Mathematical Programming 55 (1992) 129–168
North-Holland

Structural properties and decomposition of linear balanced matrices

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Received 30 June 1986
Revised manuscript received 13 July 1990

Claude Berge defines a (0, 1) matrix A to be linear if A does not contain a 2×2 submatrix of all ones. $A(0, 1)$ matrix A is balanced if A does not contain a square submatrix of odd order with two ones per row and column.

The contraction of a row i of a matrix consists of the removal of row i and all the columns that have a 1 in the entry corresponding to row i.

In this paper we show that if a linear balanced matrix A does not belong to a subclass of totally unimodular matrices, then A or A^{T} contains a row i such that the submatrix obtained by contracting row i has a block-diagonal structure.

Key words: Polyhedral combinatorics, integrality of polytopes, decomposition.

1. Introduction

1.1. *The main result and its implications*

Let G be a bipartite graph with no parallel edges. A cycle of G is odd if its length is congruent to 2 modulo 4 and even if its length is congruent to 0 modulo 4. A star cutset of G is a node v such that the removal of v and its adjacent nodes disconnects G.

In this paper we prove the following result:

Let G be a bipartite graph having odd cycles but no cycle of length four. Then either G has an odd chordless cycle or G has a star cutset.

The bipartite representation of a $(0, 1)$ matrix A is the bipartite graph $G(V^+, V^-; E)$ where V^+, V^- are the two sets of nodes representing the columns and rows of A. For each entry $a_{ij}=1$ there exists an edge (i, j) of E whose end nodes are the two nodes representing row i and column j.

Partial support from NSF grant DMS 8606188, ECS 8800281 and DDM 8800281.

A chordless cycle or hole of length $2k$ of G corresponds to a circulant matrix of order k with two ones per row and column. A hole is odd if k is odd. Matrices not containing such an odd circulant are called balanced matrices and they play an important role in integrality issues associated with combinatorial packing and covering problems. This is discussed further in Section 1.2.

Matrices whose bipartite representation has no odd cycles are a subclass of Totally Unimodular matrices and are well understood from a structural and algorithmic point of view, see Yannakakis (1985) and Conforti and Rao (1987a).

Claude Berge defines a $(0, 1)$ matrix to be linear if it does not contain a 2×2 submatrix of all ones. This is equivalent to stating that the bipartite representation of a linear $(0, 1)$ matrix does not contain a cycle of length four. Therefore our theorem for bipartite graphs yields the following result in terms of $(0, 1)$ matrices:

Let A be a linear balanced matrix containing odd cycles. Then either A or A^{T} has the following structure:

$$\left(\begin{array}{c|c|c} 1111111111111 & \multicolumn{2}{c}{0,1 \text{ vector}} \\ \hline & A_1 & 0 \\ \cline{2-3} \begin{array}{c}0 \text{ or unit}\\ \text{row vectors}\end{array} & 0 & A_2 \end{array}\right).$$

The above result has the following implications for a class of simple undirected graphs (not to be confused with the above defined bipartite graph).

The clique node incidence matrix A of a graph $H(V, E)$ is the matrix whose set of rows is the set of incidence vectors of all maximal cliques of H. We say that the graph H is balanced (linear) if A is a balanced (linear) matrix.

It is easy to see that A is linear if and only if no two maximal cliques of H have two common nodes. This is equivalent to saying that H is linear if and only if H does not contain as induced subgraph, the graph of Figure 1, which is referred to as a $K_4 - e$ or diamond graph.

Given the bipartite representation $G(V^+, V^-; E)$ of the matrix A, a node $V \in V^-$ which is a star cutset corresponds to a clique of H with the property that the removal of the nodes in the clique disconnects H.

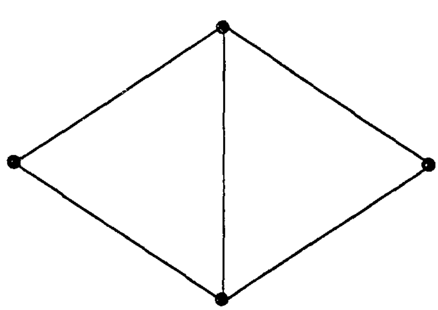


Fig. 1.

A node $v \in V^+$ which is a star cutset of G corresponds to a node x of H with the property that the removal of x together with the set of maximal cliques containing x disconnects H. This operation of removing maximal cliques does not have an easy interpretation on the graph H unless H is $K_4 - e$ free.

If H is $K_4 - e$ free, then every edge of H belongs to exactly one maximal clique. Therefore the removal of x and the set of maximal cliques containing x corresponds to the removal of node x in H and all the edges of H connecting two nodes in the maximal cliques that are removed.

1.2. Balanced matrices

Given an $m \times n$ $(0, 1)$ matrix A, we define the polytopes associated with the linear programming relaxation of the set packing and covering problems as follows:

$$P = \{x \in \mathbb{R}^n \mid x \geqslant 0;\ Ax \leqslant 1\},$$

$$Q = \{y \in \mathbb{R}^m \mid 1 \geqslant y \geqslant 0;\ yA \geqslant 1\}.$$

Berge (1972) has shown that if A is balanced, the polytopes P and Q have only integral vertices. Furthermore, when A is balanced, if some rows and columns of A are deleted, the polytopes associated with the linear programming relaxation of the corresponding set packing and covering problems have only integral vertices. The polytope P is sometimes referred to as the matching polytope and Q as the corresponding transversal polytope.

Berge and Las Vergnas (1970) have shown that a matrix A is balanced if and only if for any submatrix A' of A, the maximum rank of a 0,1 vector in the associated matching polytope is equal to the minimum rank of a 0,1 vector in the corresponding transversal polytope. This property is known as the Konig property.

The q-matching polytope P_q is defined as follows:

$$P_q = \{x \in \mathbb{R}^n \mid x \geqslant 0;\ Ax \leqslant q\}$$

where q is an $n \times 1$ vector of non-negative integers. If A is balanced, Fulkerson, Hoffman and Oppenheim (1974) have shown that the following two linear programs:

$$\text{Max} \quad \{1^{\mathrm{T}}x \mid x \in \mathbb{R}^n, x \geqslant 0, Ax \leqslant q\},$$

$$\text{Min} \quad \{yq \mid y \in \mathbb{R}^m, y \geqslant 0, yA \geqslant 1\}$$

have integral solutions with the same optimal objective function value. This property is called the Menger property. If we define $\mathrm{TR}(A)$ as the matrix whose columns are all the minimal $(0, 1)$ vectors in the polytope Q, then Berge (1984) has shown that if A is balanced then $\mathrm{TR}(A)$ also has the Menger property.

The above fundamental results show the crucial role of balanced matrices in the study of integrality issues associated with combinatorial packing and covering problems. Some other integrality properties of balanced matrices are discussed in Berge (1980), Frank (1979) and Padberg (1975). A survey of totally unimodular, balanced and perfect matrices can be found in Padberg (1975).

A question that is of considerable interest is whether there exists a "good" characterization of balanced matrices that would lead to a polynomial recognition algorithm. Although this remains an open question, for some classes of balanced matrices a complete characterization has been given, the most notable result being that of Seymour (1980) for totally unimodular matrices.

An approach to the characterization and the recognition of matrix in some class is to exhibit a sequence of compositions that produces the given matrix starting from "elementary" matrices which can be recognized easily. Also, one has to show that the composition operations cannot create submatrices that do not belong to the class of matrices under consideration. Seymour's (1980) characterization of totally unimodular matrices, the result of Yannakakis (1985) for restricted unimodular (equivalently restricted balanced) matrices, i.e., matrices that are totally unimodular (balanced) and remain totally unimodular (balanced) even if an arbitrary number of ones in the matrix are turned to zero, and the result of Conforti and Rao (1987a) for strongly unimodular (equivalently strongly balanced) matrices, i.e., matrices that are totally unimodular (balanced) and remain totally unimodular (balanced) even if any single entry in the matrix is turned to zero, are all along this line of argument. The recognition problem for totally unimodular matrices has been solved by Seymour (1980). Recognition algorithms for restricted unimodular matrices are given by Conforti and Rao (1987a) and Yannakakis (1985). A recognition algorithm for strongly unimodular matrices is contained in Conforti and Rao (1987a).

Another class of balanced matrices that has been studied extensively in the class of totally balanced matrices that arises in location theory, see Anstee and Farber (1984), Hoffman, Kolen and Sakarovitch (1985).

In this paper, we study the properties of linear balanced matrices, that is, balanced matrices which do not contain a 2×2 submatrix with all ones. We give a decomposition property for this class of matrices. In a subsequent paper, see Conforti and Rao (1988), give polynomial algorithms to test balancedness and perfection of linear matrices.

2. Definitions and notation

Throughout the rest of the paper, we are concerned only with bipartite graphs. Henceforth, unless otherwise stated, a graph will refer to a bipartite graph. Given a bipartite graph $G(V^+, V^-; E)$, a *path* of G is a sequence of distinct nodes $v_1, v_2, \ldots, v_n$ such that $(v_i, v_{i+1}) \in E$, for all $1 \leq i \leq n-1$. The edges (v_i, v_{i+1}) are the edges of the path and an edge (v_i, v_{i+l}), $l \geq 2$ is a *chord* of the path. A path with nodes v_1, v_n as end nodes is said to be a v_1v_n-path. If v_i and v_l are two nodes of a path P, the path $v_i, v_{i+1}, \ldots, v_l$ is said to be the v_iv_l-subpath of P with end nodes v_i, v_l. The length of a path is the number of edges in the path. If both end nodes v_1, v_n of P belong to either V^+ or V^-, then the length of P is congruent to 0 to

2 mod 4. If only one end of P belongs to V^+, the length of P is congruent to 1 or 3 mod 4. For sake of brevity, the word "congruent" will be often omitted. Although a path is a sequence of nodes, we frequently use P to denote the set of nodes in the path. Thus we assume that all set operations are defined with respect to P.

A *cycle* is a sequence of nodes $v_1, v_2, \ldots, v_{n+1}$ in which nodes v_1, v_{n+1} coincide but all other nodes are distinct. An edge $(v_i, v_{i+1}) 1 \leq i \leq n$, is an edge of the cycle. An edge of G connecting two non-consecutive nodes of a cycle is a chord of the cycle. A chordless cycle is a hole. A cycle is referred to as an odd cycle if its length is congruent to 2 modulo 4 and as an even cycle if its length is congruent to 0 modulo 4. Again, although a cycle C is a sequence of nodes, we frequently use C to denote the set of nodes in the cycle. Thus we assume that all set operations are defined with respect to C. For the sake of convenience, we frequently denote a cycle C as $v_1, v_2, \ldots, v_i, P, v_j, v_{j+1}, \ldots, v_n$ where P is a path from v_i to v_j. Note that while P itself denotes a sequence of nodes starting with v_i and ending with v_j, it is understood that in the cycle C, nodes v_i and v_j appear only once and are not repeated. However, by the definition of a cycle, P would not contain any of the nodes v_k, $k = 1, 2, \ldots . i-1$ and $k = j+1, j+2, \ldots, n$.

A node y is adjacent to (or is a neighbor of) node x if edge $(x, y) \in E$. Two nodes x, y are adjacent in a path or a cycle if edge (x, y) is an edge of the path on the cycle.

Let G' be a subgraph of G. A node x not belonging to G' is said to be *strongly adjacent* to G' if x is adjacent to at least two nodes of G'. A graph G is said to contain a subgraph G' if G' is an induced subgraph of G.

We say that a subset V of nodes of G is an articulation set if the subgraph of G induced by $(V^+ \cup V^-) \setminus V$ is disconnected.

The set $N(x)$ consists of node x and its neighbors. If $N(x)$ is an articulation set, it is referred to as a *star cutset*.

Given a set S, we indicate its cardinality as $|S|$. In this paper we focus on congruence relationships ($\equiv$) modulo 4.

3. Odd cycles and starred cycles

An odd cycle of a bipartite graph is said to be *minimal* if no proper subset of its nodes induces an odd cycle. In this section we characterize a property of minimal odd cycles of a linear balanced graph.

Lemma 3.1. *Every minimal odd cycle of a linear balanced graph has a unique chord.*

Proof. Conforti and Rao (1987b) have shown that, for every pair of chords (u_1, v_1) and (u_2, v_2) of a minimal odd cycle C with $u_1, u_2 \in V^+$ and $v_1, v_2 \in V^-$, the nodes must appear in the order: v_1, u_2, u_1, v_2 when C is traversed clockwise (or counterclockwise). Moreover, nodes u_1 and v_2 as well as nodes u_2 and v_1 must be adjacent in C. Hence nodes v_1, u_1, v_2, u_2 induce a cycle of length 4, which is a contradiction since G is linear. □

Given a cycle C, let $C^+ = \{v_i \mid v_i \in V^+ \cap C\}$ and $C^- = \{v_i \in V^- \cap C\}$. An odd (even) cycle C is said to be *node-minimal* if there exists no other odd (even) cycle H such that $H^+ \subset C^+$ or $H^- \subset C^-$. Recall that a node v not in G is strongly adjacent to C if v has two or more neighbors in C. We now state the following property of node-minimal odd cycles:

Lemma 3.2. *Let G be a linear balanced graph containing an odd cycle C with a unique chord. Then C is a node-minimal cycle if and only if no node is strongly adjacent to C.*

Proof. Conforti and Rao (1987b) have shown that if node v not in C has two neighbors x, y in C, and C is node-minimal, then there exists a node z of C that is adjacent to both x, y in C. This implies that nodes v, x, y, z induce a cycle of length four which is a contradiction. On the other hand if there are no strongly adjacent nodes, clearly C is node-minimal. □

Definition 3.3. A cycle C is *starred* if its set of chords satisfies the following properties:

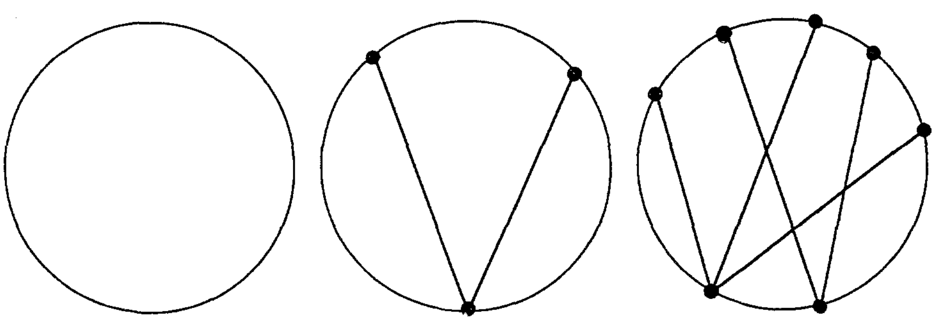
(a) There exists two nodes x and y of C, called the *star nodes* of C such that every chord of C has either node x or node y, but not both, as its end node. A chord with node x (y) as star node is a x-chord. (y-chord).

(b) No other node of C is the end node of two distinct chords.

(c) No two end nodes of chords are adjacent.

Note that, according to Definition 3.3, all the chords of a starred cycle C can have the same star node, or C can be chordless. Figure 2 shows three starred cycles.

Remark 3.4. Let x, y be two non-adjacent nodes of a cycle C in a linear graph G, with the property that every chord of C has x or y as end node. If one of the two xy-paths in C has length less than four and C does not contain an odd hole of length six then, properties (a)–(c) of Definition 3.3 are satisfied.



(a) (b) (c)

Fig. 2.

