cycle containing e. The cycle $H - (v, w) + (v, u_{n+1}) + (w, u_{n+1})$ belongs to $\mathscr{H}_{c^*}^=$ and hence (v, u_{n+1}) and (w, u_{n+1}) are c^*-adjacent and (a) follows.

Consider two c-adjacent edges $e = (v, w)$ and $f = (v, z)$ in $\delta(v)$, with $w \neq z$. Then there exists a Hamiltonian cycle $H \in \mathscr{H}_c^=$ containing them. If neither e nor f belongs to F, H must contain an edge in F and it can be extended to a Hamiltonian cycle in $\mathscr{H}_{c^*}^=$ containing e and f. If both e and f belong to F they are c^*-connected because of the two Hamiltonian cycles $H - (v, z) + (v, u_{n+1}) + (z, u_{n+1})$ and $H - (v, w) + (v, u_{n+1}) + (w, u_{n+1})$ in $\mathscr{H}_{c^*}^=$. If only one of the two edges, say e, belongs to F, by (ii) of Theorem 4.4 there exist $e' = (v', w') \in \Delta_{c^*}(u_{n+1})$, with $e' \neq e$, and $H' \in \mathscr{H}_c^=$ containing e and e'. Let $y \neq w$ be the second neighbor of v in H'. Then e and f are c^*-connected because of the Hamiltonian cycles $H - (v, w) + (v, u_{n+1}) + (w, u_{n+1})$, $H' - (v, w) + (v, u_{n+1}) + (w, u_{n+1})$ and $H' - (v', w') + (v', u_{n+1}) + (w', u_{n+1})$, which all belong to $\mathscr{H}_{c^*}^=$ and (b) follows. □

A TT inequality of $cx \geqslant c_0$ defined on $\mathbb{R}^{E_n}$ with $c_e > 0$ for all $e \in E_n$ is called *simple.*

Suppose we are given a TT inequality $cx \geqslant c_0$ which is not simple. It is easy to see that V_n can be partitioned into h sets $V^1, \ldots, V^h$ such that:

(a) $c_e = 0$ for all $e \in \gamma(V^i)$, $i = 1, \ldots, h$;

(b) $c_e = c_f$ for all $e, f \in (V^i : V^j)$ and for all $i \neq j \in \{1, \ldots, h\}$,

where by $(U : W)$ we denote the edge set $(U : W) = \{(u, w) \mid u \in U, w \in W\}$. The *simple inequality associated with* $cx \geqslant c_0$ is the inequality $\bar{c}\bar{x} \geqslant c_0$ defined on $\mathbb{R}^{E_h}$ with

$$\bar{c}(u_i, u_j) = c_e, \quad e \in (V^i : V^j), \qquad \text{for all } 1 \leqslant i < j \leqslant h.$$

From Theorem 4.5 it follows that a TT inequality $cx \geqslant c_0$ is facet-defining for GTSP(n) if its associated simple inequality $\bar{c}\bar{x} \geqslant c_0$ is facet-defining for GTSP(h).

For STSP(n) we have the following theorem that can be easily proven by inductively applying Theorem 4.7.

Theorem 4.9. *A* TT *inequality* $cx \geqslant c_0$ *is facet-defining for* STSP(n) *if its associated simple inequality* $\bar{c}\bar{x} \geqslant c_0$ *is nontrivial and facet-defining for* STSP(h) *and for every* $v \in V_h$ *the set* $\delta(v)$ *in* K_h *is* $\bar{c}$*-connected.* □

An example of inequality which is not simple is the general comb inequality, whose associated simple inequality is the 2-matching inequality.

Theorems 4.5 and 4.9 suggest to restrict the study of the polyhedral structure of GTSP(n) and STSP(n) only to simple inequalities. In fact once a new simple inequality $cx \geqslant c_0$ in TT form is proven to define a facet of STSP(n) the only additional work to be done is to prove that $\delta(v)$ is c-connected for every node v in V_n. Thus all inequalities having $cx \geqslant c_0$ as associated simple inequalities are automatically proven to be facet-defining for GTSP(n^*) and STSP(n^*).

