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Convex composite multi-objective nonsmooth programming

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This paper examines nonsmooth constrained multi-objective optimization problems where the objective function and the constraints are compositions of convex functions, and locally Lipschitz and Gâteaux differentiable functions. Lagrangian necessary conditions, and new sufficient optimality conditions for efficient and properly efficient solutions are presented. Multi-objective duality results are given for convex composite problems which are not necessarily convex programming problems. Applications of the results to new and some special classes of nonlinear programming problems are discussed. A scalarization result and a characterization of the set of all properly efficient solutions for convex composite problems are also discussed under appropriate conditions.

Key words: Composite functions, nonsmooth programming, multi-objective problems, necessary and sufficient conditions, scalarizations.

1. Introduction

Consider the composite multi-objective programming problem

$$\text{(P)} \qquad \text{V-minimize} \quad (f_1(F_1(x)), \ldots, f_p(F_p(x)))$$

$$\text{subject to} \quad x \in C, \quad g_j(G_j(x)) \leq 0, \; j = 1, 2, \ldots, m,$$

where C is a convex subset of a Banach space X, f_i, $i = 1, 2, \ldots, p$, g_j, $j = 1, 2, \ldots, m$, are real valued locally Lipschitz functions on $\mathbb{R}^n$, and F_i and G_j are locally Lipschitz and Gâteaux differentiable functions from X into $\mathbb{R}^n$ with Gâteaux derivatives $F_i'(\cdot)$ and $G_j'(\cdot)$ respectively, but are not necessarily continuously Fréchet differentiable or strictly differentiable [6]. Note here that the symbol "V-minimize" stands for vector minimization. The model problem (P) with $p = 1$ (single objective function) and continuously (Fréchet) differentiability conditions has recently received a great deal of attention in the literature, e.g., [1, 3, 8, 9, 13]. It is known that the scalar

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composite programming model prolem provides a unified framework for studying convergence behaviour of various algorithms and Lagrangian conditions, e.g., see [2, 8, 27]. More recently, various first order optimality conditions of Lagrangian type were given in Jeyakumar [17] for *single* objective composite model problems of the form (P) without the continuously Fréchet differentiability or the strict differentiability restrictions using an approximation scheme.

Problems of multi-objective optimization are widespread in mathematical modelling of real world systems for a very broad range of applications. For instance, multi-objective optimization problems which arise in mechanical engineering are discussed in Stadler [31]; applications of multi-objective optimization techniques for the design of aircraft control systems are given in Schy and Giesy [29]; various other applications of multi-objective optimization in resource planning and management, in mathematical biology,and in welfare economics can be found in Stadler [30]. We are rarely asked to make decisions based on only one criterion; most often, decisions are based on several conflicting criteria. Multi-objective optimization provides the mathematical framework to deals with these situations.

The composite model problem (P) is broad and flexible enough to cover many common types of multi-objective problems, seen in the literature. Moreover, the model obviously includes the wide class of convex composite single objective problems, which is now recognized as fundamental for theory and computation in scalar nonsmooth optimization. To illustrate the nature of the model (P), let us look at some examples.

Example 1.1. Define $F_i, G_j : X^n \to \mathbb{R}^{p+m}$ by

$$F_i(x) = (0, 0, \dots, l_i(x), 0, \dots, 0), \quad i = 1, 2, \dots, p,$$

$$G_j(x) = (0, 0, \dots, h_j(x), 0, \dots, 0), \quad j = 1, 2. \dots, m,$$

where $l_i(x)$ and $h_j(x)$ are locally Lipschitz and Gâteaux differentiable functions on a Banach space X. Define $f_i, g_j : \mathbb{R}^{p+m} \to \mathbb{R}$ by

$$f_i(x) = x_i, \quad i = 1, 2, \dots, p,$$

$$g_j(x) = x_{p+j}, \quad j = 1, 2, \dots , m.$$

Let $C = X$. Then the composite problem (P) is the problem

$$\text{(NP)} \quad \text{V-minimize} \quad (l_1(x), \dots, l_p(x))$$
$$\text{subject to} \quad x \in X^n, \quad h_j(x) \leq 0, \ j = 1, 2, \dots, m,$$

which is a standard multi-objective differentiable nonlinear programming problem. Lagrangian optimality conditions, duality properties, and scalarization techniques for the standard multi-objective nonlinear programming problem have been extensively studied in the literature under convex and generalized convex conditions, see, e.g., [4, 5, 14, 28].

Example 1.2. The penalty representation of the standard multi-objective nonlinear programming problem (NP), examined in White [34], is the multi-objective problem:

$$\text{V-minimize} \quad (l_1(x), \ldots, l_p(x)) + \mu \sum_{j=1}^{m} (\max(h_j(x), 0))^2 e$$

$$\text{subject to} \quad x \in \mathbb{R}^n,$$

where $e = (1, 1, \ldots, 1) \in \mathbb{R}^p$ and $\mu > 0$ is the penalty parameter. The penalty problem is the case of (P), where $F_i : \mathbb{R}^n \to \mathbb{R}^{m+1}$ and $f_i : \mathbb{R}^{m+1} \to \mathbb{R}$ are given by

$$F_i(x) = (l_i(x), h_1(x), \ldots, h_m(x)), \quad i = 1, 2, \ldots, p,$$

$$f_i(\alpha_0, \alpha_1, \ldots, \alpha_m) = \alpha_0 + \mu \sum_{j=1}^{m} (\max(\alpha_j, 0))^2, \; i = 1, 2, \ldots, p,$$

respectively, and $G_j : \mathbb{R}^n \to \mathbb{R}^n$ and $g_j : \mathbb{R}^n \to \mathbb{R}$ are given by $G_j(x) = x$ and $g_j(x) = 0$. White [34] considered a more general penalty function representation for the problem (NP) and presented a penalty function scheme, generalizing the methods of Zangwill [35].

Example 1.3. Consider the vector approximation (model) problem:

$$\text{V-minimize} \quad (\|F_1(x)\|_1, \ldots, \|F_p(x)\|_p)$$

$$\text{subject to} \quad x \in X,$$

where X is a Banach space, $\|\cdot\|_i$, $i = 1, 2, \ldots, p$, are norms in $\mathbb{R}^m$, and for each $i = 1, 2, \ldots, p$, $F_i : X \to \mathbb{R}^m$ is a Fréchet differentiable (error) function. This problem is also the case of (P), where for each $i = 1, 2, \ldots, p, f_i : \mathbb{R}^n \to \mathbb{R}$ is given by $f_i(x) = \|x\|_i$, and conditions on G_j and g_j are the same as in Example 1.2. Various examples of vector approximation problems of this type that arise in simultaneous approximation are given in [14, 15]. For a numerical example of the above convex composite vector approximation problem, see Example 3.3 in [15].

The idea is that by studying the composite model problem (P) a unified framework can be given for the treatment of many questions of theoretical and computational interest in multi-objective optimization.

In this paper we present first order Lagrangian optimality and duality results for (convex) composite problems (P) and show that the model problem (P) allows us to present a unified framework for studying multi-objective optimization problems in regard to first order optimality conditions and duality theory. In particular, we show that the results for the multi-objective model problem (P) cover the corresponding results for convex and many classes of nonconvex problems, commonly encountered in the study of first order global optimality and duality theory. The results are based on a method given recently in Jeyakumar [17] that emphasizes a new connection with the Clarke subdifferential, Gâteaux differentiability property and composite functions. Moreover, we provide some scalarization properties for the composite model problem under appropriate conditions. We present results mainly for properly efficient solutions of the composite model problem (P).

