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MIXED-INTEGER QUADRATIC PROGRAMMING*

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This paper considers mixed-integer quadratic programs in which the objective function is quadratic in the integer and in the continuous variables, and the constraints are linear in the variables of both types. The generalized Benders' decomposition is a suitable approach for solving such programs. However, the program does not become more tractable if this method is used, since Benders' cuts are quadratic in the integer variables. A new equivalent formulation that renders the program tractable is developed, under which the dual objective function is linear in the integer variables and the dual constraint set is independent of these variables. Benders' cuts that are derived from the new formulation are linear in the integer variables, and the original problem is decomposed into a series of integer linear master problems and standard quadratic subproblems. The new formulation does not introduce new primary variables or new constraints into the computational steps of the decomposition algorithm.

Key Words: Quadratic Programming, Mixed-Integer Quadratic Programming, Quadratic Duality Theory, Generalized Benders Decomposition, Integer Linear Programs, Generalized Inverses.

1. Introduction

This paper considers general mixed-integer quadratic programs, in which the objective function is quadratic in the integer-constrained variables, as well as in the continuous ones, the constraints are linear in the variables of both types, and the discrete variables can assume non-negative integer values (not only 0 or 1). An equivalent formulation that renders the original program more tractable is developed, and Benders' decomposition method [6] which has been generalized by Geoffrion [12] is employed to solve it. As it is shown by Lasdon [22], Benders' decomposition method and the well-known Dantzig–Wolfe decomposition method are dual pairs. The decomposition method is based on duality theory of quadratic programming, as developed by Cottle [7], Dennis [8] and Dorn [9].

A severe difficulty is encountered if the generalized Benders' method is used to solve the mixed-integer quadratic problem in *the original xy-space*, that is, without first transforming it into a simpler equivalent program. (x and y, respectively, are the vector of integer variables and the vector of continuous

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variables.) Since the dual objective function is quadratic in x, Benders' cuts are also quadratic in x, and the resulting master problem has constraints that are *quadratic in the integer variables* x. This is a difficult and untractable discrete optimization program.

The method suggested in this paper to overcome this difficulty is based on the development of a new equivalent formulation of the original program, under which the integer variables are absent from the objective function, and the constraint set is linear in the integer variables. Thus, the dual objective function is *linear*—rather than *quadratic*—in the integer variables, and the dual constraint set is independent of the integer variables. As a result, Benders' cuts that are derived from the new formulation upon implementing the generalized Benders' method are *linear in the integer variables* x. The original mixed-integer quadratic problem is decomposed into a series of relaxed master problems which are *integer linear programs*, and subproblems which are standard quadratic programs. The concept of *generalized inverse* is employed for developing the new formulation. Furthermore, the equivalence between the original and the new formulation holds in the almost general case.

Balas [2] developed a partitioning algorithm for solving all-integer and mixed-integer quadratic programs, based on his generalization of the dual symmetric quadratic programs studied by Cottle [7]; his algorithm introduces a vector of new variables. All-integer quadratic programs (especially 0–1 quadratic programs) have received a considerable amount of attention in the literature. *Linearization* is a widely used approach for solving problems of this type: by adding new variables and constraints, the original problem is transformed into an equivalent integer linear program. Watters' [27] algorithm for 0–1 quadratic programs replaces the cross-product terms in the objective function by introducing $n(n-1)/2$ additional 0–1 variables and $n(n-1)$ additional auxiliary constraints (n is the number of original 0–1 variables). Other linearization schemes that yield substantially more compact formulations have been suggested; for instance, Glover's [18] formulation of 0–1 quadratic programs requires the addition of n new *continuous* variables and $4n$ new constraints (see also [16, 17]). Bazaraa and Sherali [3] have recently suggested a new formulation of the quadratic assignment problem, which is a 0–1 quadratic program with a highly specialized constraint set. By introducing new continuous variables and linear constraints, they replace the cross-product terms in the objective function, and transform the original problem into a mixed-integer linear program; they then employ Benders' method to solve this program. The linearization process of all-integer quadratic programs in which the discrete variables can assume non-negative integer values (not only 0 or 1) requires the addition of a much larger number of new variables and constraints, since each integer variable is first expressed as a linear combination of several new 0–1 variables. Glover [18] demonstrated that the same approach can be used to reformulate mixed-integer quadratic programs. The transformation of a program with the objective function $f(x, y) = x'Qy$ ($x \in \mathbf{R}^n$ is a vector of integer variables, and $y \in \mathbf{R}^m$ is a vector of continuous variables) into a mixed-integer linear program requires a total of nr new continuous variables and $4nr$ new constraints, *after* the initial replacement of each x_i by a linear combination involving r new 0–1 variables.