We do not know of any simple inequality in TT form that is facet-defining for STSP(n) for which the conditions of Theorem 4.9 do not hold.

4.3. 2-Node lifting

We introduce here a new kind of node lifting that cannot be obtained by sequentially applying the 1-node lifting described before. In fact in this lifting a pair of nodes is added at once to the graph K_n and the right-hand side c_0^* is greater than c_0.

Definition 4.10. Let $cx \geqslant c_0$ be a TT inequality defined on $\mathbb{R}^{E_n}$ and e be an edge in E_n. We say that the inequality $c^*x^* \geqslant c_0^*$ defined on $\mathbb{R}^{E_{n+2h}}$, with $h \geqslant 1$, is obtained from $cx \geqslant c_0$ by *cloning the edge e* (h times) in the following sense. The inequality is obtained by node-lifting of $cx \geqslant c_0$ and defined as follows, where we assume without loss of generality that $e = (u_{n-1}, u_n)$:

$$c_0^* = c_0 + 2hc_e,$$

$$c^*(u_i, u_{n+j}) = \begin{cases} c(u_i, u_{n-1}), & \text{for } 1 \leqslant i \leqslant n-2, 1 \leqslant j \leqslant 2h-1 \text{ and } j \text{ odd}, \\ c(u_i, u_n), & \text{for } 1 \leqslant i \leqslant n-2, 2 \leqslant j \leqslant 2h \text{ and } j \text{ even}, \end{cases}$$

$$c^*(u_{n+i}, u_{n+j}) = \begin{cases} 2c_e, & \text{for } -1 \leqslant i < j \leqslant 2h \text{ and } j-i \text{ even}, \\ c_e, & \text{for } -1 \leqslant i < j \leqslant 2h \text{ and } j-i \text{ odd}. \end{cases}$$

The inequality $c^*x^* \geqslant c_0^*$ of Definition 4.10 can be alternatively obtained from $cx \geqslant c_0$ by iterating the process of cloning the edge e (one time). Therefore the following proofs are given for $h = 1$ and then can be easily completed by induction on h.

Let $cx \geqslant c_0$ be a TT inequality valid for GTSP(n); we call an edge $e = (u, v)$ *c-clonable* if the c-length of every tour T of K_n is at least $c_0 + (d-2)c_e$, where d is the minimum of the degrees of u and v in T.

Lemma 4.11. *Let $cx \geqslant c_0$ be a* TT *inequality valid for* GTSP(n) *and $c^*x^* \geqslant c_0^*$ be the inequality of $\mathbb{R}^{E_{n+2h}}$ obtained from $cx \geqslant c_0$ by cloning $e = (u_{n-1}, u_n)$ (h times). Then the following two propositions are equivalent*:

(a) *e is c-clonable*;

(b) *the inequality $c^*x^* \geqslant c_0^*$ is valid for* GTSP($n+2h$) *and every edge of the set* $(\{u_{n-1}, u_{n+1}, \ldots, u_{n+2h-1}\} : \{u_n, u_{n+2}, \ldots, u_{n+2h}\})$ *is c^*-clonable.*

Proof. We prove the lemma for $h=1$. Then a complete proof can be obtained by induction on h. We first prove that (a) implies (b). For every tour $T \in \mathcal{T}^*_{n+2}$ we define $\mathcal{S}(T) \in \mathcal{T}^*_n$ to be the tour obtained by contracting each of the sets $\{u_{n-1}, u_{n+1}\}$ and $\{u_n, u_{n+2}\}$ into a single node. Observe that by definition $c(\mathcal{S}(T)) \leq c^*(T)$. Assume that either the inequality $c^*x^* \geq c^*_0$ is not valid or that some edge of the set $(\{u_{n-1}, u_{n+1}\}:\{u_n, u_{n+2}\})$ is not c^*-clonable. Due to the symmetry of the coefficients of the pair $\delta(u_{n-1})$ and $\delta(u_{n+1})$ and of the pair $\delta(u_n)$ and $\delta(u_{n+2})$ we can assume without loss of generality that (u_{n-1}, u_n) is not c^*-clonable. It follows that there exists $T^* \in \mathcal{T}^*_{n+2}$, where u_{n-1} and u_n have degree at least d, with $c^*(T^*) < c^*_0 + (d-2)c^*_e$. We consider two cases:

Case (i). Both the families $T^* \cap \delta(\{u_{n-1}, u_{n+1}\})$ and $T^* \cap \delta(\{u_n, u_{n+2}\})$ have at least $d+2$ edges. Then the tour $\mathcal{S}(T^*)$ has both u_{n-1} and u_n with degree at least $d+2$; hence its c-length is at least $c_0 + dc_e = c^*_0 + (d-2)c^*_e$. This leads to a contradiction.