The outline of the paper is as follows. In Section 2, we present some preliminary results and obtain necessary optimality conditions of the Kuhn–Tucker type for the composite model problem (P) by using an approximation scheme. In Section 3, we present new sufficient conditions for feasible points which satisfy Kuhn–Tucker type conditions to be efficient and properly efficient solutions of convex composite problems in which the functions f_i and g_j are assumed to be convex. These sufficient conditions are shown to hold for various classes of nonconvex programming problems. In Section 4, multi-objective duality results are presented for convex composite problems. Finally, in Section 5, we provide various characterizations of the set of properly efficient soutions for convex composite problems.

2. Efficient solutions and necessary optimality conditions

In this section, we introduce various notions of efficient solutions and present some preliminary results for locally Lipschitz functions that will be used throughout the paper. Then, we obtain necessary optimality conditions for (P) that extend the necessary conditions presented in [17] for a scalar problem (cf. [6]). Let us begin with the definition of an efficient solution for the multi-objective problem (P). A feasible point x_0 for (P) is said to be an *efficient solution* [28, 33] if there exists no feasible x for (P) such that $f_i(F_i(x)) \leqslant f_i(F_i(x_0))$, $i=1,2,\ldots,p$, and $f_i(F_i(x)) \neq f_i(F_i(x_0)))$ for some i. The feasible point x_0 is said to be a *properly efficient solution* [10] for (P) if there exists a scalar $M>0$ such that for each i,

$$\frac{f_i(F_i(x_0)) - f_i(F_i(x))}{f_j(F_j(x)) - f_j(F_j(x_0))} \leqslant M,$$

for some j such that $f_j(F_j(x)) > f_j(F_j(x_0))$ whenever x is feasible for (P) and $f_i(F_i(x)) < f_i(F_i(x_0))$. The feasible point x_0 is said to be a *weakly efficient solution* [28] for (P) if there exists no feasible point x for which $f_i(F_i(x_0)) > f_i(F_i(x))$, $i=1,2,\ldots,p$. It is clear from the definitions that a properly efficient solution is an efficient solution which is a weakly efficient solution. To see the nature of these different efficient solutions, let us look at some simple examples:

Example 2.1. Consider the problem

$$\text{V-minimize} \quad \left(\frac{x_1}{x_2}, \frac{x_2}{x_1}\right)$$

$$\text{subject to} \quad (x_1, x_2) \in \mathbb{R}^2, \quad 1 - x_1 \leqslant 0, \quad 1 - x_2 \leqslant 0.$$

It is easy to check that (1, 1) is an efficient solution for the problem, but it is not a properly efficient solution.

Example 2.2 [22]. Consider the problem

$$\text{V-minimize} \quad (x_1, x_2)$$
$$\text{subject to} \quad (x_1, x_2) \in \mathbb{R}^2, \quad (x_1, x_2) \in C,$$

where

$$C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leqslant 1,\ x_2 \leqslant 0\}$$
$$\cup \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geqslant 0,\ 0 \geqslant x_2 \geqslant -1\}.$$

Then, we see that the feasible point $(1, -1)$ is a weakly efficient solution, but it is not an efficient solution.

We note that if $F: X \to \mathbb{R}^n$ is locally Lipschitz near a point $x \in X$ and Gâteaux differentiable at x and if $f: \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz near $F(x)$ then the continuous sublinear function, defined by

$$\pi_x(h) := \max\left\{ \sum_{k=1}^{n} w_k F_k'(x) h \;\middle|\; w \in \partial^0 f(F(x)) \right\},$$

satisfies the inequality

$$(f \circ F)_+'(x, h) \leqslant \pi_x(h) \quad \forall h \in X. \tag{A}$$

Recall that

$$q_+'(x, h) = \limsup_{\lambda \downarrow 0} \lambda^{-1}(q(x + \lambda h) - q(x))$$

is the upper Dini-directional derivative of $q: X \to \mathbb{R}$ at x in the direction of h, and $\partial^0 f(F(x))$ is the Clarke subdifferential of f at $F(x)$. The function $\pi_x(\cdot)$ in (A) is called upper convex approximation of $f \circ F$ at x, see [17, 18]. Moreover, it is worth observing that the inequality (A) does not follow from the generalized chain rule for differentiation of locally Lipschitz functions in Clarke [6]. However, the following generalized chain rule formula [17] follows under the above mentioned assumptions on F and f:

$$\partial^{\diamond}(f \circ F)(x) \subset \partial^0 f(F(x)) F'(x) := \{w^{\mathrm{T}} F'(x) \mid w \in \partial^0 f(F(x))\}.$$

The equality holds, in particular, when f is convex. Here, $\partial^{\diamond}(f \circ F)(x)$ denotes the Michel-Penot subdifferential of $f \circ F$ at x, see [25]. Note that for a set C, int C denotes the interior of C, and

$$C^+ = \{v \in X' \mid v(x) \geqslant 0\ \forall x \in C\},$$

denotes the dual cone of C, where X' is the topological dual space of X. It is also worth noting that, for a convex set C, the closure of the cone generated by the set C at a point a, $\operatorname{cl}\operatorname{cone}(C - a)$, is the tangent cone of C at a and, the dual cone $-(C - a)^+$ is the normal cone of C at a in the sense of convex analysis, see [6, 26].

Using upper convex approximations, we derive necessary optimality conditions which hold at a weakly efficient solution of our model problem (P). This extends the necessary conditions presented in [17] for a scalar composite problem to the multi-objective problem (P).

Theorem 2.1. *For the model problem* (P), *assume that f_i and g_j are locally Lipschitz functions, and that F_i and G_j are locally Lipschitz and Gâteaux differentiable functions. If $a \in C$ is a weakly efficient solution for* (P), *then there exist Lagrange multipliers $\tau_i \geq 0$, $i = 1, 2, \ldots, p$, and $\lambda_j \geq 0$, $j = 1, 2, \ldots m$, not all zero, satisfying*

$$0 \in \sum_{i=1}^{p} \tau_i \partial^0 f_i(F_i(a))F_i'(a) + \sum_{j=1}^{m} \lambda_j \partial^0 g_j(G_j(a))G_j'(a) - (C-a)^+$$

and $\lambda_j g_j(G_j(a)) = 0$, $j = 1, 2, \ldots, m$.