The major shortcoming of linearization methods is that the linear relaxation is achieved at the expense of a substantial increase in problem size. Also, McBride and Yormark [25] report that computational experience with such formulations indicates slow convergence. Another approach used to solve 0–1 quadratic programs is implicit enumeration (e.g., [24, 20, 25]). Kunzi and Oettli [21] developed an interesting new procedure for solving all-integer quadratic programs.

The new equivalent formulation of the mixed-integer quadratic program that is developed in this paper departs entirely from previous formulations: it is *not* based on replacing cross-product terms with new variables and constraints. One of the characteristics of this formulation is that it does not introduce new primary variables or new constraints into the computational steps of the decomposition algorithm: new variables or constraints are not added to the subproblems and relaxed master problems that must be solved in each iteration. Therefore, the objective of rendering tractable the original problem is not achieved at the expense of increased programs size. However, the new formulation requires the solution of an additional standard linear program in each iteration. (Further research into the nature of the formulation may eliminate the need to solve this program.)

2. Preliminaries

Denote by P the following mixed-integer quadratic program

$$\text{(P)} \qquad \begin{aligned} &\max \quad f(x, y) = q_1'x + q_2'y + \tfrac{1}{2}x'Q_1x + x'Q_2y + \tfrac{1}{2}y'Q_3y, \\ &\text{s.t.} \quad A_1x + A_2y \le b; \quad x \in X; \quad y \ge 0, \end{aligned}$$

where q_1', q_2', x, and y are vectors of order $1 \times n_1$, $1 \times n_2$, $n_1 \times 1$ and $n_2 \times 1$, respectively; Q_1 and Q_3 are symmetric matrices of order $n_1 \times n_1$ and $n_2 \times n_2$, respectively, and Q_2 is an $n_1 \times n_2$-matrix, such that the $(n_1 + n_2) \times (n_1 + n_2)$-matrix

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{bmatrix}$$

is symmetric negative semi-definite; A_1 and A_2 are matrices of order $m \times n_1$ and $m \times n_2$, respectively; b is an $m \times 1$-vector; and X is the set of all non-negative n_1-vectors with integer components. (More generally, $x_1, \dots, x_{n_1}$ can be any vector of *complicating* variables, in the sense that P is a difficult optimization program in x and y jointly, but for a fixed $\bar{x} \in X$ it becomes much easier to solve [12].) It will also be assumed that the matrix $[Q_2, \frac{1}{2}Q_1]$ has full row rank. Note that the objective function $f(x, y)$ is *quadratric* in the 'complicating' variables $x_1, \dots, x_{n_1}$.

The difficulties that arise if the generalized Benders' decomposition method [12] is implemented to solve program P *in the original xy-space* will be pointed out. By fixing $x = \bar{x} \in X$ in the program P, subproblem $\mathrm{P}(\bar{x})$ is obtained:

$$(\mathrm{P}(x)) \qquad \begin{aligned} \max \quad & f(\bar{x}, y) = q_1' + \tfrac{1}{2}\bar{x}'Q_1\bar{x} + (q_2' + \bar{x}'Q_2)y + \tfrac{1}{2}y'Q_3Y, \\ \text{s.t.} \quad & A_2y \le b - A_1\bar{x}; \quad y \ge 0. \end{aligned}$$