Case (ii). Without loss of generality we assume that $T^* \cap \delta(\{u_n, u_{n+2}\})$ contains d edges. Then $(u_n, u_{n+2}) \in T^*$ and $c(\mathcal{S}(T^*)) = c^*(T^*) - 2c_e$. Consequently, e is not c-clonable.

Now we prove that (b) implies (a). Suppose that there is a tour $T \in \mathcal{T}^*_n$ where u_{n-1} and u_n have degree at least $d \geq 4$ and that $c(T) < c_0 + (d-2)c_e$. We consider 3 cases.

Case (i). There are at least three edges different from e incident with u_{n-1} and u_n, respectively. Then there are two neighbors v_1 and v_2 of u_{n-1} in T and two neighbors w_1 and w_2 of u_n in T such that the family

$$T^* = T - (v_1, u_{n-1}) - (v_2, u_{n-1}) - (w_1, u_n) - (w_2, u_n)$$

$$+ (v_1, u_{n+1}) + (v_2, u_{n+1}) + (w_1, u_{n+2}) + (w_2, u_{n+2})$$

is a tour of K_{n+2}.

Case (ii). T contains two copies of e. Then there is a neighbor v of u_{n-1} in T and a neighbor w of u_n in T such that the family

$$T^* = T - (v, u_{n-1}) - (u_{n-1}, u_n) - (w, u_n)$$

$$+ (v, u_{n+1}) + (u_{n+1}, u_{n+2}) + (w, u_{n+2})$$

is a tour of K_{n+2}.

Case (iii). T contains at least three copies of e. Then let T^* be the tour of K_{n+2} defined by

$$T^* = T - 3\{(u_{n-1}, u_n)\} + (u_{n-1}, u_{n+2}) + (u_{n+1}, u_{n+2}) + (u_n, u_{n+1}).$$

In all the three cases we have $c^*(T^*) = c(T) < c^*_0 + (d-4)c_e$. Since u_{n-1} and u_n have both degree at least $d-2$ in T^* it follows that either $d=4$ and $c^*x^* \geq c^*_0$ is not valid or $d>4$ and e is not c^*-clonable. □

Theorem 4.12. *Let $cx \geqslant c_0$ be a* TT *inequality facet-defining for* GTSP(n) *and* $e = (u_{n-1}, u_n)$ *be a c-clonable edge such that u_{n-1} and u_n are $2c_e$-critical for $cx \geqslant c_0$. The following hold:*

(a) *the inequality $c^*x^* \geqslant c_0^*$ obtained from $cx \geqslant c_0$ by cloning e (h times) is facet-defining for* GTSP($n+2h$);

(b) *the nodes u_{n-1}, u_n, $u_{n+1}, \ldots, u_{n+2h}$ are $2c_e^*$-critical;*

(c) *if $f = (z_1, z_2) \neq e$ is an edge in E_n such that z_1 and z_2 are $2c_f$-critical for $cx \geqslant c_0$, then z_1 and z_2 are $2c_f^*$-critical for $c^*x^* \geqslant c_0^*$.*