Proof. Let $I = \{1, \ldots, p\}$, $J_p = \{j+p \mid j = 1, \ldots, m\}$, $J_p(a) = \{j+p \mid g_j(G_j(a)) = 0, j = 1, \ldots, m\}$. For convenience, we define

$$h_k(x) = \begin{cases} (f_k \circ F_k)(x), & k = 1, \ldots, p, \\ (g_{k-p} \circ G_{k-p})(x), & k = p+1, \ldots, p+m. \end{cases}$$

Suppose that the following system has a solution

$$d \in \text{cone}(C-a), \quad \pi_a^k(d) < 0, \quad k \in I \cup J_p(a), \tag{$*$}$$

where $\pi_a^k(d)$ is given by

$$\pi_a^k(d) = \begin{cases} \max\left\{ \sum_{i=1}^{n} v_i F_{k_i}'(a)d \mid v \in \partial^0 f_k(F_k(a)) \right\}, & k \in I, \\ \max\left\{ \sum_{i=1}^{n} w_i G_{k_i}'(a)d \mid w \in \partial^0 g_{k-p}(F_{k-p}(a)) \right\}, & k \in J_p(a). \end{cases}$$

Then the system

$$d \in \text{cone}(C-a), \quad h_k^+(a; d) < 0, \quad k \in I \cup J_p(a),$$

has a solution. So, there exists $\alpha_1 > 0$ such that $a + \alpha d \in C$, $h_k(a+\alpha d) < h_k(a)$, $k \in I \cup J_p(a)$, whenever $0 < \alpha \leq \alpha_1$. Since $h_k(a) < 0$ for $k \in J_p \backslash J_p(a)$ and h_k is continuous in a neighbourhood of a, there exists $\alpha_2 > 0$, such that $h_k(a+\alpha d) < 0$, whenever $0 < \alpha \leq \alpha_2$, $k \in J_p \backslash J_p(a)$. Let $\alpha^* = \min\{\alpha_0, \alpha_1\}$. Then $a + \alpha d$ is a feasible solution for (P) and $h_k(a+\alpha d) < h_k(a)$, $k \in I$ for sufficiently small α such that $0 < \alpha \leq \alpha^*$. This contradicts the weak efficiency of (P) at $x = a$. Hence, ($*$) has no solution.

Since, for each k, $\pi_a^k(\cdot)$ is sublinear and cone$(C-a)$ is convex, it follows from a separation theorem [7, 18] that there exist $\tau_i \geq 0$, $i = 1, \ldots, p$, $\lambda_j \geq 0$, $j \in J_p(a)$, not all zero, such that

$$\sum_{i=1}^{p} \tau_i \pi_a^i(x) + \sum_{j \in J_p(a)} \lambda_j \pi_a^j(x) \geq 0 \quad \forall x \in \text{cone}(C-a).$$

Then, by applying standard arguments of convex analysis (see [14, 23]) and choosing $\lambda_j = 0$ whenever $j \in J_p \backslash J_p(a)$, we have

$$0 \in \sum_{i=1}^{p} \tau_i \partial \pi_a^i(0) + \sum_{j=1}^{m} \lambda_j \partial \pi_a^{j+p}(0) - (C-a)^+.$$

So, there exist $v_i \in \partial^0 f_i(F_i(a))$, $w_j \in \partial^0 g_j(G_j(a))$ satisfying

$$\sum_{i=1}^{p} \tau_i v_i^{\mathrm{T}} F_i'(a) + \sum_{j=1}^{m} \lambda_j w_j^{\mathrm{T}} G_j'(a) \in (C-a)^+.$$

Hence, the conclusion holds. □

Necessary conditions of Kuhn–Tucker type follow from Theorem 2.1 under a suitable constraint qualification [17, 24] that guarantees $\tau = (\tau_1, \tau_2, \ldots, \tau_p) \neq 0$. For instance, the following generalized Slater condition will do this:

$$\exists x_0 \in \operatorname{cone}(C-a), \quad v^{\mathrm{T}} G_j'(a) x_0 < 0, \quad \forall v \in \partial^0 g_j(G_j(a)), \ \forall j \in J(a),$$

where $J(a) = \{ j \mid g_j(G_j(a)) = 0, j = 1, \ldots, m \}$.

Choosing $q \in \mathbb{R}^p$, $q > 0$ with $\tau^{\mathrm{T}} q = 1$ and defining $\Lambda = q q^{\mathrm{T}}$, we can select the multipliers $\bar{\tau} = \Lambda \tau = q q^{\mathrm{T}} \tau = q > 0$ and $\bar{\lambda} = \Lambda \lambda = q q^{\mathrm{T}} \lambda \geqslant 0$. Hence, the following Kuhn–Tucker type optimality conditions (KT) for (P) are obtained:

$$\text{(KT)} \qquad \tau \in \mathbb{R}^p, \ \tau_i > 0, \quad \lambda \in \mathbb{R}^m, \ \lambda_j \geqslant 0, \quad \lambda_j g_j(G_j(a)) = 0,$$

$$0 \in \sum_{i=1}^{p} \tau_i \partial^0 f_i(F_i(a)) F_i'(a) + \sum_{j=1}^{m} \lambda_j \partial^0 g_j(G_j(a)) G_j'(a) - (C-a)^+.$$

3. Sufficient optimality conditions for convex composite programs

In this section, we present new conditions under which the optimality conditions (KT) become sufficient for efficient and properly efficient solutions. The sufficient conditions in this section are significant even for scalar composite problems as these conditions are weaker than the conditions given in [17] and apply to more general scalar composite problems (cf. [17]).

Let $x, a \in X$. Define $K : X \to \mathbb{R}^{n(p+m)} := \prod \mathbb{R}^n$ by

$$K(x) = (F_1(x), \ldots, F_p(x), G_1(x), \ldots, G_m(x)).$$

For each $x, a \in X$, the linear mapping $A_{x,a} : X \to \mathbb{R}^{n(p+m)}$ is given by

$$A_{x,a}(y) = (\alpha_1(x,a) F_1'(a) y, \ldots, \alpha_p(x,a) F_p'(a) y,$$

$$\beta_1(x,a) G_1'(a) y, \ldots, \beta_m(x,a) G_m'(a) y),$$

where $\alpha_i(x,a)$, $i = 1, 2, \ldots, p$ and $\beta_j(x,a)$, $j = 1, \ldots, m$, are real *positive* constants. Let us denote the null space of a function H by $N[H]$.

Recall, from the generalized Farkas lemma [7], that $K(x) - K(a) \in A_{x,a}(X)$ if and only if $A_{x,a}^{\mathrm{T}}(u) = 0 \Rightarrow u^{\mathrm{T}}(K(x) - K(a)) = 0$. This observation prompts us to define the following general *null space condition*:

For each $x, a \in X$, there exist real constants $\alpha_i(x, a) > 0$, $i = 1, 2, \ldots, p$, and $\beta_j(x, a) > 0$, $j = 1, 2, \ldots, m$, such that

$$N[A_{x,a}] \subset N[K(x) - K(a)], \tag{NC}$$

where

$$A_{x,a}(y) = (\alpha_1(x, a)F_1'(a)y, \ldots, \alpha_p(x, a)F_p'(a)y,$$
$$\beta_1(x, a)G_1'(a)y, \ldots, \beta_m(x, a)G_m'(a)y).$$

Equivalently, the null space condition means that for each $x, a \in X$, there exist real constants $\alpha_i(x, a) > 0$, $i = 1, 2, \ldots, p$, and $\beta_j(x, a) > 0$, $j = 1, 2, \ldots, m$, and $\mu(x, a) \in X$ such that $F_i(x) - F_i(a) = \alpha_i(x, a)F_i'(a)\mu(x, a)$ and $G_j(x) - G_j(a) = \beta_j(x, a)G_j'(a)\mu(x, a)$. For our general problem (P), we assume the following generalized null space condition (GNC):

For each x, $a \in C$, there exist real constants $\alpha_i(x, a) > 0$, $i = 1, 2, \ldots, p$, and $\beta_j(x, a) > 0$, $j = 1, 2, \ldots, m$, and $\mu(x, a) \in (C - a)$ such that $F_i(x) - F_i(a) = \alpha_i(x, a)F_i'(a)\mu(x, a)$ and $G_j(x) - G_j(a) = \beta_j(x, a)G_j'(a)\mu(x, a)$.