Theorem 3.5. *Let C be a starred cycle of a balanced graph $G(V^+, V^-; E)$ and T its set of chords. Then C and T satisfy the following relationship*:

$$2|T| \equiv |C|.$$

Proof. By induction on the cardinality $|T|$ of the set of chords of C. The theorem is obviously true for $|T| \leq 1$. We consider the following two cases:

Case 1. C contains a chord (u, x) such that every chord of C other than (u, x) has both its end nodes contained in $C_1 = u, P_1, x, u$ or $C_2 = u, P_2, x, u$ where P_1, P_2 are the two disjoint paths connecting u and x in C, see Figure 3(a). Let T_1 and T_2 be the sets of chords of C_1 and C_2 respectively. Then we have that $|T_1| < |T|$ and $|T_2| < |T|$ since $T_1 \cap T_2 = \emptyset$ and $T_1 \cup T_2 = T \setminus \{(u, x)\}$. Therefore by the induction hypothesis, we have

$$2|T_1| \equiv |C_1| \quad \text{and} \quad 2|T_2| \equiv |C_2|.$$

Furthermore, we have

$$2 + |C| = |C_1| + |C_2| \quad \text{and} \quad 1 + |T| = |T_1| + |T_2|.$$

These relationships imply

$$2|T| \equiv |C|.$$

Case 2. No chord satisfying the assumption of Case 1 exists. This implies that there exists two chords (u, x) and (v, y) of C, such that their end nodes appear in the order x, v, u, y when C is traversed in one direction, see Figure 3(b).

Let P_1, P_2, P_3 and P_4 be respectively the paths connecting v to u, u to y, y to x, and x to v, as shown in Figure 3(b). Let T_1 and T_2 be the sets of chords of the cycles

$$C_1 = v, P_1, u, x, P_3, y, v \quad \text{and} \quad C_2 = v, P_4, x, u, P_2, y, v.$$

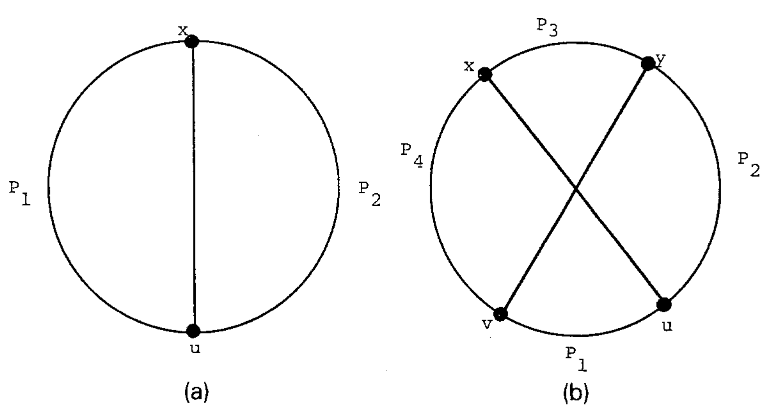


Fig. 3.

As a consequence of properties (a)-(c) of Definition 3.3, we have that none of the following edges: (v, u), (u, y), (y, x) or (x, v) is an edge or a chord of C. This implies that $T_1 \cap T_2 = \emptyset$ and $T_1 \cup T_2 = T \backslash \{(u, x), (v, y)\}$. Therefore we have

$$|T_1| + |T_2| = |T| - 2 \quad \text{and} \quad |C_1| + |C_2| = |C| + 4.$$

Since $|T_1| < |T|$ and $|T_2| < |T|$, by the induction hypothesis we have

$$2|T_1| \equiv |C_1| \quad \text{and} \quad 2|T_2| \equiv |C_2|.$$

The above relationships imply

$$2|T| \equiv |C|. \qquad \square$$

4. A two-star cutset theorem

In this section we prove the following result:

Theorem 4.1. *Let (u, v) be the chord of a node-minimal odd cycle in a linear balanced graph G. Then $N(u) \cup N(v)$ is an articulation set of G.*

Proof. Let C be a node-minimal odd cycle with chord (u, v). Let P_1 and P_2 be the two uv-paths in C. Moreover, let a (b) be the neighbor of u in P_1 (P_2) and let c (d) be the neighbor of v in P_1 (P_2). With $N(u) \cup N(v)$ removed, let P be a shortest path from P_1 to P_2 with end nodes $x \in P_1$ and $y \in P_2$. Let P_{xa} (P_{xc}) be the xa-subpath (xc-subpath) of P_1. Similarly define P_{by} and P_{dy} subpaths of P_2, see Figure 4.

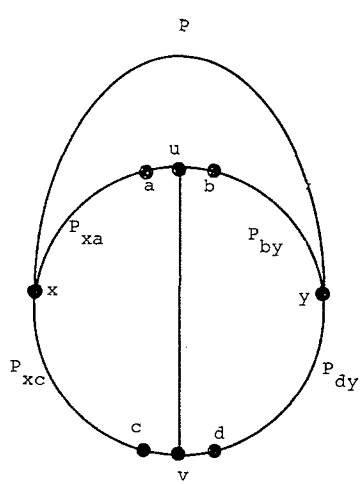


Fig. 4.

Finally, we define the four cycles

$$C_{ab} = x,\ P_{xa},\ a,\ u,\ b,\ P_{by},\ y,\ P,\ x,$$

$$C_{cd} = x,\ P_{xc},\ c,\ v,\ d,\ P_{dy},\ y,\ P,\ x,$$

$$C_{ad} = x,\ P_{xa},\ a,\ u,\ v,\ d,\ P_{dy},\ y,\ P,\ x,$$

$$C_{cb} = x,\ P_{xc},\ c,\ v,\ u,\ b,\ P_{by},\ y,\ P,\ x.$$

As C is a node-minimal odd cycle, by Lemma 3.2, the neighbor of x in P has only one neighbor in P_1, viz. x itself. Similarly for y. All intermediate nodes of P are adjacent to at most one node in $\{a, b, c, d,\}$ since G is linear. As a consequence of Remark 3.4, the four cycles defined above are starred cycles with nodes a, b; c, d; a, d; and c, b respectively as the star nodes. Let T_a, T_b, T_c, T_d be the set of edges having one end node in $P\backslash\{x, y\}$ and nodes a, b, c, d respectively as the other end node. Note that $T_a \cup T_b$, $T_c \cup T_d$, $T_a \cup T_d$ and $T_c \cup T_b$ are the sets of chords of the starred cycles C_{ab}, C_{cd}, C_{ad} and C_{cb} respectively. Now by Theorem 3.5 we have

$$|C_{ab}| \equiv 2(|T_a| + |T_b|), \qquad |C_{cd}| \equiv 2(|T_c| + |T_d|),$$

$$|C_{ad}| \equiv 2(|T_a| + |T_d|), \qquad |C_{cb}| \equiv 2(|T_c| + |T_b|).$$

Hence

$$|C_{ab}| + |C_{cd}| + |C_{ad}| + |C_{cd}| \equiv 4(|T_a| + |T_b| + |T_c| + |T_d|) \equiv 0.$$

On the other hand

$$|C_{ab}| + |C_{cd}| + |C_{ad}| + |C_{cb}| \equiv 4|P| + 2|C| + 2 \equiv 2,$$

which yields a contradiction. □

5. Expanded cycle and starred wheel

In order to prove the star cut set theorem, we need some properties associated with linear balanced graphs. These properties which are given in Lemma 5.1 and Theorem 5.3 below are used in the next two sections to prove that if a linear balanced graph contains a given induced subgraph, then the induced subgraph must contain a star cutset of G. Finally in Section 8, the Star Cutset Theorem is proved by applying the Two Star Cutset Theorem of Section 4 to show that if G contains an odd cycle with a unique chord (u, v), and $N(u)$ is not a star cutset, then G contains one of the given induced subgraphs with $N(v)$ as a star cutset of G.

5.1. *Expanded cycle*

Consider a graph G consisting of a starred cycle C plus two additional nodes b and d that are adjacent to C. Suppose C contains just one star node s and b is adjacent to d and s but not adjacent to any other node of C. We define the triple (C, b, d) to be an *expanded cycle*, see Figure 5. Let T be the set of chords of C and T_d be the set of edges joining d to a node of C.

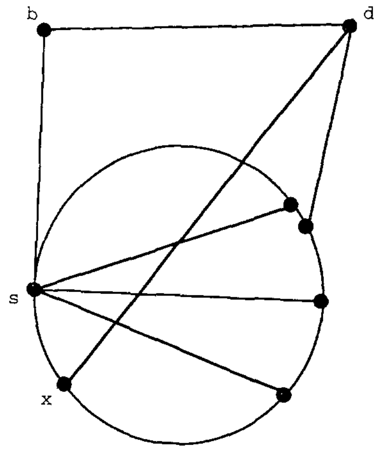


Fig. 5.

Lemma 5.1. *Let G be a linear balanced graph containing an expanded cycle* (C, b, d) *with star node s. Then* $|T_d|$ *is even.*

Proof. Suppose $|T_d|$ is odd. Then there exists a node $x \in C$ adjacent to d. Let P_1 and P_2 be the two xs-paths forming C consider the two cycles $C_1 = x, P_1, s, b, d, x$ and $C_2 = x, P_2, s, b, d, x$. All the possible chords of C_1 and C_2 must have either node s or d as an end node. By Remark 3.4, it follows that C_1 and C_2 are starred cycles with star nodes s and d. Let T_1 and T_2 be the chords of C_1 and C_2 respectively. Then by Theorem 3.5 we have $2|T_1| \equiv |C_1|$, $2|T_2| \equiv |C_2|$ and $2|T| \equiv |C|$. Clearly $T \cap T_d = \emptyset$ and $|C| = |C_1| + |C_2| - 6$. As T_1, T_2 and (x, d) partition $T \cup T_d$, it follows that $2|T_d| = 2|T \cup T_d| - 2|T| = 2|T_1| + 2|T_2| + 2 - 2|T| \equiv |C_1| + |C_2| - |C| + 2 = 6 + 2 \equiv 0$. Hence $|T_d|$ cannot be odd and the lemma follows. □

5.2. Starred wheel

A cycle C and a node $v \notin C$ form a *wheel* (C, v) if v is adjacent to at least two nodes of C. Node v is the *hub* of the wheel and edges (v, v_i) where $v_i \in C$ are the *spokes* of the wheel. The $v_{i-1}v_i$-path of C that does not contain any other neighbor of v is called a *sector* S_i of the wheel. The corresponding cycle C_i is formed by $v_{i-1}, S_i, v_i, v, v_{i-1}$. Two sectors are said to be adjacent if they both contain the same neighbor of v. Two spokes (v_i, v) and (v_j, v) are adjacent if (C, v) contains a sector with end nodes v_i and v_j.

If C is chordless then (C, v) is a *chordless wheel.* Clearly a balanced chordless wheel must have an even number of spokes.

Definition 5.2. A wheel (C, v) is a *starred wheel* if C is a starred cycle with star nodes s and t satisfying the following two conditions, see Figure 6.

(a) The star nodes s and t are adjacent to v_0 where (v_0, v) is a spoke of the wheel.
(b) No s-chord (t-chord) of C has an end node in the sector containing $t(s)$.

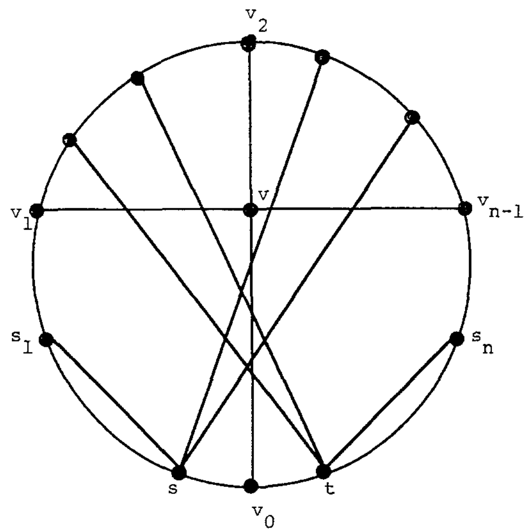


Fig. 6.

Theorem 5.3. *Let G be a linear balanced bipartite graph containing a starred wheel* (C, v) *with star nodes s and t. Let* S_i, $i = 1,2,\dots,n$ *be the n sectors of the starred wheel with* $s \in S_1$, $t \in S_n$. *Then*:

(i) *The number n of sectors (spokes) is even.*

(ii) *All the s-chords (t-chords) have end nodes in odd (even) numbered sectors only.*

(iii) *Let w be a node strongly adjacent to C and having* v_0 *as a neighbor. Node w has neighbors in only odd numbered or in only even numbered sectors but not both. If w has neighbors in only odd (even) numbered sectors, then there exists a path from s (t) to w, not containing a node in* $N(v)$.

Proof. We first prove the following two claims:

Claim 1. Each sector S_j, $1 < j < n$ has an even number of end nodes of s-chords and an even number of end nodes of t-chords.

Proof. Suppose sector S_j with end nodes v_{j-1} and v_j contradicts Claim 1. Without loss of generality, suppose S_j contains an odd number of end nodes of s-chords. The cycle $C_j = v, v_{j-1}, S_j, v_j, v$ defines an expanded cycle (C_j, v_0, s) with star node v. Now by Lemma 5.1, the number of end nodes of s-chords in S_j must be even and not odd. Hence the claim follows.

Claim 2. No sector $S_j, j = 1, 2, \dots, n$, contains end nodes of both s-chords and t-chords.

Proof. By condition (b) of Definition 5.2, S_1 and S_n satisfy the above claim. Suppose sector S_j, $1 < j < n$, contradicts Claim 2. Then S_j must contain two nodes x and y which are the end nodes of a s-chord and a t-chord respectively. Furthermore, the xy-subpath, P, of S_j does not contain an intermediate node which is an end node of a s-chord or a t-chord. The length of P must be 0 mod 4 since x, P, y, t, v_0, s, x is a chordless cycle. In S_1 (S_n) let u (w) be the node closest to v_1 (v_{n-1}) such that

u (w) is adjacent to s (t). Let P_1 and P_2 denote the uv_1 and wv_{n-1}-subpaths of S_1 and S_n respectively. Now both P_1 and P_2 must be of length 0 mod 4 since $s, u, P_1, v_1, v, v_0, s$ and $t, w, P_2, v_{n-1}, v, v_0, t$ are chordless cycles. But then $s, u, P_1, v_1, v, v_{n-1}, P_2, w, t, y, P, x, s$ is an odd hole. Thus the claim follows.

We now prove the three properties stated in the theorem.

(i) Let T_s and T_t be the set of s-chords and t-chords respectively of C. Similarly let T_s^i and T_t^i be the sets of s-chords and t-chords respectively of C with one end node in S_i. Then

$$T_s = \bigcup_{i=1}^{n} T_s^i \quad \text{and} \quad T_s^i \cap T_s^j = \emptyset \quad \text{for all } i \neq j.$$

Similarly we have that

$$T_t = \bigcup_{i=1}^{n} T_t^i \quad \text{and} \quad T_t^i \cap T_t^j = \emptyset \quad \text{for all } i \neq j.$$

As a consequence of Theorem 3.5 it follows that

$$2|T_s \cup T_t| \equiv |C|, \qquad 2|T_s^1| \equiv |C_1| \quad \text{and} \quad 2|T_t^n| \equiv |C_n|.$$

By Claim 1, we have

$$2|T_s^i| \equiv 2|T_t^i| \equiv 0 \quad \text{for } 1 < i < n.$$

Furthermore, by condition (b) of Definition 5.2, $2|T_s^n| = 2\ |T_t^1| = 0$. Since C_i, $i = 2, \ldots, n-1$ are chordless, we have $|C_i| \equiv 0$, $i = 2, \ldots, n-1$. Then

$$|C| = \sum_{i=1}^{n} |S_i| \equiv \sum_{i=1}^{n} |C_i| - 2n \equiv \sum_{i=1}^{n} 2(|T_s^i| + |T_t^i|) - 2n \equiv |C| - 2n.$$

Hence $2n \equiv 0$ and n is even.