Proof. We prove the theorem for $h = 1$. Then the proof can be completed by induction on h. The inequality $c^*x^* \geqslant c_0^*$ is valid for GTSP($n+2$) by Lemma 4.11 and supporting since, as it is shown in the following, $\mathcal{T}_{c^*}^{=}$ is nonempty. Consequently, it defines a face of GTSP($n+2$). Let $f^*x^* \geqslant f_0^*$ be an inequality defining a facet of GTSP($n+2$) that contains the face defined by $c^*x^* \geqslant c_0^*$. Therefore we have $\mathcal{T}_{c^*}^{=} \subseteq \mathcal{T}_{f^*}^{=}$. We construct a subset of $\mathcal{T}_{c^*}^{=}$ and we prove that it is a basis of $c^*x^* \geqslant c_0^*$. In fact by using only vectors in this subset it can be shown that $f^* = \pi c^*$, for some $\pi > 0$. For notational convenience we name the following 6 edges in E_{n^*} as: $e = (u_{n-1}, u_n)$, $a = (u_{n-1}, u_{n+1})$, $b = (u_{n-1}, u_{n+2})$, $c = (u_n, u_{n+1})$, $d = (u_n, u_{n+2})$ and $g = (u_{n+1}, u_{n+2})$ (see Figure 6). By assumption and by Lemma 2.5 there exist two Hamiltonian cycles Γ_1 and Γ_2 of $K_n - \{u_n\}$ and $K_n - \{u_{n-1}\}$, respectively, of length $c_0 - 2c_e$. Let v_1 be one of the two neighbors of u_{n-1} in Γ_1 and v_2 one of the two neighbors of u_n in Γ_2. We set $\mathcal{B} = \{T_1, T_2, T_3, T_4, H_1, H_2, H_3\}$, where T_i, $i = 1, \ldots, 4$ and H_i, $i = 1, 2, 3$, belong to $\mathcal{T}_{c^*}^{=}$ and are defined as (see also Figure 7):

$$T_1 = \Gamma_1 - (v_1, u_{n-1}) + (v_1, u_{n+1}) + 2\{e\} + b + g,$$

$$T_2 = \Gamma_1 - (v_1, u_{n-1}) + (v_1, u_{n+1}) + 2\{c\} + b + g,$$

$$T_3 = \Gamma_2 - (v_2, u_n) + (v_2, u_{n+2}) + 2\{e\} + c + g,$$

$$T_4 = \Gamma_2 - (v_2, u_n) + (v_2, u_{n+2}) + 2\{b\} + c + g,$$

$$H_1 = \Gamma_1 - (v_1, u_{n-1}) + (v_1, u_{n+1}) + e + d + g,$$

$$H_2 = \Gamma_1 - (v_1, u_{n-1}) + (v_1, u_{n+1}) + b + d + c,$$

$$H_3 = \Gamma_2 - (v_2, u_n) + (v_2, u_{n+2}) + b + a + c.$$

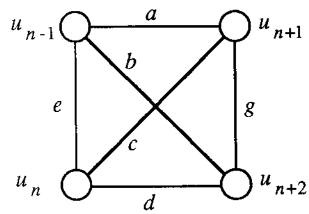

Fig. 6.

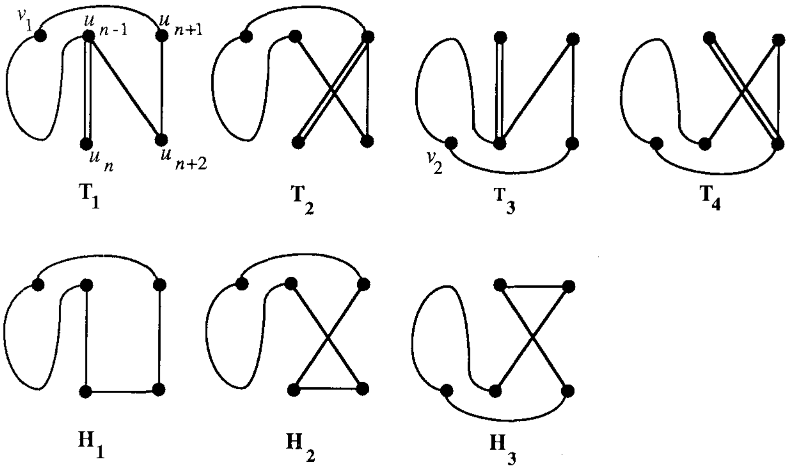

Fig. 7.