A condition of this type, called representation condition, has been used in the study of Chebyshev vector approximation problems in Jahn and Sachs [16]. Note that when $C = X$ the generalized null space condition (GNC) reduces to (NC). We shall show later in this section that the generalized null space condition (GNC) is easily verified for convex problems and various classes of nonconvex programming problems, such as convex composite pseudo linear programming problems, pseudo linear programming problems [4] and fractional linear programming problems [5]. Observe that (NC) trivially holds when F_i's and G_j's are affine functions.

In [17] a related, but restricted, null space condition without the real constants α_i, and β_j was used for a special class of scalar composite problem. The present null space condition (NC) allows us to treat various classes of nonconvex problems that cannot be handled by the theory presented in [17].

Theorem 3.1. *For the problem (P), assume that f_i and g_j are convex functions, and F_i and G_j are locally Lipschitz and Gâteaux differentiable functions. Let a be feasible for (P). Suppose that the optimality conditions* (KT) *hold at a. If the generalized null space condition* (GNC) *holds at each feasible point x of* (P) *then a is an efficient solution of* (P).

Proof. From the optimality conditions (KT), there exist $v_i \in \partial^0 f_i(F_i(a))$ and $w_j \in \partial^0 g_j(G_j(a))$ such that

$$\sum_{i=1}^{p} \tau_i v_i^{\mathrm{T}} F_i'(a) + \sum_{j=1}^{m} \lambda_j w_j^{\mathrm{T}} G_j'(a) \in (C - a)^+.$$

Suppose that a is not an efficient solution of (P). Then, there exists a feasible $x \in C$ for (P) with

$$f_i(F_i(x)) \leqslant f_i(F_i(a)) \quad \text{for all } i$$

and

$$f_r(F_r(x)) < f_r(F_r(a)) \quad \text{for some } r \in \{1, 2, \ldots, p\}.$$

Now, by the generalized null space condition, there exits $\eta(x, a) \in (C - a)$, same for each F_i and G_j, such that $F_i(x) - F_i(a) = \alpha_i(x, a) F_i'(a) \eta(x, a)$, $i = 1, 2, \ldots, p$, and $G_j(x) - G_j(a) = \beta_j(x, a) G_j'(a) \eta(x, a)$, $j = 1, 2, \ldots, m$. Hence,

$$\begin{aligned}
0 &\geqslant \sum_{j=1}^{m} \frac{\lambda_j}{\beta_j(x, a)} (g_j(G_j(x)) - g_j(G_j(a))) \quad \text{(by feasibility)} \\
&\geqslant \sum_{j=1}^{m} \frac{\lambda_j}{\beta_j(x, a)} w_j^{\mathrm{T}}(G_j(x) - G_j(a)) \quad \text{(by subdifferentiability)} \\
&= \sum_{j=1}^{m} \lambda_j w_j^{\mathrm{T}} G_j'(a) \eta(x, a) \quad \text{(by (GNC))} \\
&\geqslant -\sum_{i=1}^{p} \tau_i v_i^{\mathrm{T}} F_i'(a) \eta(x, a) \quad \text{(by a hypothesis)} \\
&= -\sum_{i=1}^{p} \frac{\tau_i v_i^{\mathrm{T}}}{\alpha_i(x, a)} (F_i(x) - F_i(a)) \quad \text{(by (GNC))} \\
&\geqslant \sum_{i=1}^{p} \frac{\tau_i}{\alpha_i(x, a)} (f_i(F_i(a)) - f_i(F_i(x))) \quad \text{(by subdifferentiability)} \\
&> 0.
\end{aligned}$$

This is a contradiction and hence a is an efficient solution for (P). □

The following example illustrates that the generalized null space condition (GNC) may not be sufficient for a feasible point which satisfies the optimality conditions (KT) to be a properly efficient solution for (P). Consider the simple multi-objective problem

$$\text{V-minimize} \quad \left(\frac{x_1}{x_2}, \frac{x_2}{x_1} \right)$$

$$\text{subject to} \quad (x_1, x_2) \in \mathbb{R}^2, \quad 1 - x_1 \leqslant 0, \quad 1 - x_2 \leqslant 0.$$

It is easy to check that $(1, 1)$ is an efficient solution for the problem, but it is not properly efficient. The generalized null space condition (GNC) holds at every feasible point (x_1, x_2) with $\alpha_1((x_1, x_2), (1, 1)) = 1/x_2$, $\alpha_2((x_1, x_2), (1, 1)) = 1/x_1$, $\beta_i((x_1, x_2), (1, 1)) = 1$, for $i = 1, 2$.

This example leads us to strengthen our generalized null space condition by constraining $\alpha_i(x, a) = \beta_j(x, a) = 1\ \forall i, j$, in order to get sufficient conditions for *properly efficient* solutions for (P). From the generalized Farkas lemma [7], it is easy to see that this strengthened null space condition holds when $C = X$ and $N[K'(a)] \subset N[K(x) - K(a)]$, for each $x, a \in X$.

Theorem 3.2. *Assume that the conditions on* (P) *in Theorem* 3.1 *hold. Let a be feasible for* (P). *Suppose that the optimality conditions* (KT) *hold at a. If the generalized null space condition* (GNC) *holds with* $\alpha_i(x, a) = \beta_j(x, a) = 1\ \forall i, j$, *for each feasible x of* (P) *then a is a properly efficient solution of* (P).

Proof. Let x be feasible for (P). Then, x is feasible for the scalar problem

$$(\mathrm{P}_\tau) \qquad \text{minimize} \quad \sum_{i=1}^{p} \tau_i f_i(F_i(x))$$

$$\text{subject to} \quad x \in C, \quad g_j(G_j(x)) \leqslant 0, \ j = 1, 2, \ldots, m.$$

From the convexity property of f_i,

$$\sum_{i=1}^{p} \tau_i f_i(F_i(x)) - \sum_{i=1}^{p} \tau_i f_i(F_i(a)) \geqslant \sum_{i=1}^{p} \tau_i v_i^{\mathrm{T}}(F_i(x) - F_i(a)).$$

Now, by the strengthened null space condition, there exists $\mu(x, a) \in (C - a)$ such that $F_i(x) - F_i(a) = F_i'(a)\mu(x, a)$ and $G_j(x) - G_j(a) = G_j'(a)\mu(x, a)$. Hence,

$$\begin{aligned}
&\sum_{i=1}^{p} \tau_i f_i(F_i(x)) - \sum_{i=1}^{p} \tau_i f_i(F_i(a)) \\
&\quad \geqslant \sum_{i=1}^{p} \tau_i v_i^{\mathrm{T}} F_i'(a)\mu(x, a) \\
&\quad \geqslant -\sum_{j=1}^{m} \lambda_j w_j^{\mathrm{T}} G_j'(a)\mu(x, a) \\
&\quad \geqslant -\sum_{j=1}^{m} \lambda_j g_j(G_j(x)) + \sum_{j=1}^{m} \lambda_j g_j(G_j(a)) \\
&\quad \geqslant 0,
\end{aligned}$$

and so, a is minimum for the scalar problem (P_τ). Since $\tau \neq \mathbb{R}^p$, $\tau > 0$, it follows from Theorem 1 [10] that a is a properly efficient solution of (P). □

The following numerical example provides a nonsmooth convex composite problem for which our sufficiency Theorem 3.1 holds.