(ii) It is sufficient to prove that if edge (t, w) is a chord and $w \in S_j$, then j is even. By condition (b) of Definition 5.2, we have that $j \neq 1$. Let P be the wt-subpath of C containing s. Let $C^* = w, P, t, w$. Then by Claim 2 (C^*, v) is a starred wheel, therefore (C^*, v) must have an even number of spokes as proved in (i) above. Hence j is even.

(iii) Consider a node $w \notin C$ such that $w \neq v$ but w is adjacent to v_0 and strongly adjacent to C. Suppose w has a neighbor b in an even numbered sector and a neighbor d in an odd numbered sector. Let P be the bd-path along C not containing v_0, see Figure 7. Let $C^* = w, b, P, d, w$.

Then (C^*, v_0, v) is an expanded cycle with star node w. But v has an odd number of neighbors in C^* which contradicts Lemma 5.1. Hence w has neighbors in only odd numbered or in only even numbered sectors, but not both. In order to complete the proof of the theorem, we have to show that if w has neighbors in only odd (even) numbered sectors, then there exists a path from s (t) to w, not containing a node in $N(v)$.

We consider the case in which w has neighbors in odd numbered sectors only. The other case follows by symmetry. Let i be the smallest index such that in S_i, w

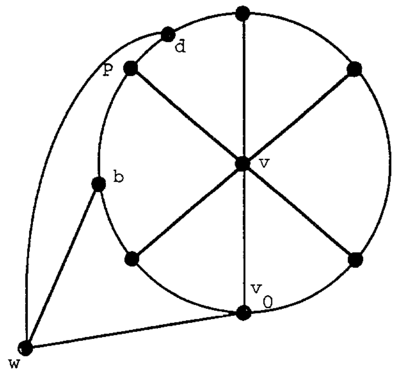


Fig. 7.

has a neighbor, say $w_i \neq v_0$. Note that $w_i \notin N(v)$ since $N(v) \cap N(w) = v_0$. If $i = 1$, the theorem follows since in S_1, node s has a neighbor different from v_0. In fact, if s has a neighbor in S_i, the theorem follows. Suppose now s has no neighbors in S_i. Let $j < i$ be the largest index such that S_j contains a neighbor of s. Note that j may equal 1. In S_j, let s_1 be the neighbor of s closest to v_j and let P be the $v_j s_1$-subpath of S_j. In S_i, let w_1 be the neighbor of w closest to v_{i-1} and let Q be the $w_1 v_{i-1}$-subpath of S_i. Let R be the $v_{i-1} v_j$-subpath of C not containing v_0. Consider the cycle $C^* = w_1, Q, v_{i-1}, R, v_j, P, s_1, s, v_0, w, w_1$. Now (C^*, v) is a chordless wheel with an odd number of spokes greater than or equal to 3, which yields a contradiction. This completes the proof of the theorem. □

6. Odd cycle with a unique chord

In this section, given a linear balanced graph G containing an odd cycle C with a unique chord (u, v), we first study the structure of nodes that are strongly adjacent to C. Let P_1, P_2 be the two uv-paths in C and C_1 and C_2 be the chordless cycles formed by edge (u, v) with P_1 and P_2. The main result in this section, Theorem 6.6, is that if there exists a strongly adjacent node that has neighbors both in $P_1 \backslash \{u, v\}$ and $P_2 \backslash \{u, v\}$, then node u or v is a star cutset of G.

Classification

Consider the following three mutually exclusive classifications of nodes that are strongly adjacent to C.

A node x strongly adjacent to C is a:

Type 1 node if $|N(x) \cap C|$ is positive and even and $(N(x) \cap C) \subset C_i$ for either $i = 1$ or 2, see Figure 8(a).

Type 2 node if $|N(x) \cap C|$ is odd and $|N(x) \cap C_i|$ is positive and even for $i = 1, 2$. In this case, either u or $v \in N(x)$, see Figure 8(b).

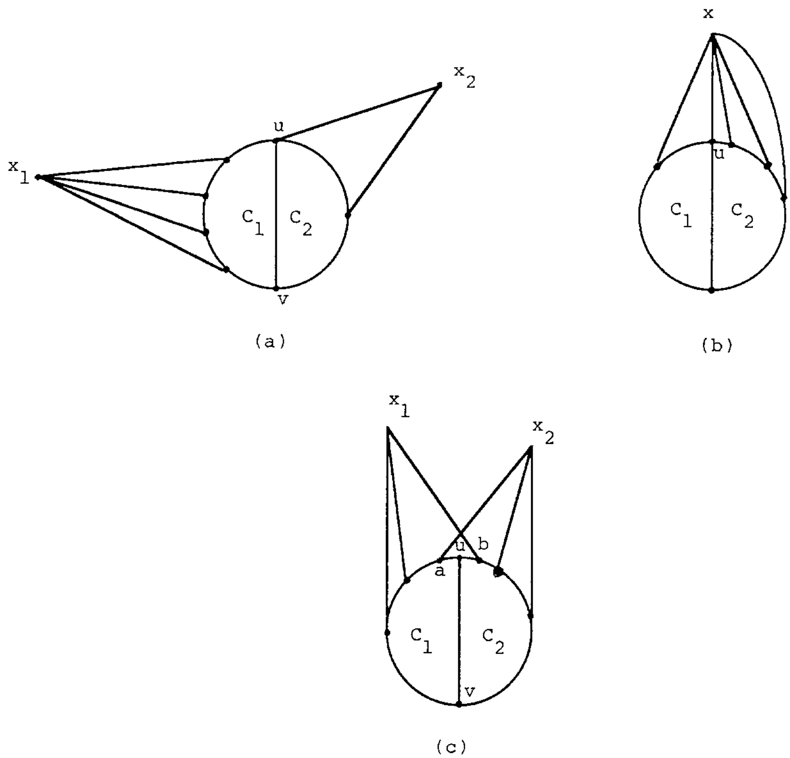


Fig. 8.

Type 3 node if $|N(x) \cap C|$ is odd and either (i) $|N(x) \cap C_1| = 1$ and $|N(x) \cap C_2|$ is positive and even or (ii) $|N(x) \cap C_2| = 1$ and $|N(x) \cap C_1|$ is positive and even. Furthermore, the unique neighbor of x in $C_j, j = 1, 2$, is also a neighbor of u or v, see Figure 8(c).

We now prove the above classification to be complete for linear balanced graphs:

Theorem 6.1. *Let $G(V^+, V^-; E)$ be a linear balanced graph containing an odd cycle C with a unique chord (u, v). A node t that is strongly adjacent to C is a type* 1 *or a type* 2 *or a type* 3 *node.*

Proof. As defined at the beginning of this section let C_i, $i = 1, 2$ be the two chordless cycles formed by (u, v). Since for $i = 1, 2$, the cycles C_i are chordless, we have that if $|N(t) \cap C_i|$ is odd, then it is equal to 1. We now divide the proof into the following claims.

Claim 1. If $|N(t) \cap C|$ is even, t must be a type 1 node.

Proof. If $|N(t) \cap C|$ is even, then one of the sectors of (C, t) together with t forms an odd cycle, say C'. Now C' must have a chord. However, the only candidate is (u, v). This yields the claim.

Claim 2. If $|N(t) \cap C|$ is odd, t must be a type 2 or type 3 node.

Proof. Clearly t cannot be a type 1 node. If $t \in N(u)$ or $t \in N(v)$, then t must be a type 2 node since for $i = 1, 2$, $|N(t) \cap C_i|$ is positive and even. Suppose $t \notin N(u) \cup N(v)$. We now show that t is a type 3 node. Because of symmetry, we may assume that $|N(t) \cap C_1|$ is positive and even and that $|N(t) \cap C_2| = 1$. If t is not a type 3 node then its unique neighbor w in C_2 is not a neighbor of u or v. We can assume without loss of generality that $u \in V^+$ and $w \in V^-$. Let x be the neighbor of t in C_1 such that the xu-path, Q, in C_1 does not contain any other neighbor of t. Let P_1 and P_2 be the two uw-paths in C_2. Since C_2 is an even cycle and $u \in V^+$, $w \in V^-$ it follows that P_1 and P_2 are paths of lengths $1 \bmod 4$ and $3 \bmod 4$ (or $3 \bmod 4$ and $1 \bmod 4$) respectively. Consequently either $C^* = t, x, Q, u, P_1, w, t$ or $C^* = t, x, Q, u, P_2, w, t$ is an odd hole. Hence t must be a type 3 node and Claim 2 is proved. This completes the proof of the theorem. □

We next prove the following lemma.

Lemma 6.2. *Let G be a linear balanced graph containing an odd cycle C with a unique chord (u, v). Let a, b (c, d) be the neighbors of u (v) in C. Then there cannot exist two nodes x and y of type 2 or 3 with $N(x) \cap \{a, b, u\} \neq \emptyset$ and $N(y) \cap \{c, d, v\} \neq \emptyset$.*

Proof. Suppose the lemma is false. Let x and y be two nodes contradicting the lemma, see Figure 9. There are three cases to consider.

Case 1. Both x and y are type 2 nodes, see Figure 9(a).

Let P_1 (P_2) be the shortest path from x to y containing only nodes in $C_1 \setminus \{u, v\}$ $(C_2 \setminus \{u, v\})$ as intermediate nodes. Let the cycle $C^* = x, P_1, y, P_2, x$. Then (C^*, v, u) is an expanded cycle with star node y. But since u is adjacent to only one node x of C^*, Lemma 5.1 is violated. Consequently x and y cannot both be type 2 nodes.

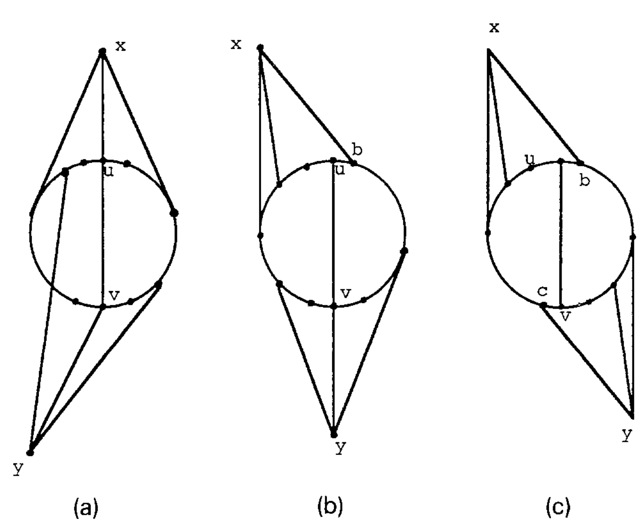


Fig. 9.

Case 2. x is a type 3 node and y is a type 2 node, see Figure 9(b).

The proof is identical to that of case 1 except that u is adjacent to only one node b of C^*.

Case 3. Both x and y are type 3 nodes, see Figure 9(c).

Without loss of generality we assume that x is adjacent to b. The proof is identical to that of case 2 except that now, if y is adjacent to d (c) then d (c) is the star of the expanded cycle C^*.

This completes the proof of the lemma. □

Definition 6.3. Let G be a linear bipartite graph containing an odd cycle C with a unique chord (u, v). Let Q_1 and Q_2 be the two uv-paths in C. Then C_1 and C_2 are two chordless cycles formed by (u, v). Suppose a and b (c and d) are the neighbors of u (v) in C. We define a chordless path P_u from $s \in C_1\backslash\{u, v, a\}$ to $t \in C_2\backslash\{u, v, b\}$ to be a minimal path if it satisfies the following two conditions.

(i) No intermediate node of P_u is in $C \cup N(u)$

(ii) Let $x(y)$ be the neighbors of $s(t)$ in P_u. Let $N(P_u)$ be the set of nodes in $C\backslash\{s, t\}$ adjacent to intermediate nodes of P_u, other than x and y. Then $N(P_u) \subseteq \{a, b, v\}$.

Note that $N(u)$ is not a star cutset of G separating c and d if and only if there exists a minimal path P_u connecting a node in $C_1\backslash\{u, v, a\}$ to a node in $C_2\backslash\{u, v, b\}$.

Definition 6.4. Let the cycles C, C_1, C_2, paths Q_1, Q_2 and nodes u, v, a, b, c, d be as in Definition 6.3. Let P_u be a minimal path from $s \in C_1\backslash\{u, v, a\}$ to $t \in C_2\backslash\{u, v, b\}$, see Definition 6.3, that does not contain a type 2 or type 3 node. If the neighbor x (y) of s (t) in P_u is a type 1 node, let x_1 (y_1) be the neighbor of x (y) in C_1 (C_2) such that the x_1a-path, T_1, (y_1b-path, T_2) in C_1 (C_2) does not contain any other neighbor of x (y). Similarly, when x (y) is a type 1 node, let x_{2n} (y_{2m}) be the neighbor of x (y) in C_1 (C_2) such that the $x_{2n}v$-path, R_1, ($y_{2m}v$-path, R_2) in C_1 (C_2) does not contain any other neighbor of x (y). (Note that if $x_1 = a$, then $T_1 = a$ and $x_{2n} \neq c$). If x (y) is not strongly adjacent to C, let $x_1 = s$ $(y_1 = t)$. Let P_1 be the xy-subpath of P_u and let the cycle $C^* = u, a, T_1, x_1, x, P_1, y, y_1, T_2, b, u$, see Figure 10. (Note that x and y may be adjacent in which case $P_1 = x, y$).

Lemma 6.5. *Let G be a balanced graph containing an odd cycle C with a unique chord (u, v). Let C^* be the cycle as in Definition 6.4. Then (C^*, v) is a starred wheel.*

Proof. We first prove the following claim.

Claim 1. (C^*, v) is a wheel.

Proof. In order to prove the claim we have to show that C^* contains at least one other neighbor of v besides node u. Suppose not. Then $x_1, x, y, y_1 \notin N(v)$. Now consider the subgraph G' induced by the nodes in $C \cup P_u$. If C is a node-minimal odd cycle in G', applying Theorem 4.1 to G', we have that C^* must contain at least two neighbors of v and the claim follows. Suppose now C is not node-minimal in G'. Then x or y or both must be type 1 nodes. We consider the case in which both of them are type 1 nodes, The other case is similar and hence omitted. Consider the odd cycle $\hat{C} = u, a, T_1, x_1, x, x_{2n}, R_1, v, R_2, y_{2m}, y, y_1, T_2, b, u$.

Now we want to show that $\hat{C}$ has only one chord, (u, v). Suppose not. Then the only other chord of $\hat{C}$ must be (x, y). But note that both x and y are not in $N(u) \cup N(v)$. Let Q_1 and Q_2 (Q_3 and Q_4) be the uy and ux-subpaths (vy and vx-subpaths) of $\hat{C}$. Then it follows that either u, Q_1, y, x, Q_4, v, u or u, v, Q_3, y, x, Q_2, u is an odd hole. If $\hat{C}$ is a node-minimal odd cycle in G' let P_1 as defined earlier be the xy-subpath of P_u. Then applying Theorem 4.1 to G' since $\hat{C}$ is node-minimal we have that C^* must contain at least two neighbors of v, thereby proving the claim. Suppose now $\hat{C}$ is not node-minimal in G'. Then the neighbor w of x in P_1 or the neighbor z of y in P_1 or both are in $N(a) \cup N(b)$. First note that $w \neq z$ for otherwise it would imply that $w \in N(a) \cup N(b)$ is strongly adjacent to $\hat{C}$ but w is not a type 1 or a type 2 or a type 3 node. Now if w or z is in $N(v)$, C^* contains two neighbors of v and the claim follows. Suppose both w and z are not in $N(v)$. We now show that $w \notin N(b)$. Suppose $w \in N(b)$. Then the x_1a-path, T_1, in C_1 must be of length 3 mod 4 for otherwise the cycle $w, x, x_1, T_1, a, u, b, w$ would be an odd hole. This implies that the $x_{2n}v$-path, R_1, in C_1 must be of length 1 mod 4. Then the cycle $w, x, x_{2n}, R_1, v, u, b, w$ is an odd hole. Hence $w \notin N(b)$. By symmetry, $z \notin N(a)$. Now since $\hat{C}$ is not node-minimal in G', we must have $w \in N(a)$ or $z \in N(b)$ or both. We consider the case in which $w \in N(a)$ and $z \in N(b)$. The case in which only $w \in N(a)$ or $z \in N(b)$ but not both is similar and hence omitted. Clearly $w \notin N(z)$. Let $\tilde{C} = u, a, w, x, x_{2n}, R_1, v, R_2, y_{2m}, y, z, b, u$ and $P_2 = P_1 \backslash \{x, y\}$. Now $\tilde{C}$ is a node-minimal odd cycle in G' since a neighbor of w or z in P_2 can not be in $N(a) \cup N(b)$ and moreover w and z can not have a common neighbor in P_2. Applying Theorem 4.1 to G', since $\tilde{C}$ is node-minimal it follows that $\tilde{C}$ and hence C^* must contain at least two neighbors of v. Thus the claim follows.