The dual $\bar{D}(\bar{x})$ to subproblem $\bar{P}(\bar{x})$ is [9]:

$$(\mathrm{D}(\bar{x})) \qquad \begin{aligned} \min \quad & (\psi(\bar{x}, u, v) = q_1'\bar{x} + \tfrac{1}{2}\bar{x}'Q_1\bar{x} - \tfrac{1}{2}u'Q_3u + (b - A_1\bar{x})'v, \\ \text{s.t.} \quad & A_2'v - Q_3u \ge q_2 + Q_2'\bar{x}; \quad v \ge 0, \end{aligned}$$

where u and v are vectors of dual variables of order $n_2 \times 1$ and $m \times 1$, respectively.

Let u^i and v^i be fixed points, and x a vector of variables. Clearly, $\psi(x, u^i, v^i)$ is *quadratic* in the integer variables x. Since $\psi(x, u^i, v^i)$, $i = 1, 2, \dots$, are used to form Benders' cuts, the implementation of the generalized Benders' method yields relaxed master problems that have constraints that are quadratic in x, as will be seen.

Let S_x be the set of all *admissible solutions* x to $\mathrm{P}(x)$: for every $x \in S_x$, there exists at least one feasible solution y to $\mathrm{P}(x)$. Let r^j, $j = 1, \dots, n_r$ (n_r is finite), be the generators of the convex polyhedron

$$R = \{r \in \mathbf{R}^m \mid A_2'r \ge 0; r \ge 0\}. \tag{1}$$

Using Farkas' Lemma, one can show that $x \in X$ is admissible (i.e., $x \in S_x$) if and only if it satisfies the finite system$\{(r^j)'(b - A_1x) \ge 0, j = 1, \dots, n_r\}$. Therefore,

$$S_x = \{x \in \mathbf{R}^{n_1} \mid (r^j)'(b - A_1x) \ge 0, j = 1, \dots, n_r; x \in X\}. \tag{2}$$

Using (2), program P can be written as:

$$\underset{x \in S_x}{\text{maximize}} \begin{bmatrix} \text{maximize} & f(x, y) = q_1'x + q_2'x + \tfrac{1}{2}x'Q_1x + x'Q_2y + \tfrac{1}{2}y'Q_3y \\ & \text{s.t.} \quad A_2y \le b - A_1x; \quad y \ge 0 \end{bmatrix}. \tag{3}$$

The decomposition strategy is clearly seen from (3). For a fixed $x \in S_x$, the quadratic subproblem $\mathrm{P}(x)$ in the brackets of (3) is solved, and a new and 'better' $x \in S_x$ is generated, and so forth.

Let $\{U, V\} = \{(u, v) \mid A_2'v + Q_3u \geq q_2 + Q_2'x;\ v \geq 0\}$ be the feasible set of the dual $D(x)$. The generators r^j, $j = 1, \ldots, n_r$ of the convex polyhedron R (see (1)) are the extreme rays of $\{v \mid A_2'v \geq q_2 + Q_2'x;\ v \geq 0\}$.

Using duality theory of quadratic programming [9], the primal $\mathrm{P}(x)$ in the brackets of (3) can be replaced by its dual $\mathrm{D}(x)$, to yield the following equivalent program:

$$\underset{x \in S_x}{\text{maximize}}\{\underset{(u,v)}{\text{minimize}}[\psi(x, u, v) \mid (u, v) \in \{U, V\}]\}. \tag{4}$$

By using definition (2) of S_x and writing $\psi(x, u, v)$ explicitly, program (4) can be written as the following master problem MP:

$$\begin{array}{lll} & \max & \theta, \\ \text{(MP)} & \text{s.t.} & (1)\ \theta \leq q_1'x + \tfrac{1}{2}x'Q_1x + v'(b - A_1x) - \tfrac{1}{2}u'Q_3u, \quad \text{all } (u, v) \in \{U, V\}, \\ & & (2)\ (r^j)'(b - A_1x) \geq 0, \quad j = 1, \ldots, n_r, \\ & & (3)\ x \in X. \end{array}$$