Example 3.1. Consider the multi-objective problem

$$\text{V-minimize} \quad \left(\left| \frac{2x_1 - x_2}{x_1 + x_2} \right|, \frac{x_1 + 2x_2}{x_1 + x_2} \right)$$

$$\text{subject to} \quad x_1 - x_2 \leq 0, \quad 1 - x_1 \leq 0, \quad 1 - x_2 \leq 0,$$

Let $F_1(x) = (2x_1 - x_2)/(x_1 + x_2)$, $F_2(x) = (x_1 + 2x_2)/(x_1 + x_2)$, $G_1(x) = x_1 - x_2$, $G_2(x) = 1 - x_1$, $G_3(x) = 1 - x_2$, $f_1(y) = |y|$, $f_2(y) = y$, and $g_1(y) = g_2(y) = g_3(y) = y$. Then, the problem becomes a convex composite problem with an efficient solution $(1, 2)$. It is easy to see that the null space condition holds at each feasible point of the problem with $\alpha_i(x, a) = 1$, for $i = 1, 2$, $\beta_j(x, a) = \frac{1}{3}(x_1 + x_2)$, for $j = 1, 2, 3$ and $\mu(x, a) = (3(x_1 - 1)/(x_1 + x_2), 3(x_2 - 2)/(x_1 + x_2))^{\mathrm{T}}$. The optimality conditions (KT) hold with $v_1 = v_2 = 1$, $\tau_1 = 1$, $\tau_2 = 3$, and $\lambda_j = 0$, $w_j = 1$, for $j = 1, 2, 3$.

Let us now give some classes of nonlinear problems which satisfy our sufficient conditions.

Example 3.2 (Pseudolinear programming problem [4]). Consider the multi-objective pseudolinear programming problem

(PLP) $\quad$ V-minimize $\quad (l_1(x), \ldots, l_p(x))$

$\qquad$ subject to $\quad x \in \mathbb{R}^n, \quad h_j(x) - b_j \leq 0, \; j = 1, 2, \ldots, m,$

where $l_i : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ are differentiable and pseudolinear, i.e., pseudoconvex and pseudoconcave [4], and $b_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$. It should be noted that a real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ is pseudolinear if and only if for each $x, y \in \mathbb{R}^n$, there exists a real constant $\alpha(x, y) > 0$ such that

$$h(y) = h(x) + \alpha(x, y) h'(x)(y - x).$$

Moreover, any fractional linear functions of the form $(ax + b)/(cx + d)$ on $\mathbb{R}^n$ are pseudolinear functions. Define $F_i : \mathbb{R}^n \to \mathbb{R}^{p+m}$ by

$$F_i(x) = (0, 0, \ldots, l_i(x), 0, 0, \ldots, 0), \quad i = 1, 2, \ldots, p,$$

and

$$G_j(x) = (0, 0, \ldots, h_j(x) - b_j, 0, \ldots, 0), \quad j = 1, 2, \ldots, m.$$

Define $f_i, g_j : \mathbb{R}^{p+m} \to \mathbb{R}$ by

$$f_i(x) = x_i, \quad i = 1, 2, \ldots, p,$$

$$g_j(x) = x_{p+j}, \quad j = 1, 2, \ldots, m.$$

Then, we can rewrite (PLP) as the following convex composite multi-objective problem

$$\text{V-minimize} \quad (f_1(F_1(x)), \ldots, f_p(F_p(x))$$

$$\text{subject to} \quad x \in \mathbb{R}^n, \quad g_j(G_j(x)) \leq 0, \; j = 1, 2, \ldots, m.$$

Now, our generalized null space condition (GNC) is verified at each feasible point by the pseudolinearity property of the functions involved. It follows from Theorem 3.1 that if the optimality conditions

$$\sum_{i=1}^{p} \tau_i l_i'(a) + \sum_{j=1}^{m} \lambda_j g_j'(a) = 0, \quad \lambda_j(g_j(a) - b_j) = 0,$$

hold with $\tau_i > 0$, $i = 1, 2, \ldots, p$, and $\lambda_j \geqslant 0$, $j = 1, 2, \ldots, m$, at the feasible point $a \in \mathbb{R}^n$ of (PLP) then a is an efficient solution for (PLP).

We now see that our sufficient conditions in Theorem 3.1 hold for a class of convex composite pseudolinear programming problems.

Example 3.3. Consider the problem

$$\begin{aligned} &\text{V-minimize} \quad (f_1((h \circ \psi)(x)), \ldots, f_p((h \circ \psi)(x))) \\ &\text{subject to} \quad x \in X, \quad g_j((h \circ \psi)(x)) \leqslant 0, \ j = 1, 2, \ldots, m, \end{aligned}$$

where $h = (h_1, h_2, \ldots, h_n)$ is a pseudolinear vector function from X to $\mathbb{R}^n$, that is, each component h_i is pseudolinear, ψ is a Fréchet differentiable mapping from X onto X such that $\psi'(a)$ is surjective for each $a \in X$, and f_i, g_j are convex for each i, j. For this class of nonconvex problems, the null space condition (GNC) holds. To see this, let x, $a \in \mathbb{R}^n$, $u = \psi(x)$, $v = \psi(a)$. Then, by the pseudolinerity condition, we get

$$\begin{aligned} h_i(\psi(x)) - h_i(\psi(a)) &= h_i(u) - h_i(v) \\ &= \alpha_i(u, v) h_i'(v)(u - v). \end{aligned}$$

Since $\psi'(a)$ is onto, $u - v = \psi'(a)\mu(x, a)$ is solvable for some $\mu(x, a) \in \mathbb{R}^n$. Hence,

$$\begin{aligned} h_i(\psi(x)) - h_i(\psi(a)) &= \alpha_i(u, v) h_i'(v) \psi'(a) \mu(x, a) \\ &= \alpha_i^{\wedge}(x, a)(h_i \circ \psi)'(a) \mu(x, a), \end{aligned}$$

where $\alpha_i^{\wedge}(x, a) = \alpha_i(\psi(x), \psi(a)) > 0$; thus, (GNC) holds.

We finish this section by observing that any finite dimensional convex programming problem can also be rephrased as a composite problem (P) with F_i's and G_j's identity mappings, and that it clearly satisfies the generalized null space condition. For certain other sufficient conditions for global optimality of some classes of differentiable finite dimensional nonconvex problems, see [11].

4. Duality for convex composite programs

It is known that duality results have played a crucial role in the development of multi-objective programming [12, 22, 28]. Following the success of multi-objective linear programming duality, various generalizations of the duality theory have been given for multi-objective nonlinear programming problems, e.g., see [22, 28]. Here, we consider the primal problem

$$\text{(P)} \qquad \text{V-minimize} \quad (f_1(F_1(x)), \ldots, f_p(F_p(x)))$$
$$\text{subject to} \quad x \in C, \quad g_j(G_j(x)) \leqslant 0, \; j = 1, 2, \ldots, m,$$

with f_i, g_j convex and F_i, G_j locally Lipschitz and Gâteaux differentiable, and the dual problem

$$\text{(D)} \qquad \text{V-maximize} \quad (f_1(F_1(u)), \ldots, f_p(F_p(u)))$$
$$\text{subject to} \quad 0 \in \sum_{i=1}^{p} \tau_i \partial^0 f_i(F_i(u)) F_i'(u) + \sum_{j=1}^{m} \lambda_j \partial^0 g_j(G_j(u)) G_j'(u) - (C-u)^+,$$
$$\lambda_j g_j(G_j(u)) \geqslant 0, \; j = 1, 2, \ldots, m,$$
$$u \in C, \quad \tau \in \mathbb{R}^p, \quad \tau_i > 0, \; \lambda \in \mathbb{R}^m, \; \lambda_i \geqslant 0,$$