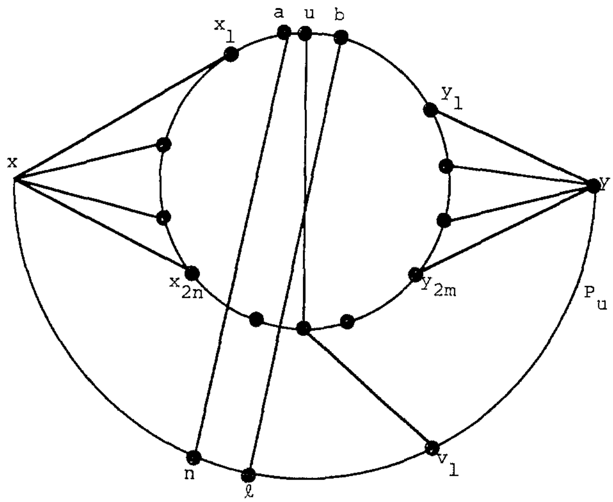
Now, all the chords of C^* have a or b as one of the end nodes, see Figure 10. Thus condition (a) of Definition 5.2 is satisfied. It remains to show that condition (b) of the same definition is satisfied. Let S_i, $i = 1, 2, \ldots, n$, be the sectors of the wheel with $a \in S_1$ and $b \in S_n$. We next show that S_1 does not contain an end node of a b-chord and by symmetry it follows that S_n does not contain an end node of an a-chord. Note that if $x_1 = s = c$, then S_1 has c and u as end nodes and hence S_1 does not contain an end node of a b-chord. On the other hand if x is adjacent to v, then sector S_1 has x and u as end nodes and hence S_1 does not contain an end node of a b-chord. Suppose none of the above holds. Let v_1 and u be the end nodes of S_1, see Figure 10. We now divide the proof into the following two claims:

Claim 2. S_1 cannot contain both a-chord and end nodes of b-chords.

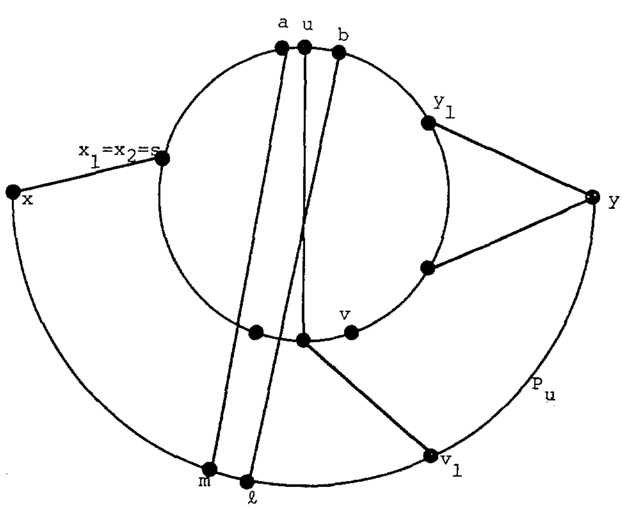
Proof. Suppose the contrary. Let (a, m) be a chord in S_1 and $l \in S_1 \cap N(b)$ be such that the ml-subpath, P_2 of P_u does not contain any other neighbor of a or b. Next we show that $x \neq m, l$. If $x = m$, then (a, m) is an edge of S_1 and not a chord of S_1. If $x = l$, then x would be a type 3 node. But since P_u contains no type 3 node, it follows that $x \neq m, l$. Now P_2 must be of length 0 mod 4 for otherwise the cycle a, m, P_2, l, b, u, a would be an odd hole. Let R be the ba-path in C not containing u. Then the cycle a, m, P_2, l, b, r, a is an odd hole. Consequently the claim follows.

Claim 3. S_1 cannot contain end nodes of b-chords.

Proof. In view of Claim 1 it is sufficient to show that S_1 does not contain end nodes of b-chords only. If x is a type 1 node, let x_{2n} be the neighbor of x in C_1 such that the $x_{2n}v$-subpath, R_1, of C_1 does not contain any other neighbor of x, see Figure 10(a). If x is not a type 1 node, let $x_{2n} = s$, see Figure 10(b). Let P_2 be the xv_1-subpath of P_u. Then $H = x, P_2, v_1, v, R_1, x_{2n}, x$ is even hole. Therefore (H, u, b)



(a)



(b)

Fig. 10.

is an expanded cycle and Lemma 5.1 implies that b must have a positive even number of neighbors. Now the cycle $\hat{C} = u, S_1, v_1, v, u$ is chordless since S_1 does not contain any chords. But node b has an odd number of ($\geqslant 3$) of neighbors in $\hat{C}$. Thus Claim 3 follows and the proof of the lemma is complete. □

We next prove the main result of this section.

Theorem 6.6. *Let $G(V^+, V^-; E)$ be a linear balanced graph containing an odd cycle C with a unique chord (u, v). Let a and b (c and d) be the neighbors of u (v) in C. Suppose a node w strongly adjacent to C is a type 2 or type 3 node and w is adjacent to one of the nodes u, a or b (or v, c or d). Then $N(u)$ ($N(v)$) is a star cut set of G, separating c and d (a and b).*

Proof. Let Q_1 and Q_2 be the two uv-paths in C. Then let C_1 and C_2 be the two chordless cycles formed by (u, v) with Q_1 and Q_2 respectively. Assume node w is adjacent to one of the nodes u, a or b. Suppose the theorem is false. Then there exists a minimal path P_u from $s \in C_1 \backslash \{u, v, a\}$ to $t \in C_2 \backslash \{u, v, b\}$ as in Definition 6.3. There are two cases to consider.

Case 1. The first (x) or the last (y) intermediate node of P going from s to t is a type 3 node.

Without loss of generality, assume that x is a type 3 node. Node y may be a type 3 or a type 1 node or neither, see Figure 11(a).

Without loss of generality we assume that $u \in V^+$. Then $\{x, c, d\} \subset V^+$ and $\{a, v, b\} \subset V^-$. Let $x_i \in C_1$, $i = 1, 2, \ldots, 2n$, be the neighbors of x. Then $x_i \in V^-$. Furthermore $x_1 \neq a$ and $x_{2n} \neq v$ for otherwise we would have a cycle of length 4. Let x_1 (x_{2n}) $\in C_1$ be the neighbor of x such that the $x_1 a$-subpath ($x_{2n} c$-subpath), T_1 (T_2), of C_1 not containing v (u) also does not contain any other neighbor of x. Now consider the subgraph G' induced by the nodes in $\{C \cup P\} \backslash \{b \bigcup_{i=1}^{2n} x_i\}$. In G', let P' be the shortest path connecting x to a or v whichever is closest. Note that if y is a type 3 node, P' would be from x to a. Otherwise P' would be from x to v. No intermediate node of P', except possibly node y is adjacent to a node in C_1. Moreover y is not adjacent to any node in $C_1 \backslash \{a\}$. Now $C^* = x, x_1, T_1, a, u, v, c, T_2, x_{2n}, x$ is a chordless cycle and hence C^* must be of length 0 mod 4. Since $x \in V^+$ and $a \in V^-$, the two xa-paths as well as the two xv-paths in C^* must be such that one of them is of length 1 mod 4 and the other is of length 3 mod 4. If P' is from x to a (v) it would close an odd hole with one of the xa-paths (xv-paths) in C^*. Consequently, x and y are not type 3 nodes. So we must have Case 2 below.

Case 2. x as well as y is not a type 3 node.

Note that x (y) may be a type 1 node. We now have two sub-cases.

Case 2.1. w is a type 3 node, see Figure 11(b).

Define x_1, y_1, T_1, T_2 and P_1 as in Definition 6.4. Now by Lemma 6.5, we have that (C^*, v) is a starred wheel, where $C^* = u, a, T_1, x_1, x, P_1, y, y_1, T_2, b, u$. Hence v must have a positive even number of neighbors in C^*. Let R be a shortest path from w to y with intermediate nodes contained in $C_2 \backslash \{u\}$. Now consider the cycle

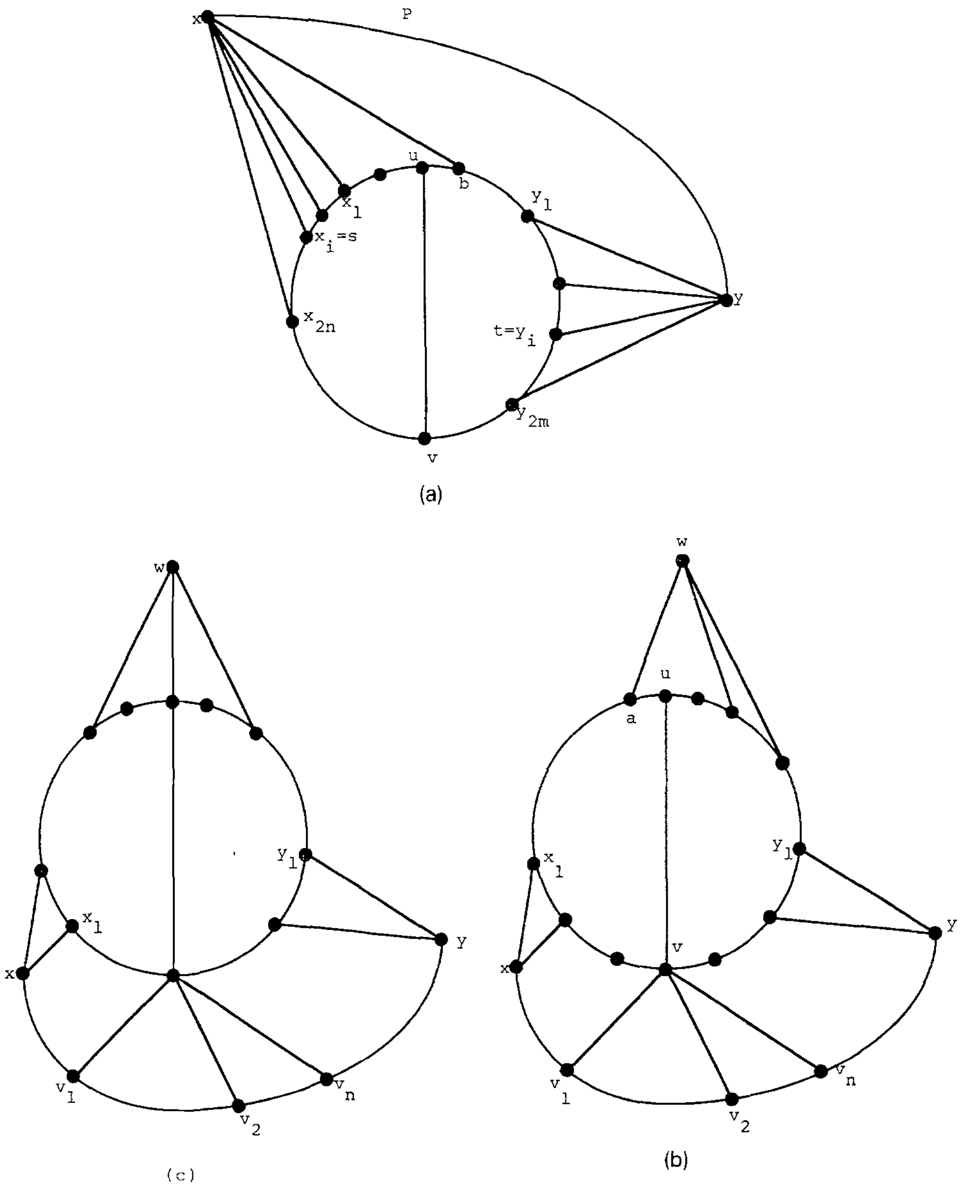


Fig. 11.

$\hat{C} = w, R, y, P_1, x, x_1, T_1, a, w$. Then $(\hat{C}, u, v)$ is an expanded cycle with node a as star node. But v has an odd number of neighbors in $\hat{C}$ thereby contradicting Lemma 5.1. Hence a minimal path P_u as in Definition 6.3 can not exist. Consequently $N(u)$ must be a star cutset separating c and d.

Case 2.2. w is a type 2 node, see Figure 11(c).

The wheel (C^*, v) as defined above in Case 2.1 is still a starred wheel and hence node v must have an even ($\geqslant 2$) number of neighbors in C^*. Let R_1 (R_2) be a shortest path from w to x (y) with intermediate nodes contained in $C_1\backslash u$ $(C_2\backslash u)$. Now consider the chordless cycle $\hat{C} = w, R_1, x, P_1, y, R_2, w$. Then $(\hat{C}, u, v)$ is an expanded cycle with w as the star node. But v has an odd number of neighbors in $\hat{C}$ thereby contradicting Lemma 5.1. Again this implies that a minimal path P_u as in Definition 6.3 can not exist.

This completes the proof of the theorem. □

Lemma 6.7. *Let G be a linear balanced graph containing an odd cycle C with a unique chord (u, v). Then either*:

(i) *$N(u)$ is a star cutset of G separating c and d,*

(ii) *$N(v)$ is a star cutset of G separating a and b,*

(iii) *there exists a node-minimal odd cycle, say C^* with edge (u, v) as the unique chord. Suppose a^* and b^* (c^* and d^*) are the neighbors of u (v) in C^*. Then from a to a^* and b to b^* (c to c^* and d to d^*), there exists a path not containing a node in $N(v)$ $(N(u))$.*

Proof. If there exists a type 2 or type 3 node that is strongly adjacent to C, then by Theorem 6.6 either $N(u)$ or $N(v)$ is a star cutset of G and the lemma follows. Suppose there are no type 2 or type 3 nodes. If there is a type 1 node, then no node is strongly adjacent to C. Moreover $C^* = C$, $a^* = a$, $b^* = b$ $(c^* = c, d^* = d)$ and the lemma follows by Lemma 3.2. Suppose now there is a type 1 node x. Let P_1, P_2, C_1, C_2 be as defined at the beginning of this section. Without loss of generality assume that x_i, $i = 1,2,\ldots, 2n$ are the neighbors of x in C_1. Suppose x_1 (x_{2n}) is the neighbor of x in C_1 such that the x_1u-path, R ($x_{2n}v$-path, Q) does not contain any other neighbor of x. Then the odd cycle $\hat{C} = u, R, x_1, x, x_{2n}, Q, v, P_2, u$ has (u, v) as the unique chord. Moreover the length of $\hat{C}$ is less than the length of C. If $x_1 = u$ $(x_{2n} = v)$ then $x_{2n} \notin N(v)$ $(x_1 \notin N(v))$ and $\hat{a} = x$ $(\hat{c} = x)$ is a neighbor of u (v) in $\hat{C}$. Clearly, there is a path from a to $\hat{a}$ (c to $\hat{c}$) with intermediate nodes contained in $C_1\backslash N(v)$ $(C_1\backslash N(u))$. Now if $\hat{C}$ has a type 2 or type 3 node, the lemma follows with $C = \hat{C}$. Otherwise repeating the above process must result in either a node-minimal odd cycle C^* with (u, v) as the unique chord or we must have a type 2 or type 3 node. In the former case, let a^* and b^* (c^* and d^*) be the neighbors of u (v) in C^*. In this case, clearly, the process of redefining the odd cycle is such that (iii) holds. In the latter case, by Theorem 6.6, either (i) or (ii) holds. □

7. Parachutes and star cutsets

In this section we first define a parachute and prove that a parachute in a linear balanced graph G must contain a star cut set of G. The proof of the Star Cutset Theorem requires this result.