From $f^*(T_1)=f^*(T_2)$, $f^*(T_3)=f^*(T_4)$, $f^*(H_1)=f^*(H_2)$, $f^*(H_1)=f^*(T_1)$ and $f^*(H_3)=f^*(T_4)$ it follows:

$$f_e^*=f_b^*=f_c^*=f_g^* \quad \text{and} \quad f_a^*=f_d^*=2f_e^*. \tag{4.6}$$

For every tour $T\in\mathcal{T}_c^=$ containing the edge (w, u_{n-1}) we define two "extended" tours $\mathscr{E}_1(T)$ and $\mathscr{E}_1'(T)$ belonging to $\mathcal{T}_{c^*}^=$ in the following way:

Case (i). $e\in T$ (see Figure 8):

$$\mathscr{E}_1(T)=T-e+b+c+g,$$

$$\mathscr{E}_1'(T)=T-(w, u_{n-1})+(w, u_{n+1})+b+g.$$

Case (ii). $e\notin T$. Let v be any neighbor of u_n in T (see Figure 9):

$$\mathscr{E}_1(T)=T-(v, u_n)+(v, u_{n+2})+c+g,$$

$$\mathscr{E}_1'(T)=T-(w, u_{n-1})-(v, u_n)+(w, u_{n+1})+(v, u_{n+2})+b+c.$$

If $\mathscr{E}_1'(T)$ is not a tour we take the edges e and g instead of b and c.

For every tour $T\in\mathcal{T}_c^=$ containing the edge (w, u_n) we analogously define the two extended tours $\mathscr{E}_2(T)$ and $\mathscr{E}_2'(T)$ in $\mathcal{T}_{c^*}^=$.

For every node $w\in V_n-\{u_{n-1}, u\}$ we call H_w^1 and H_w^2 two Hamiltonian cycles in $\mathscr{H}_c^=$ containing (w, u_{n-1}) and (w, u_n), respectively. By $f^*(\mathscr{E}_1(H_w^1))=f^*(\mathscr{E}_1'(H_w^1))$, $f^*(\mathscr{E}_2(H_w^2))=f^*(\mathscr{E}_2'(H_w^2))$ and (4.6) it follows that

$$\begin{aligned} &f^*(w, u_{n-1})=f^*(w, u_{n+1}), \quad w\in V_n-\{u_{n-1}\},\\ &f^*(w, u_n)=f^*(w, u_{n+2}), \quad w\in V_n-\{u_n\}. \end{aligned} \tag{4.7}$$

We add to $\mathscr{B}$ the set

$$\{\mathscr{E}_1(H_w^1),\ \mathscr{E}_1'(H_w^1),\ \mathscr{E}_2(H_w^2),\ \mathscr{E}_2'(H_w^2)\mid w\in V_n-\{u_{n-1}, u_n\}\}.$$

Observe that, besides the 4 almost Hamiltonian tours T_1, T_2, T_3, and T_4, the set $\mathscr{B}$ contains only Hamiltonian cycles.

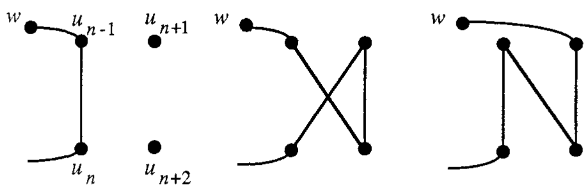

Fig. 8.

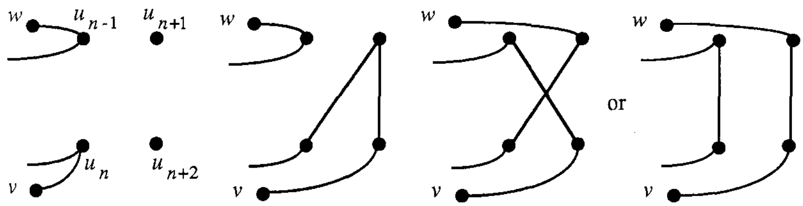

Fig. 9.