It is worth observing that, unlike the Wolfe dual pair (see [7]), the primal and the dual problems here have the same form of objective functions. Duality results for this kind of dual pairs have recently been examined in [32, and other references therein]. It has been established that these dual pairs allow one to relax the standard convexity requirements, used in duality theorems in the literature. In this section, we examine duality properties for the convex composite problem that include the corresponding results for convex problems and other related generalized convex prolems. Note that the problem (D) is considered as a dual to (P) in the sense that

(i) (zero duality gap property) if $\bar{x}$ is a properly efficient solution of (P) then, for some $\bar{\tau} \in \mathbb{R}^p$, $\bar{\lambda} \in \mathbb{R}^m$, $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a properly efficient solution of (D), and the objective values of (P) and (D) at these points are equal;

(ii) (weak duality property) if x is feasible for (P) and (u, τ, λ) is feasible for (D) then

$$(f(F_1(x)), \ldots, f_p(F_p(x)))^{\mathrm{T}} - (f_1(F_1(u)), \ldots, f_p(F_p(u)))^{\mathrm{T}} \notin -\mathbb{R}^p_+ \backslash \{0\}.$$

Theorem 4.1. (weak duality). *Let x be feasible for* (P) *and let* (u, τ, λ) *be feasible for* (D). *Assume that the generalized null space condition* (GNC) *holds. Then*

$$(f_1(F_1(x)), \ldots, f_p(F_p(x)))^{\mathrm{T}} - (f_1(F_1(u)), \ldots, f_p(F_p(u)))^{\mathrm{T}} \notin -\mathbb{R}^p_+ \backslash \{0\}.$$

Proof. Since (u, τ, λ) is feasible for (D), there exist $\tau > 0$, $\lambda \geqslant 0$, $v_i \in \partial^0 f_i(F_i(u))$, $w_j \in \partial^0 g_j(G_j(u))$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, m$, satisfying $\lambda_j g_j(G_j(u)) \geqslant 0$, for $j = 1, 2, \ldots, m$, and

$$\sum_{i=1}^{p} \tau_i v_i^{\mathrm{T}} F_i'(u) + \sum_{j=1}^{m} \lambda_j w_j^{\mathrm{T}} G_j'(u) \in (C-u)^+.$$

Suppose that $x \neq u$ and

$$(f_1(F_1(x)), \ldots, f_p(F_p(x)))^{\mathrm{T}} - (f_1(F_1(u)), \ldots, f_p(F_p(u)))^{\mathrm{T}} \in -\mathbb{R}^p_+ \backslash \{0\}.$$

Then,

$$\sum_{i=1}^{p} \frac{\tau_i}{\alpha_i(x, u)} (f_i(F_i(x)) - f_i(F_i(u))) < 0,$$

since $\tau_i / \alpha_i(x, u) > 0$. Now, by the convexity of f_i and by the generalized null space condition (GNC), we get

$$\sum_{i=1}^{p} \tau_i v_i^{\mathrm{T}} F_i'(u) \eta(x, u) < 0.$$

From the feasibility conditions, we get

$$\lambda_j g_j(G_j(x)) \leqslant 0, \qquad \lambda_j g_j(G_j(u)) \geqslant 0,$$

and so,

$$\sum_{i=1}^{m} \frac{\lambda_j}{\beta_j(x, u)} (g_j(G_j(x)) - g_j(G_j(u)) \leqslant 0.$$

Similarly, by the convexity of g_j, positivity of $\beta_j(x, u)$, and by the generalized null space condition,

$$\sum_{j=1}^{m} \lambda_j w_j^{\mathrm{T}} G_j'(u) \eta(x, u) \leqslant 0.$$

Hence,

$$\left[\sum_{i=1}^{p} \tau_i v_i^{\mathrm{T}} F_i'(u) + \sum_{j=1}^{m} \lambda_j w_j^{\mathrm{T}} G_j'(u) \right] \eta(x, u) < 0.$$

This is a contradiction. The proof is completed by noting that when $x = u$ the conclusion trivially holds. □

Theorem 4.2 (strong duality). *For the problem* (P), *assume that the generalized Slater constraint qualification in Section* 2 *holds and that the generalized null space condition* (GNC) *is verified at each feasible point of* (P) and (D). *If* a *is a properly efficient solution for* (P), *then there exist* $\tau \in \mathbb{R}^p$, $\tau_i > 0$, $\lambda \in \mathbb{R}^m$, $\lambda_j \geqslant 0$ *such that* (a, τ, λ) *is a properly efficient solution for* (D) *and the objective values at these points are equal.*

Proof. It follows from Theorem 2.1 that there exist $\tau \in \mathbb{R}^p$, $\tau_i > 0$, $\lambda \in \mathbb{R}^m$, $\lambda_j \geqslant 0$, such that

$$0 \in \sum_{i=1}^{p} \tau_i \partial^0 f_i(F_i(a)) F_i'(a) + \sum_{j=1}^{m} \lambda_j \partial^0 g_j(G_j(a)) G_j'(a) - (C - a)^+,$$

$$\lambda_j g_j(G_j(a)) = 0, \quad j = 1, 2, \ldots, m.$$

Then (a, τ, λ) is a feasible solution for (D).

Now, from the Weak Duality Theorem, the point (a, τ, λ) is an efficient solution for (D).

We shall prove that (a, τ, λ) is a properly efficient solution for (D) by the method of contradiction. Suppose that there exists (a^*, τ^*, λ^*) feasible for (D) satisfying, for some i,

$$f_i(F_i(a^*)) - f_i(F_i(a)) > M[f_j(F_j(a)) - f_j(F_j(a^*))],$$

for any $M > 0$ and all j satisfying $f_j(F_j(a)) > f_j(F_j(a^*))$. Let

$$A = \{j \in I \,|\, f_j(F_j(a) > f_j(F_j(a^*))\},$$

where $I = \{1, 2, \dots, p\}$. Let $B = I \backslash (A \cup \{i\})$. Choose $M > 0$ such that

$$M/|A| > \tau_j/\tau_i, \quad j \in A.$$

Note that, for a set L, $|L|$ denotes the number of elements in the set L. Then,

$$\tau_i(f_i(F_i(a^*)) - f_i(F_i(a))) > \sum_{j \in A} \tau_j(f_j(F_j(a)) - f_j(F_j(a^*))),$$

since $f_j(F_j(a)) - f_j(F_j(a^*)) > 0$, for $j \in A$. So,

$$\begin{aligned}\sum_{i=1}^{p} \tau_i f_i(F_i(a)) &= \tau_i f_i(F_i(a)) + \sum_{j \in A} \tau_j f_j(F_j(a)) + \sum_{j \in B} \tau_j f_j(F_j(a)) \\ &< \tau_i f_i(F_i(a^*)) + \sum_{j \in A} \tau_j f_j(F_j(a^*)) + \sum_{j \in B} \tau_j f_j(F_j(a^*)) \\ &= \sum_{i=1}^{p} \tau_i f_i(F_i(a^*)).\end{aligned}$$

This contradicts the weak duality property. Hence, (a, τ, λ) is a properly efficient solution for (D). □

As we demonstrated in Section 3, it can easily be shown that our duality theorems include corresponding duality results for convex, pseudolinear and other related composite problems.