7.1. Chordless wheel with two spokes

Definition 7.1. A *parachute* (C, v, v_3, P_3) is a chordless cycle C and a node v which has two neighbors v_1, v_2 in C and a path of length $\geqslant 2$ from v to a node z of C such that no intermediate node of this path is adjacent to a node in C. Let v_3 be the neighbor of v in this path, see Figure 12.

Let P_0, P_1, P_2 be respectively the paths connecting v_1 and v_2, v_1 and z, z and v_2, and making up the cycle C. Let P_3 be the path connecting v_3 to z, as in Figure 12. The length of both paths P_1, P_2 is greater than one.

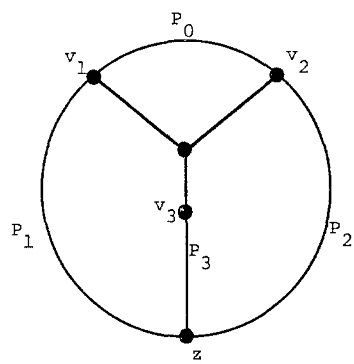


Fig. 12.

Theorem 7.2. *Let G be a linear balanced graph containing a parachute (C, v, v_3, P_3) with two spokes (v, v_1), (v, v_2) and corresponding paths P_0, P_1, P_2, see Definition 7.1. Let $a \in P_1$, $b \in P_0$ ($c \in P_2$, $d \in P_0$) be the neighbors of v_1 (v_2) in C. Suppose every node $w \notin N(v)$ which is strongly adjacent to $C \cup P_3$ has exactly two neighbors in $C \cup P_3$, both of which are in P_0 or in $P_1 \cup P_2 \cup P_3$. Then $N(v)$ is a star cutset of G separating $\{a, c\}$ and $\{b, d\}$.*

Proof. With nodes of $N(v)$ removed, let P_v be a shortest path from $s \in P_0 \backslash \{v_1, v_2\}$ to $t \in P_1 \cup P_2 \cup P_3 \backslash \{v_1, v_2, v_3\}$, see Figure 13. Let x (y) be the neighbor of s (t) in P_v. Let P_{xy} be the xy-subpath of P_v. If x is strongly adjacent to P_0, let x_1 be the neighbor of x closest to v_1 in P_0. Otherwise, let $x_1 = s$. In either case, let R be the $x_1 v_1$-subpath of P_0. Now consider the two cycles $C_1 = v_2, P_0, v_1, P_1, z, P_3, v_3, v, v_2$ and $C_2 = v_1, P_0, v_2, P_2, z, P_3, v_3, v, v_1$. Both C_1 and C_2 are odd cycles with a single chord (v, v_1) and (v, v_2) respectively. Now there are two cases to consider depending upon whether y is strongly adjacent to $P_1 \cup P_2 \cup P_3$ or not.

Case 1. y is not strongly adjacent to $P_1 \cup P_2 \cup P_3$, see Figure 14.

There are two subcases depending upon whether $t \in P_1 \cup P_2$ or $t \in P_3$. The latter case is identical to the first case and hence we consider only the case $t \in P_1 \cup P_2$. Because of symmetry we need to consider only the case $t \in P_2$, see Figure 14(a).

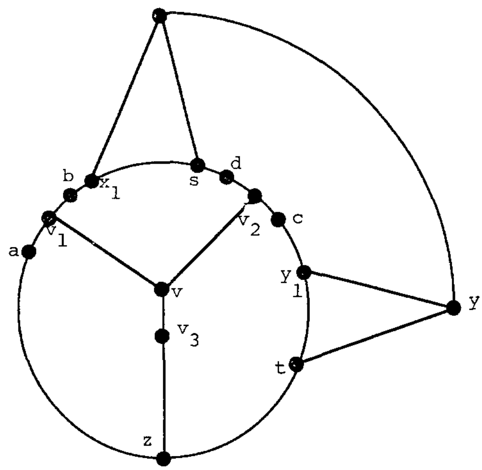


Fig. 13.

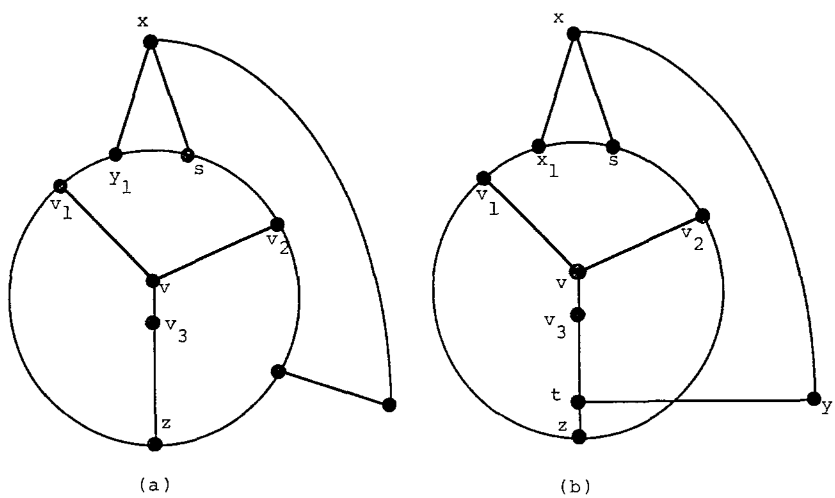


Fig. 14.

Let Q be the zt-subpath of P_2. Consider the cycle $C^* = t, y, P_{xy}, x, x_1, R, v_1, v, v_3, P_3, z, Q, t$. Now applying Lemma 6.5 to C_2 and P_v, we have that (C^*, v_2) is a starred wheel with v_1 and v_3 as the star nodes. By Theorem 5.3, v_2 must have a positive even number of neighbors in C^*. Consider the starred cycle $\tilde{C} = t, y, P_{xy}, x, x_1, R, v_1, P_1, z, Q, t$ with v_1 as the star node. Now $(\tilde{C}, v, v_2)$ is an expanded cycle. But since $N(v_2) \cap (C^* \setminus \tilde{C}) = v$, we have a contradiction to Lemma 5.1.

Case 2. y is strongly adjacent to $P_1 \cup P_2 \cup P_3$, see Figures 13 and 15 to 17. In $P_1 \cup P_2 \cup P_3$, let t and y_1 be the neighbors of y. There are four subcases. The arguments in all four subcases are identical to those in Case 1 except that in each subcase the starred wheel and the expanded cycle that are identified may be different.

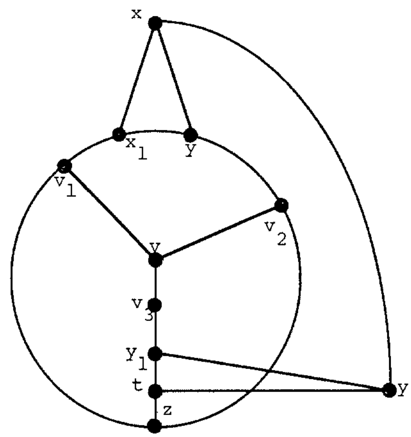


Fig. 15.

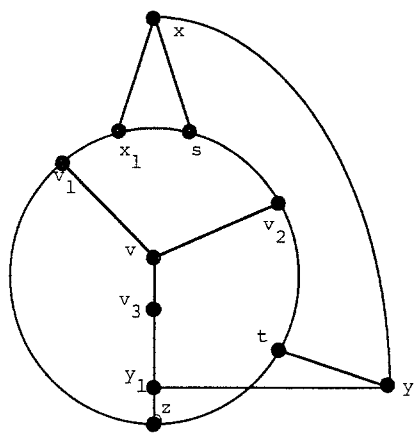


Fig. 16.

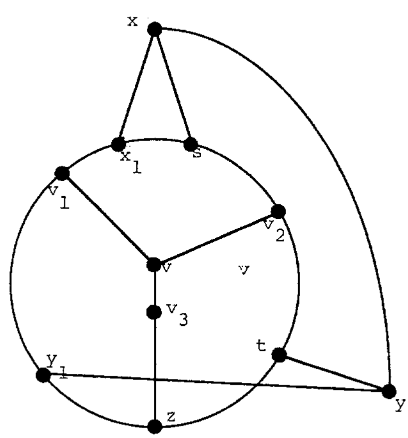


Fig. 17.

Case 2.1. Both $t, y_1 \in P_1$ or $t, y_1 \in P_2$.

Because of symmetry, we need to consider only the case $t, y_1 \in P_2$, see Figure 13. Without loss of generality, we assume that in P_2, y_1 is closer to v_2 than t. Now the starred wheel and the expanded cycle are identical to those in Case 1.

Case 2.2. Both $t, y_1 \in P_3$, see Figure 15.

Without loss of generality, we assume that in P_3, y_1 is closer to v_3 than t. Let Q_1 and Q_2 be the v_3y_1 and zt-subpaths of P_3. Now consider the cycle $C^* = y_1, y, P_{xy}, x, x_1, R, v_1, v, v_3, Q_1, y_1$. As in Case 1, it follows that (C^*, v_2) is a starred wheel and v_2 must have a positive even number of neighbors in C^*. Now $\tilde{C} = t, y, P_{xy}, x, x_1, R, v_1, P_1, z, Q_2, t$ is a starred cycle with v_1 as the star node. Then $(\tilde{C}, v, v_2)$ is an expanded cycle. But since $N(v_2) \cap (C^* \backslash \tilde{C}) = v$, we have a contradiction to Lemma 5.1.

Case 2.3. $t \in P_1 \cup P_2 \backslash \{z\}, y_1 \in P_3 \backslash \{z\}$, see Figure 16.

Without loss of generality, we assume that $t \in P_2 \backslash \{z\}$. Let Q_1 and Q_2 be the v_3y_1 and zy_1-subpaths of P_3. Now let $C^* = y_1, y, P_{xy}, x, x_1, R, v_1, v, v_3, Q_1, y_1$. Again (C^*, v_2) is a starred wheel and v_2 must have a positive even number of neighbors in C^*. If $y_1 \neq v_3$, let $\tilde{C} = y_1, y, P_{xy}, x, x_1, R, v_1, P_1, z, Q_2, y_1$. If $y_1 = v_3$, let Q_3 be the zt-subpath of P_2 and $\tilde{C} = t, y, P_{xy}, x, x_1, R, v_1, P_1, z, Q_3, t$. Note that if $y_1 = v_3$, then $t \notin N(v_2)$. In either case, $(\tilde{C}, v, v_2)$ is an expanded cycle and $N(v_2) \cap (C^* \backslash \tilde{C}) = v$. Hence we have a contradiction to Lemma 5.1.

Case 2.4. $t \in P_2 \backslash \{z\}$, $y_1 \in P_1 \backslash \{z\}$, see Figure 17.

Note that if $y_1 \in N(v_1)$ then $t \notin N(v_2)$. Conversely, if $t \in N(v_2)$, then $y_1 \notin N(v_1)$. Because of symmetry, we can assume without loss of generality that $t \notin N(v_1)$. Let Q_1 be the v_1y_1-subpath of P_1 and Q_2 be the zt-subpath of P_2. Now consider $C^* = t, y, P_{xy}, x, x_1, R, v_1, v, v_3, P_3, z, Q_2, t$. Again (C^*, v_2) is a starred wheel and v_2 must have an even number of neighbors in C^*. Let $\tilde{C} = y_1, y, P_{xy}, x, x_1, R, v_1, Q_1, y_1$. Now $(\tilde{C}, v, v_2)$ is an expanded cycle. Again since $N(v_2) \cap (C^* \backslash \tilde{C}) = v$ we have a contradiction to Lemma 5.1.

Thus in all cases we have a contradiction to Lemma 5.1. Hence the theorem follows. □

8. Wheels and star cutsets

The main result in this section is that if a linear balanced graph contains a starred wheel with at least 4 spokes, then the hub v of the wheel defines a star cutset $N(v)$.

Definition 8.1. Consider a linear graph G and a chordless wheel (C, v) with no node in $V \backslash N(v)$ strongly adjacent. Let $s, t \in C$. We define a chordless path P from s to t as a *minimal* path if it satisfies the following two conditions:

(i) No intermediate node of P belongs to $C \cup N(v)$ and

(ii) The set $N(P)$ of nodes in $C \backslash \{s, t\}$ adjacent to intermediate nodes of P is contained in $N(v)$.

8.1. Chordless wheel with at least 4 spokes

Let (v, v_i), $i=0,1,\dots,n-1$ be the n spokes of a chordless wheel (C, v). Let S_i, $i=1,2,\dots,n$ with end nodes v_{i-1} and v_i (where $v_n=v_0$) be the n sectors of the same wheel.

Lemma 8.2. *Let G be a linear balanced bipartite graph containing a chordless wheel (C, v) with at least four spokes and no node in $V\backslash N(v)$ strongly adjacent to C. Let $s, t\in C$. Suppose P is a minimal path from s to t with the additional condition: $|N(P)|\leq 1$. Then the following properties hold.*

(i) *If $s=v_k$ and $t=v_j$, $k\neq j$, then $k-j$ is odd. Moreover $N(P)=\emptyset$, see Figure* 18(a).

(ii) *If $s\in S_i\backslash N(v)$ and $t=v_j$ then $j=i$* or *$j=i-1$ and $N(P)\subset\{v_i, v_{i-1}\}$, see Figure* 18(b).

(iii) *If $s\in S_i\backslash N(v)$ and $t\in S_j\backslash N(v)$ then $i-j$ is even, see Figure* 18(c).

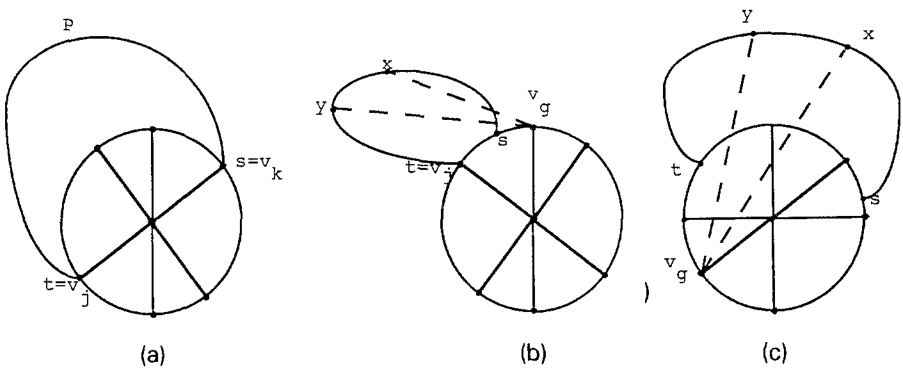


Fig. 18.

Proof. Let Q_1 and Q_2 be the two st-subpaths of C. Let $C_i=s, P, t, Q_i, s$ for $i=1,2$. Then for $i=1,2$ we have the following:

Claim 1. If $Q_i\cap N(P)=\emptyset$ then $|Q_i\cap N(v)|$ is even or equal to one.