Let us define the inequality $fx \geqslant f_0$ on $\mathbb{R}^{E_n}$ by:

$$f_g = f_g^*, \quad g \in E_n,$$

$$f_0 = f_0^* - 2f_e.$$

Let $\mathscr{B}_c$ be a basis for $cx \geqslant c_0$, T be a tour of $\mathscr{B}_c$, and $\mathscr{E}(T)$ be any of the tours $\mathscr{E}_1(T)$, $\mathscr{E}_1'(T)$, $\mathscr{E}_2(T)$, $\mathscr{E}_2'(T)$. Then $\mathscr{E}(T) \in \mathscr{T}_{c^*}^=$, and so $\mathscr{E}(T) \in \mathscr{T}_{f^*}^=$. By (4.6) and (4.7) it follows that $f^*(\mathscr{E}(T)) = f(T) + 2f_e = f_0^*$, and so $T \in \mathscr{T}_f^=$ and $\mathscr{B}_c \subseteq \mathscr{T}_f^=$. Since $cx \geqslant c_0$ defines a facet of GTSP(n) it follows that $fx \geqslant f_0$ defines the same facet and for some $\pi > 0$:

$$\begin{aligned} f &= \pi c, \\ f_0 &= f_0^* - 2\pi c_e = \pi c_0. \end{aligned} \tag{4.8}$$

From (4.6), (4.7) and (4.8) it follows that $f^* = \pi c^*$ and $f_0^* = \pi c_0^*$, with $\pi > 0$; hence $\mathscr{B} \cup \{\mathscr{E}(T) \mid T \in \mathscr{B}_c\}$ contains a basis for $c^*x^* \geqslant c_0^*$ and (a) follows. To prove (b), consider the tours $T_3 - 2\{e\}$, $T_1 - 2\{e\}$, $H_2 - b - d + e$ and $H_3 - a - c + e$, that have c^*-length $c_0^* - 2c_e^*$ and apply Remark 3.2. To prove (c), take a tour T of $K_n - \{z_1\}$ of c-length $c_0 - 2c_f$. Let $\bar{\mathscr{E}}_1(\cdot)$ and $\bar{\mathscr{E}}_2(\cdot)$ be two transformations, defined in the same way as $\mathscr{E}_1(\cdot)$ and $\mathscr{E}_2(\cdot)$, respectively, that extend a tour of K_{n-1} of c-length l into a tour of K_{n+1} of length $l + 2c_e$. If $e \in T$, call T' the tour $\bar{\mathscr{E}}_1(T)$; otherwise call T' the tour $\bar{\mathscr{E}}_2(T)$. The tour T' of K_{n+1} has c^*-length $c_0^* - 2c_f^*$, and so by Remark 3.2 the node z_1 is $2c_f^*$-critical for $c^*x^* \geqslant c_0^*$. The same holds for z_2 and (c) follows. □

Theorem 4.13. *Let $cx \geqslant c_0$ be a nontrivial* TT *inequality facet-defining for* STSP(n) *and $e = (u_{n-1}, u_n)$ be a c-clonable edge such that u_{n-1} and u_n are $2c_e$-critical for $cx \geqslant c_0$. Then the following hold*:

(a) *the inequality $c^*x^* \geqslant c_0^*$ obtained by cloning e (h times) is facet-defining for* STSP($n + 2h$);

(b) *the edge subsets $\delta(u_{n-1}), \ldots, \delta(u_{n+2h})$ of E_{n+2h} are c^*-connected*;

(c) *for* $v \in V_n - \{u_{n-1}, u_n\}$ *if* $\delta(v)$ *in* K_n *is c-connected, then* $\delta(v)$ *in* K_{n+2h} *is* c^**-connected.*

Proof. We prove the theorem for $h = 1$. Then the proof can be completed by induction on h. Since $cx \geq c_0$ is nontrivial and facet-defining for STSP(n) it has a canonical basis $\mathscr{C}_c$. Let Γ_1, Γ_2, $\mathscr{E}_1(\cdot)$, $\mathscr{E}_1'(\cdot)$, $\mathscr{E}_2(\cdot)$, $\mathscr{E}_2'(\cdot)$, $\mathscr{E}(\cdot)$ and $\mathscr{B}$ be defined as in the proof of Theorem 4.12. Weithout loss of generality we can assume that $T_{u_{n-1}} = \Gamma_1 + 2\{e\}$ and $T_{u_n} = \Gamma_2 + 2\{e\}$ belong to $\mathscr{C}_c$. Since $\mathscr{E}_1'(T_{u_{n-1}}) \in \mathscr{B}$ and $\mathscr{E}_2'(T_{u_n}) \in \mathscr{B}$, the set

$$\mathscr{B} \cup \{\mathscr{E}(T) \mid T \in \mathscr{C}_c - \{T_{u_{n-1}}, T_{u_n}\}\}$$

contains a canonical basis of $c^*x^* \geq c_0^*$ and the inequality is facet-defining for STSP($n+2$).