5. Scalarizations in composite multi-objective programming

Multi-objective optimization problems are often solved by transforming them into scalar ones. The most widely used and the simplest scalarization technique is the convex combination of the different objectives. This technique has been used successfully for solving linear and convex multi-objective problems, e.g., [12, 14, 28]. In this section we present a scalarization result for convex composite problems. These problems are not necessarily convex. As an application of the scalarization result we also characterize the set of properly efficient solutions in terms of subgradients [26] for convex problems. These conditions do not depend on a particular properly efficient solution, and differ from the conditions presented in Section 3.

For the multi-objective composite model problem

$$\text{(P)} \qquad \text{V-minimize} \quad (f_1(F_1(x)), \ldots, f_p(F_p(x)))$$
$$\text{subject to} \quad x \in C, \quad g_j(G_j(x)) \leqslant 0, \ j = 1, 2, \ldots, m,$$

the associated scalar problem (P_λ) is given by

$$(\mathrm{P}_\lambda) \qquad \text{minimize} \quad \sum_{i=1}^{p} \lambda_i f_i(F_i(x))$$
$$\text{subject to} \quad x \in C, \quad g_j(G_j(x)) \leqslant 0, \ j = 1, 2, \ldots, m,$$

where $\lambda \in \mathbb{R}^p$, $\lambda \neq 0$. The feasible set Ω of (P) is given by

$$\Omega = \{x \in C \mid g_j(G_j(x)) \leqslant 0, \ j = 1, 2, \ldots, m\}.$$

The set of all properly efficient solutions of (P) is denoted by PE. For each $\lambda \in \mathbb{R}^p$, the solution set S_λ of the scalar problem (P_λ) is given by

$$S_\lambda = \left\{ x \in \Omega \;\middle|\; \sum_{i=1}^{p} \lambda_i f_i(F_i(x)) = \min_{y \in \Omega} \sum_{i=1}^{p} \lambda_i f_i(F_i(y)) \right\}.$$

The following theorem establishes a scalarization result for (P) corresponding to a properly efficient solution.

Theorem 5.1. *For the multi-objective problem* (P), *assume that, for each* $i = 1, 2, \ldots, p$, *and* $\alpha > 0$, *the set*

$$\Gamma^i_\alpha = \{z \in \mathbb{R}^p \mid \exists x \in C, f_i(F_i(x)) < z^i_i, f_i(F_i(x)) + \alpha f_j(F_j(x)) < z^j_i, j \neq i\}$$

is convex. Then,

$$\mathrm{PE} = \bigcup_{\lambda_i > 0, \sum_{i=1}^{p} \lambda_i = 1} S_\lambda .$$

Proof. Let $a \in \mathrm{PE}$. Then, there exists a scalar $M > 0$ such that, for each $i = 1, 2, \ldots, p$, the system

$$f_i(F_i(x)) < f_i(F_i(a)),$$
$$f_i(F_i(x)) + M f_j(F_j(x)) < f_i(F_i(a)) + M f_j(F_j(a)) \quad \forall j \neq i,$$

is inconsistent. Thus,

$$0 \notin \Gamma^i_M(a) = \{z \in \mathbb{R}^p \mid \exists x \in C, f_i(F_i(x)) < f_i(F_i(a)) + z^i_i,$$
$$f_i(F_i(x)) + M f_j(F_j(x))$$
$$< f_i(F_i(a)) + M f_j(F_j(a)) + z^j_i, j \neq i\}.$$

From the assumption, $\Gamma^i_M(a)$ is convex, and so by the standard separation arguments as in the proof of Theorem 1 [10], we can show that there exists $\lambda \in \mathbb{R}^p$, $\lambda_i > 0$, $\sum_{i=1}^{p} \lambda_i = 1$ such that $a \in S_\lambda$; thus,

$$\mathrm{PE} \subset \bigcup_{\lambda_i > 0, \sum_{i=1}^{p} \lambda_i = 1} S_\lambda .$$

The converse inclusion follows from Theorem 1 [10] without any convexity conditions on the functions involved. □

Remark 5.1. It is worth noting that, for each $\alpha > 0$, and $i = 1, 2, \ldots, p$, the set Γ_α^i is convex if, for instance, f_i is convex for each $i = 1, 2, \ldots, p$, and $H(\Omega)$ is convex, where $H : X \to \prod \mathbb{R}^n$ is given by $H(x) := (F_1(x), \ldots, F_p(x))$. The set $H(\Omega)$ may be convex while F_i's are nonconvex. For instance, let $H(x) = (|x_1|, -|x_2|)^{\mathrm{T}}, x = (x_1, x_2) \in \mathbb{R}^2$, where $F_1(x) = |x_1|$ and $F_2(x) = -|x_2|$. Then, it is easy to see that H is non-convex, but $H(\mathbb{R}^2)$ is a convex set. The convexity property of Γ_α^i can also hold under rather more general conditions. For example, Γ_α^i is convex if the vector function $(f_i(\cdot))$ satisfies the convex-like (or subconvex-like) property (see [19, 21]) on $H(\Omega)$. It is also worth observing that the conclusion of Theorem 5.1 can also be interpreted as a complete characterization of the set of properly efficient solutions in terms of the solution sets of appropriate scalar problems.

Using the above scalarization Theorem 5.1 and a recent result of Mangasarian [23] we show how the set of properly efficient solutions for a *convex* problem can be characterized in terms of subgradients. This extends the characterization result of Mangasarian (see Theorem 1(a) [23]) for a scalar problem to multi-objective convex problems. In the following, we assume that the functions F_i's, and G_j's in problem (P) are linear functions from $\mathbb{R}^n$ and $\mathbb{R}^m$. Thus, we consider the composite convex problem

$$\text{(CP)} \quad \text{V-minimize} \quad (f_1(A_1(x)), \ldots, f_p(A_p(x)))$$

$$\text{subject to} \quad x \in C, \quad g_j(B_j(x)) \leqslant 0, \ j = 1, 2, \ldots, m,$$

where $A_i : \mathbb{R}^n \to \mathbb{R}^m$, $i = 1, 2, \ldots, p$, and $B_j : \mathbb{R}^n \to \mathbb{R}^m$, $j = 1, 2, \ldots, m$, are continuous linear mappings, $f_i, i = 1, 2, \ldots, p$, and $g_j, j = 1, 2, \ldots, m$, are convex functions on $\mathbb{R}^m$. Note that the feasible set

$$\Omega = \{x \in C \mid g_j(B_j(x)) \leqslant 0, j = 1, 2, \ldots, m\}$$

is now a convex subset of $\mathbb{R}^n$.

The convex scalar (parameterized) problem for (CP) is given by

$$(\lambda\text{CP}) \quad \text{minimize} \quad \sum_{i=1}^{p} \lambda_i f_i(A_i(x))$$

$$\text{subject to} \quad x \in C, \quad g_j(B_j(x)) \leqslant 0, \ j = 1, 2, \ldots, m.$$

Let the convex solution set of (λCP) be CS_λ, $\lambda \in \mathbb{R}^p$.