Proof. If v is strongly adjacent to C_i, then (C_i, v) is a chordless wheel and the number of spokes must be even.

Claim 2. If $s=v_k$ and $t=v_j$, $k\neq j$, then $k-j$ is odd.

Proof. For $i=1,2$, we have that $|Q_i\cap N(v)|\geq 2$ since nodes $v_k, v_j\in Q_i$, $i=1,2$. But $|N(P)|\leq 1$ implies that $Q_i\cap N(P)=\emptyset$ for either $i=1$ or 2. Without any loss in generality, suppose $Q_1\cap N(P)=\emptyset$. Then (C_1, v) is a chordless wheel, hence we have that $|Q_1\cap N(v)|$ is even. This implies that $k-j$ is odd.

Claim 3. If $s=v_k$ and $t=v_j$, $k\neq j$ then $N(P)=\emptyset$.

Proof. Suppose $N(P)=\{v_g\}$. Let x, y be the neighbors of v_g in P such that the sx-subpath P_1 and yt-subpath P_2 of P are shortest. Note that x and y may coincide. Now applying Claim 2 with $j=g$ $(k=g)$ and $P=v_k, P_1, x, v_g$ $(P=v_g, y, P_2, v_j)$ we get that $k-g$ $(g-j)$ is odd. But this implies that $k-j$ is even which contradicts Claim 2.

Property (i) now follows from Claims 2 and 3.

Claim 4. If $s \in S_i \backslash N(v)$, $t = v_j$ and $N(P) = \emptyset$, then $v_j \in S_i$.

Proof. Since node v_j is the end node of both Q_1 and Q_2, we have that $|Q_i \cap N(v)|$ is odd for either $i = 1$ or 2. Without loss of generality, suppose $|Q_1 \cap N(v)|$ is odd. Then by Claim 1, we have that $|Q_1 \cap N(v)| = 1$ and the claim follows.

Claim 5. If $s \in S_i \backslash N(v)$, $t = v_j$ and $N(P) = \{v_g\}$, then $\{v_j, v_g\} \subseteq S_i$.

Proof. Let x, y be the neighbors of v_g in P such that the sx-subpath P_1 of P and yt-subpath P_2 of P are shortest. Now applying Claim 4 with $j = g$ and $P = s, P_1, x, v_g$ we get that $v_g \in S_i$. Hence $N(P) = \{v_g\} \subset \{v_i, v_{i-1}\}$. Without loss of generality, assume that $v_g = v_i$. We now prove that $j = i - 1$ thereby proving the claim. Now applying Claim 2 with $k = i$ and $P = v_i, y, P_2, v_j$ it follows that $i - j$ is odd. Let Q_1 be the sv_j-subpath of C not containing v_i. Then $|Q_1 \cap N(v)|$ is odd since $s \in S_i \backslash N(v)$ and $i - j$ is odd. But $Q_1 \cap N(P) = \emptyset$. Consequently by Claim 1, we have that $|Q_1 \cap N(v)| = 1$. This implies $j = i - 1$.

Claims 4 and 5 together prove property (ii) . We now prove property (iii): Suppose $s \in S_i \backslash N(v)$, $t \in S_j \backslash N(v)$ and $i - j$ is odd. Then for $i = 1, 2$, we have that $|Q_i \cap N(v)|$ is odd. Since (C, v) has at least 4 spokes, we assume, without loss of generality, that $|Q_1 \cap N(v)| \geqslant 3$. Clearly we must have $\emptyset \neq N(P) \subset Q_1 \cap N(v)$ for otherwise C_1 is a chordless wheel with an odd number of spokes. Now, let $v_g \in N(P)$. Since $v_g \in Q_1 \cap N(v)$ and $|Q_1 \cap N(v)| \geqslant 3$, v_g can not be an end node of both S_i and S_j. We assume that v_g is not an end node of S_i. Let x be the first neighbor of v_g along P going from s to t. Then let P_1 be the sx-subpath of P. Now applying Claim 4 with $s \in S_i \backslash N(v)$, $t = v_g$ and $P = s, P_1, x, v_g$ we get that $g \in \{i-1, i\}$ which is a contradiction. Consequently $i - j$ is even. Therefore property (iii) follows and the proof of the lemma is complete. □

We now prove the Star Cutset Theorem for chordless wheels with at least four spokes.

Theorem 8.3. *Let G be a linear balanced graph containing a chordless wheel (C, v) with at least four spokes. Let (v, v_i), $i = 0, 1, \ldots, n-1$, be the spokes of the wheel. For $i = 1, 2, \ldots, n$, let $d_{i1}, d_{i2} \in S_i$ be the two neighbors of v_{i-1} and v_i respectively. Let $D_1 = \{d_{i1}, d_{i2} \mid i \text{ is odd}\}$, $D_2 = \{d_{i1}, d_{i2} \mid i \text{ is even}\}$. Suppose S is the set of nodes that are strongly adjacent to C. If $S \subseteq N(v)$, then $N(v)$ is a star cutset of G, separating D_1 and D_2.*

Proof. Suppose the theorem is false. Then there must be a minimal path Q (see Definition 8.1) between $b \in S_i \backslash N(v)$ and $d \in S_j \backslash N(v)$ where i is odd and j is even, see Figure 19. Now $|N(Q)| \geqslant 2$ for otherwise Q would satisfy all the three conditions of Lemma 8.3 and since $i - j$ is odd, property (iii) of the same lemma would be violated. Next we show that $|N(Q)| < 3$. Suppose $|N(Q)| \geqslant 3$. Then it is easily verified that $N(Q)$ contains three nodes v_k, v_g and v_h such that the following condition holds:

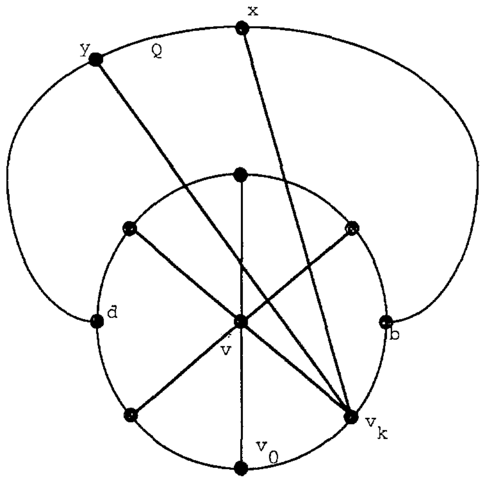


Fig. 19.

Q contains nodes $x \in N(v_k)$ and $y \in N(v_h)$ and the xy-subpath P_1 of Q is such that $P_1 \backslash \{x, y\}$ contains one or more neighbors of v_g but does not contain any other neighbor of a node in $N(v) \cap C \backslash \{v_g\}$.

But now applying Lemma 8.2 with $s = v_k$, $t = v_h$, $P = v_k, x, P_1, y, v_h$ and noting that $|N(P)| = 1$, we get a contradiction to property (i) of Lemma 8.2 which states that $N(P) = \emptyset$. Consequently $|N(Q)| = 2$.

Now let $N(Q) = \{v_k, v_g\}$. Let x (y) be the first (last) neighbor of v_k in Q going from b to d. Let P_1 (P_2) be the bx-subpath (dy-subpath) of Q. Then applying Lemma 8.2 with $s = b \in S_i \backslash N(v)$, $t = v_k$ and $P = b, P_1, x, v_k$, property (ii) implies that $k \in \{i-1, i\}$. But then applying again Lemma 8.2 with $s = d \in S_j \backslash N(v)$, $t = v_k$ and $P = d, P_2, y, v_k$, property (ii) implies that $k \in \{j-1, j\}$. Consequently $k = \{i-1, i\} \cap \{j-1, j\}$. A similar argument shows that $g = \{i-1, i\} \cap \{j-1, j\}$. But since (C, v) has at least four rays it follows that $k = g$. But this leads to a contradiction thereby proving the theorem. □

Theorem 8.4. *Let G be a linear balanced graph containing a chordless wheel (C, v) with at least four spokes. Suppose (u, v) is one of the spokes and nodes x and y are adjacent to u in C, see Figure 20. Let T be the set of nodes that are strongly adjacent to C. Suppose each $h \in T \backslash N(v)$ satisfies the following condition*:

(i) *Node h has exactly two neighbors in C with one of them being either x or y. In C, suppose h_1 is the other neighbor of h. Then every neighbor of h, not including h_1, x or y, does not have any neighbor in C.*

Then $N(v)$ is a star cutset of G, separating x and y.

Proof. If $T \backslash N(v) = \emptyset$, the result follows from Theorem 8.3. Therefore, we assume that $T \backslash N(v) \neq \emptyset$. Let $h \in T \backslash N(v)$. Note that two nodes $h, g \in N(x) \cap T$ can not have a common neighbor other than x. If $g \in N(x) \cap T$ and $h \in N(y) \cap T$, then h

and g do not have a common neighbor. Let S_i, $i=1,2,\ldots,n$, be the sectors of the chordless wheel (C, v) with $x \in S_1$ $(y \in S_n)$ having end nodes u and v_1 (u and v_{n-1}), see Figure 20.

Claim 1. If $h \in N(y) \cap T$ $(h \in N(x) \cap T)$ let h_1 the other neighbor of h in C. Then the index i of sector S_i containing h_1 is even (odd).

Proof. Suppose the claim is false and $h_1 \in S_i$, i odd. Let v_{i-1} and v_i be the end nodes of S_i. Let Q be the yh_1-subpath of C not containing u and $\hat{C} = y, Q, h_1, h, y$. Now $(\hat{C}, u, v)$ is an expanded cycle with y as the star node. But node v has an odd number of neighbors in $\hat{C}$, contradicting Lemma 5.1. Hence the claim follows.

There are two cases to consider depending upon whether both $g \in N(x) \cap T$ and $h \in N(y) \cap T$ exist or only one of them exists. We consider the case in which both g and h exist. The other case is similar and hence omitted.

Let $g \in N(x) \cap T$ and $h \in N(y) \cap T$ be chosen such that the following holds: Node g has a neighbor $g_1 \in C$, $g_1 \neq x$, and h has a neighbor $h_1 \in C$, $h_1 \neq y$, such that the g_1h_1-subpath H of C not containing u, does not contain an intermediate node which is a neighbor of a node in $T \backslash N(v)$, see Figure 20(a). If $N(x) \cap T = \emptyset$ $(N(y) \cap T = \emptyset)$, then $g = x$ $(h = y)$. Let $C^* = u, x, g, g_1, H, h_1, h, y, u$. Then (C^*, v)

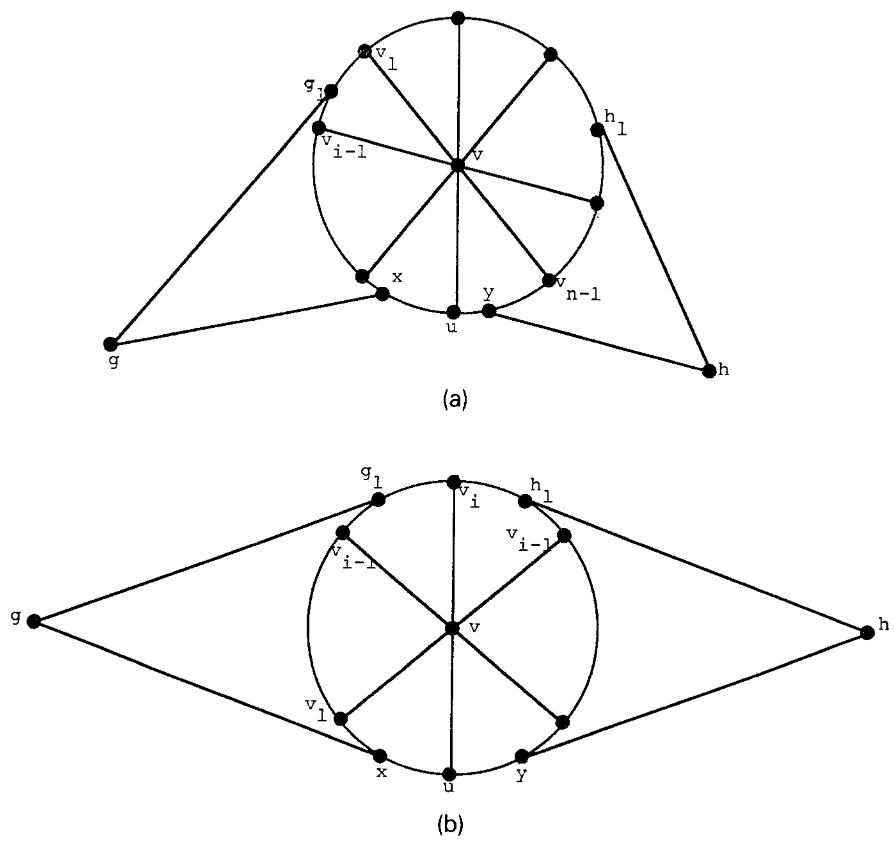


Fig. 20.

is a chordless wheel with all strongly adjacent nodes contained in $N(v)$. Note that (C^*, v) has at least two spokes with (u, v) being one of them. There are two cases to consider.

Case 1. (C^*, v) has at least four spokes, see Figure 20(a).

Let S be the set of nodes that are strongly adjacent to C^*. Since $S \subseteq N(v)$, by Theorem 8.3, $N(v)$ is a star cutset of G, separating x and y.

Case 2. (C^*, v) has two spokes, see Figure 20(b). We further divide this case into two sub-cases.

Case 2.1. Edge (v_i, v), $i \neq 1$ or $n-1$, is a spoke of (C^*, v). Let v_i and v_{i+1} be the end nodes of the sector S_{i+1}. Without loss of generality, assume that $g_1 \in S_i$ and $h_1 \in S_{i+1}$. Let Q be the $v_{i-1}g_1$-subpath of S_i. First we note that Q is of length 3 mod 4 for otherwise the cycle $u, v, v_{i-1}, Q, g_1, g, x, u$ would be an odd hole. Now we want to show that (C^*, v) together with a node v_3 and a path P_3 defines a parachute that satisfies Definition 7.1.

Let R_1 be the g_1v_i-subpath of S_i and R_2 be the v_ih_1-subpath of S_{i+1}. Suppose $P_0 = u, y, h, h_1, R_2, v_i$; $P_1 = g_1, g, x, u$; $P_2 = v_i, R_1, g_1$ and $P_3 = v, v_{i-1}, Q, g_1$. Clearly, $C^* = u, P_0, v_i, P_2, g_1, P_1, u$. Consequently (C^*, v, v_{i-1}, P_3) with the corresponding paths P_0, P_1 and P_2 is a parachute. It is easily verified that a node $x \notin N(v)$ but which is strongly adjacent to $C^* \cup P_3$ has exactly two neighbors, one of them being $x \in P_1$ and the other neighbor in P_3. Now by Theorem 7.2, $N(v)$ is a star cut set of G, separating x and y.

Case 2.2. Edge (v_i, v), $i = 1$ or $n-1$, is a spoke of (C^*, v). Because of symmetry, we need to consider only the case with $i = n-1$. The proof however, is identical to that of Case 2.1 with i replaced by $n-1$.

This completes the proof of the theorem. □

8.2. Starred wheel with at least 4 spokes

In this section, we consider a starred wheel (C, v) with at least four spokes and show that if the strongly adjacent nodes satisfy a given condition, the hub v defines a star cutset $N(v)$.