Assume now that $v \in V_n - \{u_{n-1}, u_n\}$. If two edges (w_1, v) and (w_2, v) are c-adjacent let $H \in \mathscr{H}_c^=$ be a cycle containing both of them. Then there exists a cycle $\mathscr{E}(H) \in \mathscr{H}_{c^*}^=$ containing (w_1, v) and (w_2, v). For $i = 1$, 2, there exist $w \in V_n$ and a cycle $H^i \in \mathscr{H}_c^=$ containing (v, u_{n+i-2}) and (v, w). The cycle $\mathscr{E}_i'(H^i) \in \mathscr{H}_{c^*}^=$ contains (v, w) and (v, u_{n+i}). Hence if $\delta(v)$ in K_n is c-connected, $\delta(v)$ in K_{n+2} is c^*-connected.

For $i = 1, 2$, let $v = u_{n+i-2}$. For every $w \in V_n - \{u_{n-1}, u_n\}$ there exists a cycle $H_w \in \mathscr{H}_c^=$ that contains (w, v) because $cx \geq c_0$ is a nontrivial facet-defining inequality for STSP(n). The cycle $\mathscr{E}_i(H_w) \in \mathscr{H}_{c^*}^=$ contains both (w, v) and (u_{n+1}, u_{n+2}). Moreover, the edges in $\gamma(\{u_{n-1}, u_n, u_{n+1}, u_{n+2}\})$ are c^*-connected because of the cycles H_1, H_2, and H_3 defined in the proof of Theorem 4.12. Consequently, $\delta(u_{n+i-2})$ is c^*-connected. Due to the symmetry of the coefficients of the edges in $\delta(u_{n+i-2})$ and $\delta(u_{n+i})$, $\delta(u_{n+i})$ is also c^*-connected. □

Observe that, differently from other node lifting theorems, in the lifting by edge-cloning no extra conditions are required in the case of STSP(n).

As shown in Naddef and Rinaldi (1988), Theorem 4.13 can be exploited to prove that some inequalities, generalizing the chain inequalities defined in Padberg and Hong (1980), are facet-defining for STSP(n), $n \geq 8$. These inequalities are obtained from the comb inequalities by repeated application of edge-cloning and zero-lifting operations. A proof that the chain inequalities are facet-defining is given already by Boyd (1988) and by Hartman (1988). However, the range of application of Theorem 4.13 goes beyond the comb inequalities. In fact, as shown in Naddef and Rinaldi (1988, 1992), the theorem can be applied to prove that inequalities obtained by edge-cloning from some path inequalities, that generalize comb inequalities, and from crown inequalities are facet-defining for STSP(n).

Acknowledgments

Thanks are due to two anonymous referees for their helpful comments. The authors wish also to acknowledge ARTEMIS Université Joseph Fourier de Grenoble and

Istituto di Analisi dei Sistemi ed Informatica del CNR for supporting the mutual visits they had during the writing of this paper.

References

X. Berenguer, "A characterization of linear admissible transformations for the m-travelling salesmen problem," *European Journal of Operational Research* 3 (1979) 232-249.

S. Boyd, private communication (1988).

S. Boyd and W.H. Cunningham, "Small travelling salesman polytopes," *Mathematics of Operations Research* 16 (1991) 259-271.

V. Chvátal, "Edmonds polytopes and weakly Hamiltonian graphs," *Mathematical Programming* 5 (1973), 29-40.

G. Cornuéjols, J. Fonlupt and D. Naddef, "The traveling salesman problem on a graph and some related integer polyhedra," *Mathematical Programming* 33 (1985) 1-27.