Corollary 5.1. *Consider the convex problem* (CP). *Suppose that for each* $\lambda \in \mathbb{R}^p$, $\lambda_i > 0$, $\sum_{i=1}^p \lambda_i = 1$, *the relative interior of* CS_λ, $(\mathrm{ri}(\mathrm{CS}_\lambda))$, *is non-empty. Let* $z_\lambda \in \mathrm{ri}(\mathrm{CS}_\lambda)$. *Then*

$$\mathrm{PE} = \bigcup_{\lambda_i > 0, \sum_{i=1}^p \lambda_i = 1} \left\{ x \in \Omega \,\middle|\, \exists u_i \in \partial f_i(A_i(x)),\ \sum_{i=1}^{p} \lambda_i u_i^{\mathrm{T}} A_i(x - z_\lambda) = 0 \right\}.$$

Proof. From the scalarization Theorem 5.1,

$$\mathrm{PE} = \bigcup_{\lambda_i > 0, \sum_{i=1}^{p} \lambda_i = 1} \left\{ x \in \Omega \;\middle|\; \sum_{i=1}^{p} \lambda_i f_i(A_i(x)) = \min_{y \in \Omega} \sum_{i=1}^{p} \lambda_i f_i(A_i(y)) \right\}.$$

Since $\min_{y \in \Omega} \sum_{i=1}^{p} \lambda_i f_i(A_i(y))$ is a scalar convex problem, it follows from Theorem 1(a) [23] (see also [20]) that

$$\left\{ x \in \Omega \;\middle|\; \sum_{i=1}^{p} \lambda_i f_i(A_i(x)) = \min_{y \in \Omega} \sum_{i=1}^{p} \lambda_i f_i(A_i(y)) \right\}$$

$$= \left\{ x \in \Omega \;\middle|\; \exists v \in \partial\left(\sum_{i=1}^{p} \lambda_i (f_i \circ A_i) \right)(x),\ v(x - z_\lambda) = 0 \right\}.$$

Hence, the conclusion holds by the chain rule and the properties of the subdifferentials. □

Note, in particular, that when $p = 1$ the solution set S of the scalar problem

minimize $\quad f_1(A_1(x))$,

subject to $\quad x \in \Omega$,

is characterized by the equalities

$$S = \mathrm{PE} = \{ x \in \Omega \mid \exists u \in \partial f_1(A_1(x)),\ u^{\mathrm{T}} A_1(x - z) = 0 \},$$

where $z \in \mathrm{ri}\, S$. We conclude by noting that the sufficient conditions, presented in Section 3, depend on the properly efficient solution under discussion. However, the conditions, given in this section, are independent of a particular properly efficient solution.

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References

[1] A. Ben-Tal and J. Zowe, "Necessary and sufficient optimality conditions for a class of nonsmooth minimization problems," *Mathematical Programming* 24 (1982) 70–91.

[2] J.V. Burke, "Decent methods for composite nondifferentiable optimization," *Mathematical Programming* 33 (1985) 260–279.

[3] J.V. Burke, "Second order necessary and sufficient conditions for composite NDO," *Mathematical Programming* 38 (1987) 287–302.

[4] K.L. Chew and E.U. Choo, "Pseudolinearity and efficiency," *Mathematical Programming* 28 (1984) 226–239.

[5] E.U. Choo, "Proper efficiency and linear fractional vector maximum problem," *Operations Research* 32 (1984) 216–220.

[6] F.H. Clarke, *Optimization and Nonsmooth Analysis* (Wiley, New York, 1983).
[7] B.D. Craven, *Mathematical Programming and Control Theory* (Chapman and Hall, London, 1978).
[8] R. Fletcher, *Practical Methods of Optimization* (Wiley, New York, 1987).
[9] R. Fletcher, "A model algorithm for composite nondifferentiable optimization problems," *Mathematical Programming Study* 17 (1982) 67-76.
[10] A.M. Geoffrion, "Proper efficiency and the theory of vector maximization," *Journal of Mathematical Analysis and Applications* 22 (1968) 618-630.
[11] M.A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications* 80 (1981) 644-550
[12] R. Hartley, "Vector and parametric programming," *Journal of Operations Research Society* 36 (1985) 423-432.
[13] A.D. Ioffe, "Necessary and sufficient conditions for a local minimum 2: conditions of Levitin-Milutin-Osmoloviskii type," *SIAM Journal on Control and Optimization* 17 (1979) 251-265.
[14] J. Jahn, "Scalarization in multi-objective optimization," *Mathematical Programming* 29 (1984) 203-219.
[15] J. Jahn and W. Krabs, "Applications of multicriteria optimization in approximation theory," in: W. Stadler, ed., *Multicriteria Optimization in Engineering and in the Sciences* (Plenum Press, New York, 1988) pp. 49-75.
[16] J. Jahn and E. Sachs, "Generalized quasiconvex mappings and vector optimization," *SIAM Journal on Control and Optimization* 24 (1986) 306-322.
[17] V. Jeyakumar, "Compsosite nonsmooth programming with Gâteaux differentiability," *SIAM Journal on Optimization* 1(1991) 30-41.
[18] V. Jeyakumar, "On optimality conditions in nonsmooth inequality constrained minimization," *Numerical Functional Analysis and Optimization* 9 (1987) 535-546.
[19] V. Jeyakumar, "Convexlike alternative theorems and mathematical programming," *Optimization* 16 (1985) 643-652.
[20] V. Jeyakumar, "Infinite dimensional convex programming with applications to constrained approximation," *Journal of Optimization Theory and Applications* 75 (1992).
[21] V. Jeyakumar and H. Wolkowicz, "Zero duality gaps in infinite dimensional programming," *Journal of Optimization Theory and Applications* 67 (1990) 87-108.
[22] D.T. Luc, *Theory of Vector Optimization* (Springer, Berlin, 1988).
[23] O.L. Mangasarian, "A simple characterization of solution sets of convex programs," *Operations Research Letters* 7 (1988) 21-26.
[24] R.R. Merkovsky and D.E. Ward, "Constraint qualifications in nondifferentiable programming," *Mathematical Programming* 47 (1990) 389-405.
[25] P. Michel and J.P. Penot, "Calcul sous-differentiel pour des fonctions lipschitziennes et non lipschitziennes," *Comptes Rendus de l'Académie des Sciences Paris* 298 (1984) 269-272.
[26] R.T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, NJ, 1969).
[27] R.T. Rockafellar, "First and second order epi-differentiability in nonlinear programming," *Transactions of the American Mathematical Society* 307 (1988) 75-108.
[28] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization* (Academic Press, New York, 1985).
[29] A. Schy and D.P. Giesy, "Multicriteria optimization techniques for design of aircraft control systems," in: W. Stadler, ed., *Multicriteria Optimization in Engineering and in the Sciences* (Plenum Press, New York, 1988) pp. 225-262.
[30] W. Stadler, *Multicriteria Optimization in Engineering and in the Sciences* (Plenum Press, New York, 1988).
[31] W. Stadler, "Multicriteria optimization in mechanics: A survey," *Applied Mechanics Reviews* 37 (1984) 277-286.
[32] T. Weir and B. Mond, "Generalized convexity and duality in multiple objective programming," *Bulletin of the Australian Mathematical Society* 39 (1989) 287-299.
[33] D.J. White, *Optimality and Efficiency* (Wiley, New York, 1982).
[34] D.J. White, "Multi-objective programming and penalty functions," *Journal of Optimization Theory and Applications* 43 (1984) 583-599.
[35] W.I. Zangwill, *Nonlinear Programming* (Prentice-Hall, Englewood Cliffs, NJ, 1969).