Theorem 8.5. *Let G be a linear balanced graph containing a starred wheel (C, v) with at least 4 spokes. Let a and b be the star nodes adjacent to node u where (v, u) is a spoke of the wheel. Let D be the set of nodes that are strongly adjacent to C. Suppose each $d \in D \backslash N(v)$ satisfies the following conditions*:

(i) *Suppose $d \notin N(u)$. Then d has exactly two neighbors in C with one of them being either node a or b. Furthermore, every neighbor of d, say node $w \notin C$ does not have any neighbor in C.*

(ii) *Suppose $d \in N(u) \backslash \{v\}$ and a neighbor of d, say $r \notin C$, has a neighbor, r_1, in C. Then every neighbor of r, other than d and r_1, does not have a neighbor in C.* (*Note that by* (i) *above every neighbor of d not in C has at most one neighbor in C.*)

Then $N(v)$ is a star cutset of G, separating a and b.

Proof. If node $d \in N(u) \cap D$, by property (iii) of Theorem 5.3, d has neighbors in only odd numbered or in only even numbered sectors but not both. Hence $N(u)\backslash\{v\}$ can be partitioned into three disjoint sets as follows:

$T_o = \{x \mid x \in N(u)\backslash\{v\}$ and x has a neighbor in an odd numbered sector of $(C, v)\}$,

$T_e = \{x \mid x \in N(u)\backslash\{v\}$ and x has a neighbor in an even numbered sector of $(C, v)\}$,

$T_n = \{x \mid x \in N(u)\backslash\{v\}$ and x has no neighbors in $C\}$.

Note that $a \in T_o$ and $b \in T_e$. Hence T_o and T_e are non-empty. We now construct a chordless wheel (C^*, v) from the starred wheel (C, v). Let $x \in T_o$ and $y \in T_e$ be chosen such that the following holds:

Node x has a neighbor $g \in C$ and y has a neighbor $h \in C$ such that the gh-subpath H of C not containing u does not contain an intermediate node which is a neighbor of a node in $T_o \cup T_e$, see Figure 21.

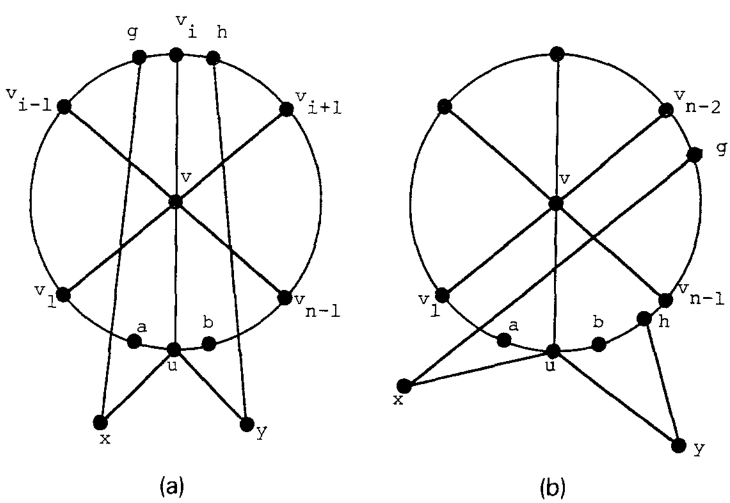


Fig. 21.

Note that such a choice of x and y is always possible. Let $C^* = u, x, g, H, h, y, u$. Then (C^*, v) is a chordless wheel. Moreover, if $x \neq a$ $(y \neq b)$, by property (iii) of Theorem 5.3, there exists a path from x to a (y to b) not containing a node in $N(v)$.

We now have two cases to consider.

Case 1. (C^*, v) has two spokes.

We further divide this case into two sub-cases.

Case 1.1. Edge (v_i, v), $i \neq 1$ or $n-1$, is a spoke of (C^*, v), see Figure 21(a).

Let S_i be the sector with end nodes v_{i-1} and v_i. Let $g \in S_i$ and $h \in S_{i+1}$. Note that x has a positive even number of neighbors in S_i. Let Q_i be the v_ih-subpath of S_{i+1}. The cycle $\hat{C} = u, v, v_{i-1}, S_i, v_i, Q_i, h, y, u$ has (v, v_i) as the unique chord, see

Figure 21(a). Let $w \in S_i$ and $z \in S_{i+1}$ be the neighbors of v_i in $\hat{C}$. Now x is a type 3 node with respect to $\hat{C}$. Consequently by Theorem 6.6, $N(v)$ is a star cutset of G, separating w and z. Now since x has a neighbor $g \in S_i$ and y has a neighbor $h \in S_i$, it follows that $N(v)$ is a star cut set of G, separating x and y. But there exists a path connecting x to a (y to b) and not containing a node in $N(v)$. Hence $N(v)$ is a star cutset of G, separating a and b.

Case 1.2. Edge (v_i, v), $i=1$ or $n-1$, is a spoke of (C^*, v), see Figure 21(b).

Because of symmetry, we need to consider only the case with $i=n-1$, see Figure 21(b). Now let Q_{n-1} be the $v_{n-1}h$-subpath of S_n. Then $\hat{C} = u, v, v_{n-2}, S_{n-1}, v_{n-1}, Q_{n-1}, h, y, u$ has the unique chord (v, v_{n-1}). Let $w \in S_{n-1}$ and $z \in S_n$ be the neighbors of v_{n-1} in $\hat{C}$. Node x has a positive even number neighbors in S_{n-1} and is a type 3 node with respect to $\hat{C}$. Consequently by Theorem 6.6, $N(v)$ is a star cut set of G separating w and z. Now by arguments identical to those in Case 1.1, it follows that $N(v)$ is a star cutset of G, separating a and b. This completes the proof for Case 1.

Case 2. (C^*, v) has at least four spokes.

Without loss of generality, assume that in (C^*, v), $x \in S_1$ and $y \in S_n$. Now if $x=a$ ($y=b$) then $x \in C$ ($y \in C$) and consequently by condition (i) of the theorem, a neighbor of x (y) which is strongly adjacent to C^* has exactly two neighbors in C^* with one of them being x (y). If $x \neq a$ ($y \neq b$), by condition (ii) of the theorem, again a neighbor of x (y) can have at most two neighbors in C^*. In either case, it is easily verified that (C^*, v) is a chordless wheel satisfying condition (i) of Theorem 8.4. Hence $N(v)$ is a star cutset of G, separating x and y. Again since there exists a path from x to a (y to b) not containing a node in $N(v)$, it follows that $N(v)$ is a star cutset of G_1 separating a and b.

This completes the proof of the theorem. □

9. Star Cutset Theorem

We first give a definition and prove a lemma before proving the Star Cutset Theorem.

Definition 9.1. Let G be a linear balanced graph containing a node minimal odd cycle C with a unique chord (u, v). Suppose a and b (c and d) are the neighbors of u (v) in C. Let C_1 and C_2 be the two chordless cycles formed by (u, v). Let $a, c \in C_1$ and $b, d \in C_2$. Suppose there exists a minimal path P_u, as in Definition 6.3, from $g \in C_1 \backslash \{u, v, a\}$ to $h \in C_2 \backslash \{u, v, b\}$. From among the collection of all such minimal paths, let ρ^* be the set of paths that have the shortest length. Let $\rho_s^* \subseteq \rho^*$ be the set of paths from $s \in C_1 \backslash \{u, v, a\}$ such that node s is closest to u in the cu-subpath of C_1 not containing v. Let $P_{st} \in \rho_s^*$ be a minimal path to $t \in C_2 \backslash \{u, v, b\}$ where t is closest to u along tu-subpath of C not containing v. Let P_1 (P_2) be the as-subpath (bt-subpath) of C_1 (C_2), not containing u. Now define the cycle $C^* = u, a, P_1, s, P_{st}, t, P_2, b, u$, see Figure 22.

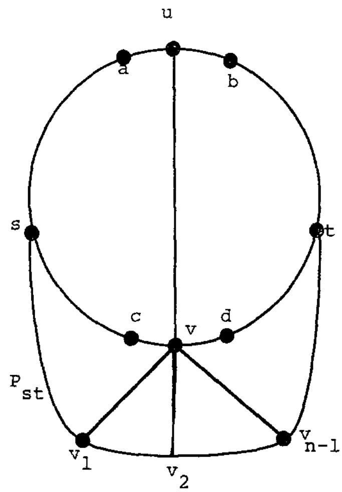


Fig. 22.

Lemma 9.2. *Let C^* be a cycle as in Definition* 9.1. *Let D be the set of nodes that are strongly adjacent to C^*. Suppose $f \in D$. Then the following two properties hold*:

(i) *Suppose $f \notin N(u)$. Then f has exactly two neighbors in C^* with one of them being either node a or b. Furthermore every neighbor of f, other than the two neighbors in C^*, does not have any neighbor in C^*.*

(ii) *Suppose $f \in N(u) \backslash \{v\}$. A neighbor of f, say $r \notin C^*$, has a neighbor, r_1, in C^*. Then every neighbor of r, other than f and r_1, does not have a neighbor in C^*.*

Proof. We first prove three claims which in turn prove property (i) of the lemma.

Claim 1. A node $w \notin C^* \cup N(u) \cup N(a) \cup N(b)$ has at most one neighbor in C^*.

Proof. Suppose a node $w \notin C^* \cup N(u) \cup N(a) \cup N(b)$ has two or more neighbors in C^*. First we show that $w \notin C \backslash C^*$. If $w \in C_1 \backslash C^*$ then it must have a neighbor, say $x \neq s$ in P_{st}. The sx-subpath, P_{sx}, of P_{st} has length ≥ 2 for otherwise x would be type 1 or type 2 or type 3 node. Let P_{xt} denote the xt-subpath of P_{st}. Then $P_{st} \notin \rho^*$ since $P_{wt} = w, x, P_{xt}, t$ is a shorter path than P_{st}. Hence $w \notin C_1 \backslash C^*$. Similarly $w \notin C_2 \backslash C^*$. Hence $w \notin C \backslash C^*$. Clearly w can not have two neighbors in $C \cap C^*$ or in P_{st} since C is node-minimal and P_{st} is a shortest path. Hence w must have one neighbor, say $x \in P_{st} \backslash \{s, t\}$ and one neighbor $y \in C \cap C^*$. Without loss of generality, assume that $y \in C_1 \cap C^*$. Note that $y \neq u, a, b$. Now the sx-subpath, P_{sx}, of P_{st} must be of length 1 or 2 for otherwise $P_{yt} = y, w, x, P_{xt}, t$ would be shorter than P_{st}. If P_{sx} is of length 2, both P_{st} and P_{wt} have the same length, contradicting the choice of P_{st} such that s is closest to u in the cu-subpath of C_1, not containing v. Consequently s must be a neighbor of x. This implies that s, x, w, y is a path of length 3 between s and $y \in C_1$. Note that since C is node-minimal, x can not have any other neighbor in C. But then one of the two sy-subpaths of C_1 would close an odd hole. Thus the claim follows.

Claim 2. If $f \in D \backslash N(u)$ then $f \in N(a) \cup N(b)$. Moreover node f has exactly two neighbors in C^*, one of them being a or b and the other one in $P_{st} \backslash \{s, t\}$.

Proof. Since $f \in D \backslash N(u)$, by Claim 1, $f \in N(a) \cup N(b)$. Clearly f can not have two neighbors in $C \cap C^*$ or in P_{st}. Hence it must have exactly one neighbor in $P_{st} \backslash \{s, t\}$.

Claim 3. Suppose $f \in D \backslash N(u)$. Then a neighbor of f, say node $w \notin C^*$ not have any neighbor in C^*, see Figure 23(a).

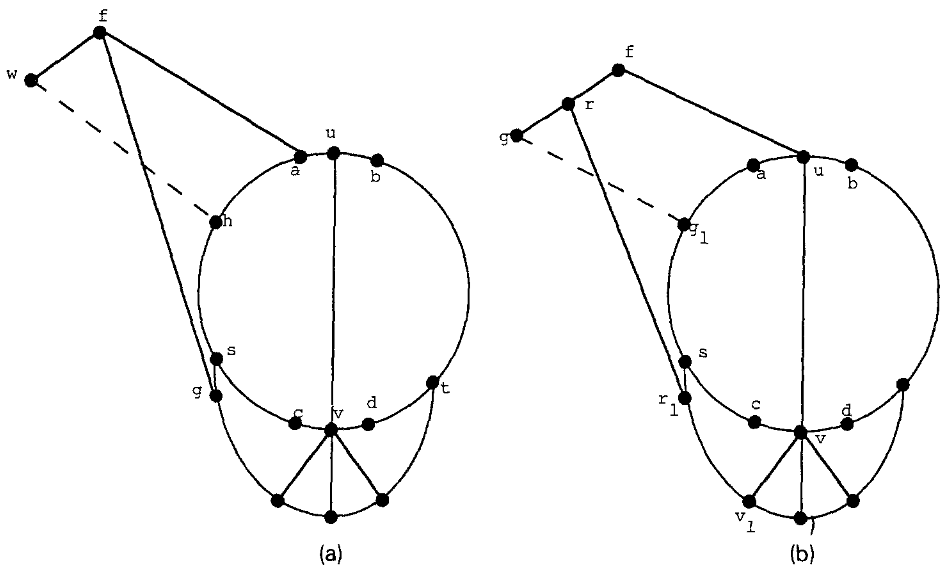


Fig. 23.

Proof. By Claim 2, $f \in N(a) \cap N(b)$ and has exactly one neighbor, say g, in $P_{st} \backslash \{s, t\}$. Without loss of generality, assume that $f \in N(a)$. Consider now a neighbor of f, say node $w \notin C^*$. Suppose w has a neighbor, say h, in C^*. Note that $h \neq a, b, u$. Now there are 3 cases to consider:

Case 1. $h \in C_1 \cap C^*$.

The path a, f, w, h is of length 3. Hence the ha-subpath of C_1, not containing u, must be of length 1 mod 4. Let P_{hv} be the hv-subpath of C_1, not containing u. Then $u, v, P_{hv}, h, w, f, a, u$ is an odd hole.

Case 2. $h \in P_{st} \backslash \{s, t\}$.

The path g, f, w, h is of length 3 and is between $g, h \in P_{st} \backslash \{s, t\}$. This implies that the length of the hg-subpath of P_{st} must be at least 5 which contradicts the choice of P_{st} such that it is the shortest path.

Case 3. $h \in C_2 \cap C^*$.

Since the path g, f, w, h is of length 3, the choice of P_{st} implies that the gt-subpath, P_{gt}, of P_{st} must be of length 1 or 2. Now we show that P_{gt} does not contain a neighbor of v. Assume the contrary. The path $P^* = v, u, a, f, g$ is of length 4. If P_{gt} is of length 1, then t must be a neighbor of v, but the hole v, P^*, g, t, v is of length 6. If P_{gt} is of length 2, then let q be the intermediate node of P_{gt}. Then q is a neighbor of v and the hole v, P^*, g, q, v has length 6. Hence P_{gt} does not contain a neighbor of v. By Lemma 6.5, (C^*, v) is a starred wheel, and by Theorem 5.3, v has an odd number of neighbors in P_{st}. This implies that f has exactly two neighbors a and g in C^* with a in an odd numbered sector and g in a even numbered sector. Let P_{ag} be the ag-subpath of C^* not containing u. Then $\hat{C} = a, P_{ag}, g, f, a$ is starred cycle with a as the star node. This implies that $(\hat{C}, u, v)$ is an expanded cycle and $\hat{C}$ has an odd number of neighbors of v, contradicting Lemma 5.1.

Thus in all cases it follows that w can not have a neighbor in C^* and the claim follows.

Claims 2 and 3 together prove property (i).