T. Christof, M. Jünger and G. Reinelt, "A complete description of the traveling salesman polytope on 8 nodes," *Operations Research Letters* 10 (1991) 497-500.

G.B. Dantzig, D.R. Fulkerson and S.M. Johnson, "Solution of a large-scale travelling salesman problem," *Operations Research* 2 (1954) 393-410.

J. Edmonds, "Maximum matching and a polyhedron with 0, 1-vertices," *Journal of Research of the National Bureau of Standards, Section B* 69 (1965) 125-130.

B. Fleischmann, "A new class of cutting planes for the symmetric travelling salesman problem," *Mathematical Programming* 40 (1988) 225-246.

M. Hartman, private communication (1988).

M. Grötschel and O. Holland, "Solution of large-scale symmetric travelling salesman problems," *Mathematical Programming* 15 (1991) 141-202.

M. Grötschel and M. Padberg, "On the symmetric travelling salesman problem," working paper, Institut für Oekonometrie und Operation Research, Universität Bonn (1975).

M. Grötschel and M. Padberg, "On the symmetric travelling salesman problem I: Inequalities," *Mathematical Programming* 16 (1979) 265-280.

M. Grötschel and M. Padberg, "On the symmetric travelling salesman problem II: Lifting theorems and facets," *Mathematical Programming* 16 (1979) 281-302.

M. Grötschel and M. Padberg, "Polyhedral theory," in: E.L. Lawler et al., eds., *The Traveling Salesman Problem* (Wiley, Chichester, 1985) pp. 251-305.

M. Grötschel and W.R. Pulleyblank, "Clique tree inequalities and the symmetric travelling salesman problem," *Mathematics of Operations Research* 11 (1986) 537-569.

E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D.B. Shmoys, eds., *The Traveling Salesman Problem* (Wiley, Chichester, 1985).

J.F. Maurras, "Some results on the convex hull of Hamiltonian cycles of symmetric complete graphs," in: B. Roy, ed., *Combinatorial Programming: Methods and Applications* (Reidel, Dordrecht, 1975) pp. 179-190.

D. Naddef and G. Rinaldi, "The symmetric traveling salesman polytope and its graphical relaxation: Composition of valid inequalities," *Mathematical Programming* 51 (1991) 359-400.

D. Naddef and G. Rinaldi, "The symmetric traveling salesman polytope: New facets from the graphical relaxation," Report R.248, IASI-CNR (Rome, 1988).

D. Naddef and G. Rinaldi, "The crown inequalities for the symmetric traveling salesman polytope," *Mathematics of Operations Research* 17 (1992) 308-326.

G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization* (Wiley, Chicester, 1988).

M. Padberg, "On the facial structure of the set packing polyhedra," *Mathematical Programming* 5 (1973) 199-216.

M. Padberg and M. Grötschel, "Polyhedral computations," in: E.L. Lawler et al., eds., *The Traveling Salesman Problem* (Wiley, Chichester, 1985) pp. 307-360.

M. Padberg and S. Hong, "On the symmetric travelling salesman problem: A computational study," *Mathematical Programming Study* 12 (1980) 78-107.

M. Padberg and G. Rinaldi, "Optimization of a 532-city symmetric traveling salesman problem by branch-and-cut," *Operations Research Letters* 6 (1987) 1-7.

M. Padberg and G. Rinaldi, "Facet identification for the symmetric traveling salesman polytope," *Mathematical Programming* 47 (1990) 219-257.

M. Padberg and G. Rinaldi, "A branch-and-cut algorithm for the resolution of large-scale symmetric traveling salesman problems," *SIAM Review* 33 (1991) 60-100.

W.R. Pulleyblank, "Polyhedral combinatorics," in: A. Bachem et al., eds., *Mathematical Programming: The State of the Art - Bonn 1982* (Springer, Berlin, 1983) pp. 312-345.

M. Queyranne and Y. Wang, private communication (1989).

H. Weyl, "Elementare Theorie der konvexen Polyeder," *Commentarii Mathematici Helvetici* 7 (1935) 509-533.