We now prove property (ii). Let $f \in D \cap N(u) \setminus \{v\}$. Let $r \notin C^*$ be a neighbor of f. Since $r \notin N(u) \cup N(a) \cup N(b)$ by Claim 1, r has at most one other neighbor in C^*. Suppose r has a neighbor $r_1 \in C^*$. Note that $r_1 \neq a, b, c, d, u$. Now $r_1 \notin C \cap C^*$ for otherwise the path u, f, r, r_1 has length 3 and is from u to r_1, which are both in C_1 or both in C_2. But that clearly implies the existence of an odd hole. Hence $r_1 \in P_{st} \setminus \{s, t\}$. Consider now a neighbor of r, say $g \notin C^*$, see Figure 23(b). Now $g \notin N(u) \cup N(a) \cup N(b)$. By Claim 1, g can have at most one neighbor in C^*. Suppose g has a neighbor, say $g_1 \in C^*$. The path r_1, r, g, g_1 is of length 3. Now there are two cases to consider:

Case 4. $g_1 \in P_{st}$.

The $r_1 g_1$-subpath, Q, of P_{st} must be of length at least 5. But then the path r_1, r, g, g_1 is shorter than Q and contradicts the choice of P_{st} such that it is the shortest.

Case 5. $g_1 \in C_1 \cap C^*$ or $g_1 \in C_2 \cap C^*$, see Figure 23(b).

Becuase of symmetry, we can assume that $g_1 \in C_1 \cap C^*$. Now, since r_1, r, g, g_1 is a path of length 3, the choice of P_{st} implies that the $r_1 s$-subpath, Q, of P_{st} must be of length 1 or 2. Then noting that u, f, r, r_1 is a path of length 3, it follows that Q can not contain a neighbor of v or a neighbor of node a. Let v_1 be the neighbor v closest to r_1 in P_{st}. Let R_1 be the $v_1 r_1$-subpath of P_{st} and let R_2 be the $r_1 a$-subpath of C^*, not containing u. If R_1 does not contain a neighbor of a, define $\hat{C} = v_1, R_1, r_1, R_2, a, u, v, v_1$. If R_1 contains a neighbor of node a, let $a_1 \in R_1$ be the neighbor of node a such that the $a_1 r_1$-subpath, Q_1, of R_1, does not contain any other neighbor of node a. Now define $\hat{C} = a_1, Q_1, r_1, R_2, a, a_1$. In both instances the path r_1, r, g, g_1 which is of length 3 closes an odd hole with one of the $r_1 g_1$-subpaths of $\hat{C}$. Hence g can not contain a neighbor in C^* and part (ii) of the lemma is proven. □

Theorem 9.3. *Let G be a linear balanced graph containing an odd cycle $\hat{C}$ with a unique chord (u, v). Let $\hat{a}$ and $\hat{b}$ ($\hat{c}$ and $\hat{d}$) be the neighbors of u (v) in $\hat{C}$. Then either*

(i) *$N(u)$ is a star cut set of G, separating $\hat{c}$ and $\hat{d}$ or*

(ii) *$N(v)$ is a star cutset of G, separating $\hat{a}$ and $\hat{b}$.*

Proof. By Lemma 6.7, we have one of the following:

(i) $N(u)$ is a star cutset of G, separating $\hat{c}$ and $\hat{d}$.

(ii) $N(v)$ is a star cutset of G, separating $\hat{a}$ and $\hat{b}$.

(iii) There exists a node-minimal odd cycle, say C, with (u, v) as the unique chord. Let a and b (c and d) be the neighbors of u (v) in C. Then there exists a path form a to $\hat{a}$ and b to $\hat{b}$ (c to $\hat{c}$ and d to $\hat{d}$), not containing a node in $N(v)$ ($N(u)$).

If (i) or (ii) holds, there is nothing to prove. Suppose (iii) holds. By Lemma 3.2, no node is strongly adjacent to C. Suppose $N(u)$ is not a star cutset of G separating c and d. Then we want to show that $N(v)$ is a star cut set of G separating a and b. Let C_1 and C_2 be the two chordless cycles formed by (u, v). Let $a, c \in C_1$ and $b, d \in C_2$. Since $N(u)$ is not a star cutset separating c and d, there exists a minimal path, P_u, as in Definition 9.1, from $g \in C_1 \backslash \{u, v, a\}$ to $h \in C_2 \backslash \{u, v, b\}$. Let the path P_{st} and the cycle C^* be as in Definition 9.1. By Lemma 6.5, (C^*, v) is a starred wheel. There are two cases to consider.

Case 1. The starred wheel (C^*, v) has four or more spokes.

By properties (i) and (ii) of Lemma 9.2, it follows that the two conditions of Theorem 8.5 are satisfied. Hence by Theorem 8.5, $N(v)$ is a star cut set of G, separating a and b. Now since (iii) holds, the theorem follows.

Case 2. The starred wheel (C^*, v) has only two spokes, see Figure 24.

Let (v, u) and (v, v_1) be the two spokes and S_1 and S_2 be the two sectors of (C^*, v) with $a \in S_1$, $b \in S_2$. Since v has only one neighbor in P_{st}, we assume, without any loss in generality, that $s \neq c$.

Now we have the following two claims:

Claim 1. If $w \in N(a)$ ($w \in N(b)$) is strongly adjacent to C^*, then w has exactly two neighbors in C^*, one of them being node a (b) and the other neighbor is in $S_1 \cap P_{st} \backslash \{s, v_1\}$ ($S_2 \cap P_{st} \backslash \{t, v_1\}$).

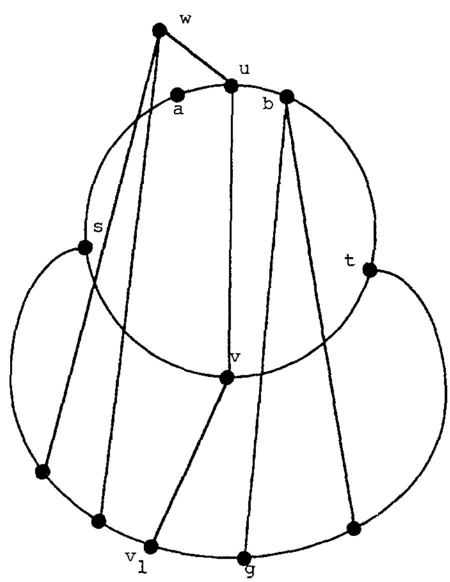


Fig. 24.

Proof. By Lemma 9.2, w has exactly two neighbors, one of them being node a. By arguments identical to those in the proof of Claim 1 which is a part of the proof of Theorem 8.4, it follows that in C^*, the other neighbor of w must be in S_1. Now noting that w can not be strongly adjacent to C_1, the claim follows.

Claim 2. If a node $w \in N(u)\backslash\{v\}$ has neighbors in $P_{st}\backslash\{s, t\}$ then $N(v)$ is a star cutset of G, separating a and b.

Proof. Without loss of generality assume that $w \in N(u)\backslash\{v\}$ has neighbors in S_1. Note that w may coincide with a and v_1 may coincide with d but not c. By property (iii) of Theorem 5.3, w can not have any neighbors in S_2. Let P_2 denote the shortest path from b to v_1 containing only nodes in $C_2\backslash\{u, v\} \cup P_{st}$. Let Q_s be the sv-subpath of C_1, not containing u and T_s be the sv_1-subpath of P_{st}. Then the cycle $\hat{C} = u, b, P_2, v_1, T_s, s, Q_s, v, u$ has only one chord (v, v_1). But then w is a type 3 node with respect to $\hat{C}$. Let $h_1 \in S_1$ and $h_2 \in S_2$ be the neighbors of v_1 in C^*. Then by Theorem 6.6, $N(v)$ is a star cutset of G separating h_1 and h_2. Note that $t = d$ implies that $v_1 = t$ and $h_2 \in C_2$. But from h_1 to a and from h_2 to b there exists a path, not containing a node in $N(v)$. Hence the claim follows.

If a node $w \in N(u)\backslash\{v\}$ has neighbors in $P_{st}\backslash\{s, t\}$, since (iii) holds by Claim 2, the theorem follows. Suppose now every node $w \in N(u)\backslash\{v\}$ has no neighbors in $P_{st}\backslash\{s, t\}$. Then C^* is chordless and a node $w \notin C^*$ but $w \in N(u)\backslash\{v\}$ is not strongly adjacent to C^*. Next we want to show the existence of a parachute that satisfies Definition 7.1.

Let P_0 be the path from u to v_1 containing only nodes in $(C_2\backslash\{v\}) \cup P_{st}$; P_1 be the us-subpath of C_1, not containing v; P_2 be the v_1s-subpath of P_{st} and P_3 be the sc-subpath of C_1, not containing u. Let $C^* = v_1, P_0, u, P_1, s, P_2, v_1$. Note that $s \neq a, c$. Furthermore v_1 is not a neighbor of s, since C is a node-minimal odd cycle. Hence it follows that both P_1 and P_2 have length greater than 1. Thus (C^*, v, c, P_3) is a parachute satisfying Definition 7.1. Now we prove the following claim.

Claim 3. Either $N(v)$ is a star cutset of G separating a and b or every node $w \notin N(v)$ that is strongly adjacent to $C^* \cup P_3$ has exactly two neighbors, in $C^* \cup P_3$, both of which are in P_0 or in $P_1 \cup P_2 \cup P_3$.

Proof. Suppose $N(v)$ is not a star cutset of G separating a and b. Then by Lemma 9.2 and Claim 2, only nodes in $N(a) \cup N(b) \cup \{v\}$ may be strongly adjacent to the chordless cycle C^*. Now there can not be a node w that is strongly adjacent to P_1 or to P_3 or to $P_1 \cup P_3$ for otherwise it would imply the existence of a node strongly adjacent to C_1. Since every node $w \in N(u)\backslash\{v\}$ has no neighbors in $P_{st}\backslash\{s, t\}$, the choice of P_{st} implies that there can not be a node strongly adjacent to P_2.

By Claim 1, a node $w \in N(b)$ that is strongly adjacent to C^* must have a neighbor in the tv_1-subpath of P_{st}. Hence w is strongly adjacent to P_0 and has exactly two neighbors in C^*. Similarly, a node $w \in N(a)$ that is strongly adjacent to C^* is strongly adjacent to $P_1 \cup P_2$ and has exactly two neighbors in C^*. There may be a node $w \notin (C^* \cup P_3)$ which is strongly adjacent to $P_2 \cup P_3$. But such a node w must have exactly one neighbor in P_3, say g, and one neighbor in P_2, say x, which is not a neighbor of s but both x and s have a common neighbor, say h, in P_2, see Figure

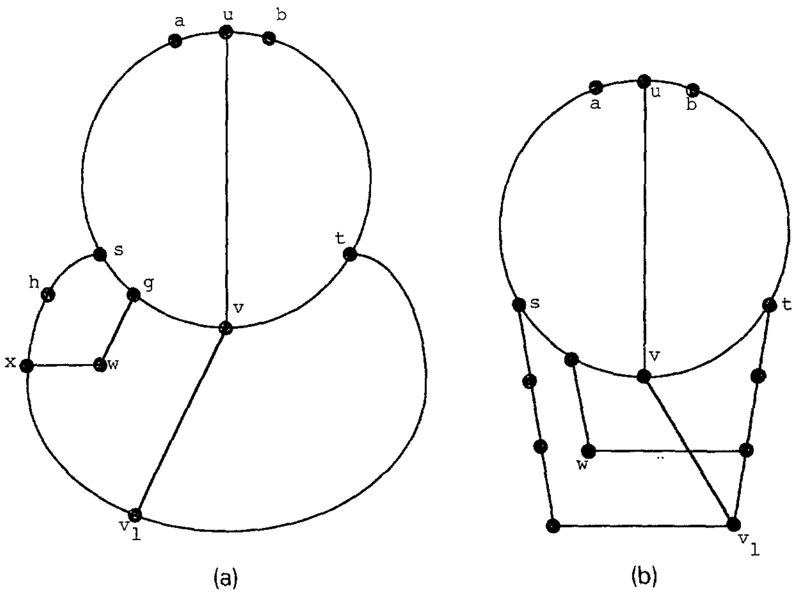


Fig. 25.

25(a). Note that $x \neq v_1$ for otherwise it would imply a path of length 3 between node v and node g in C_1.

We now show that every node $w \notin C^* \cup P_3 \cup \{v\}$ can not have neighbors in both P_0 and in $P_1 \cup P_2 \cup P_3 \backslash \{u, v_1\}$. By Lemma 9.2 and Claims 1 and 2, it follows that there cannot be a mode w which is strongly adjacent to C^* and has one neighbor in P_0 and the other in $P_1 \backslash \{u\}$ or in $P_2 \backslash \{v_1\}$. Now if $t = v_1$, node minimality of C ensures that every node $w \notin C^* \cup P_3 \cup \{v\}$ cannot have neighbors in both P_0 and P_3. If $t \neq v_1$, the vs and v_1t-subpaths of P_{st} must have length at least 3. Now the choice of P_{st} ensures that every node $w \notin C^* \cup P_3 \cup \{v\}$ can not have neighbors in both P_0 and P_3, see Figure 25(b). Hence the claim follows.

By Claims 2 and 3, either $N(v)$ is a star cutset of G separating a and b or there exists a parachute (C^*, v, c, P_3) satisfying Definition 7.1 and every node $w \notin N(v)$ that is strongly adjacent to $C^* \cup P_3$ has exactly two neighbors in $C^* \cup P_3$ both of which are in P_0 or in $P_1 \cup P_2 \cup P_3$. In the latter case, by Theorem 7.2, $N(v)$ is a star cutset of G separating a and b. Now since (iii) holds, the theorem follows.

This completes the proof of the theorem. □

10. Concluding remarks and a conjecture

The main result is simple to state and so it is natural to look for a more elementary and elegant proof. We believe that results along these lines might shed some light on the structure of the class of balanced matrices. Furthermore, in view of the nature of our result, an obvious question is whether this leads to a polynomial algorithm to test membership of a matrix in the class of linear balanced matrices. Such an algorithm is described in Conforti and Rao (1988). Finally, we propose the following conjecture:

Conjecture. Let A be a balanced matrix. Then A contains an entry $a_{ij} = 1$ such that the matrix obtained from A by setting a_{ij} to zero is still balanced.

Partial evidence for the above conjecture is the fact that a similar result holds for totally balanced and strongly balanced matrices. Totally balanced bipartite graphs have the property that every cycle of length greater than 4 has at least one chord. It is well known, see Anstee and Farber (1984), that if G is totally balanced, then it contains an edge (u, v) such that the subgraph induced by the nodes in $N(u) \cup N(v)$ is a complete bipartite graph. Hence (u, v) cannot be the only chord of a cycle of length greater than 4 containing nodes u and v. Therefore, the graph G' obtained from G by removing edge (u, v) is still totally balanced or equivalently the corresponding matrix obtained by turning the entry a_{uv} to zero is still totally balanced.

We recall that a graph is strongly balanced if every odd cycle of the graph contains at least two chords. Conforti and Rao (1987a) have shown the following:

If G is strongly balanced and contains an odd cycle, then G contains a complete bipartite subgraph K_{BD} such that the removal of the edges of K_{BD} disconnects the graph. This implies that we can sequentially remove the edges not contained in any complete bipartite articulation and still retain the strongly balanced property. When all such edges have been removed, the resulting graph is the union of disjoint complete bipartite graphs and hence is totally balanced. Then proceeding as in the case of totally balanced matrices, the matrix is reduced to the null matrix.

The conjecture is true for the class of linear balanced graphs if one could prove the following statement:

If G is linear, balanced and contains an odd cycle, then G contains a star cutset v such that the subgraph induced by the nodes of one connected component of $G - N(v)$ together with the nodes of $N(v)$ is without odd cycles.

Acknowledgement

We thank Klaus Truemper and Bert Gerards for their help and advise in the presentation and shortening of some of the proofs. We are also indebted to Gerard Cornuejols for careful reading of the manuscript and several suggestions.

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