Master problem MP is equivalent to the original problem P. However, MP is of theoretical interest only, since it has an enormous number of constraints. But, it can be solved iteratively by a process of *relaxation* [11]. At each iteration, a relaxed version of MP is solved: it includes only few of the constraints of types (1) and (2) of MP. In order to test the solution $\bar{x}$ of the relaxed master problem for feasibility in the *unrelaxed master problem* MP, the subproblem $\mathrm{P}(\bar{x})$ is solved, and new Benders' cuts are added to the relaxed problem as needed (see [6, 12, 22], and the discussion in Section 4 for more details).

Benders' cuts that form the constraints of type (1) of the master problems are *quadratic in the integer variables* $x_1, \ldots, x_{n_1}$. This makes the relaxed master problems difficult and untractable discrete optimization programs, especially if n_1 is large (as is the case in most practical applications).

Next, the equivalent formulation of problem P under which *Benders' cuts are linear, rather than quadratic, in the integer variables*, is developed.

3. Equivalent formulations

Consider the objective function $f(x, y) = q_1'x + q_2'y + x'(Q_2y + \tfrac{1}{2}Q_1x) + \tfrac{1}{2}y'Q_3y$ of program P. Define $Q \equiv [Q_2, \tfrac{1}{2}Q_1]$: Q is an $n_1 \times (n_1 + n_2)$-matrix of rank r (the assumption $r = n_1$ will be introduced later). Let $Q^\dagger$ be a *generalized inverse of* Q, i.e., $Q^\dagger$ is an $(n_1 + n_2) \times n_1$-matrix satisfying $QQ^\dagger Q = Q$. (See Ben-Israel and Greville [5], who denote this type of generalized inverse $\{1\}$-*inverse*. Such a matrix always exists, and it is not unique. Also, $Q^\dagger = Q^{-1}$ if Q is nonsingular; rank $Q^\dagger \geq$ rank Q; and $QQ^\dagger$ and $Q^\dagger Q$ are idempotent and have the same rank as Q.) Let Q^0 be an $(n_1 + n_2) \times (n_1 + n_2 - r)$-basis for the null space $N(Q)$ of Q. In [5], the *Hermite normal form* of Q is used to construct $Q^\dagger$; an important advantage of this method is that a basis Q^0 is easily obtained as a by-product of computing $Q^\dagger$.

Hence,

$$Q^{\dagger} = F\begin{bmatrix} I_r & 0 \\ 0 & L \end{bmatrix}E, \qquad Q^0 = F\begin{bmatrix} -K \\ I_{n_1+n_2-r} \end{bmatrix}, \tag{5}$$

where E is the product of the elementary row matrices that transform Q into its *Hermite normal form* $H = EQ$; F is a permutation matrix such that a postmultiplication of EQ by F rearranges the columns of EQ so that its first r columns are the unit vectors; L is an arbitrary matrix of order $(n_1+n_2-r)\times(n_1-r)$; K is an $r\times(n_1+n_2-r)$-submatrix of the column-permuted form EQF; and 0 is a null matrix. (E and F are nonsingular matrices of order $n_1\times n_1$ and $(n_1+n_2)\times(n_1+n_2)$, respectively.)

Lemma 1. *If $Q^{\dagger}$ is an $n_1\times(n_1+n_2)$-matrix of rank r, then:*
(1) *$QQ^{\dagger} = I_{n_1}$ iff $r = n_1$ (i.e., Q has full row rank),*
(2) Rank $Q^{\dagger} = r + \text{rank } L$.
(3) Rank $Q^0 = n_1 + n_2 - r$.
(4) *If Q has full row rank, then* rank $Q^{\dagger} = n_1$, *and* rank $Q^0 = n_2$.

Part (1) follows from the fact that $QQ^{\dagger}$ is idempotent and nonsingular; the remaining parts follow from definition (5) and from the fact that E and F are nonsingular. If Q has full row rank (i.e., $r = n_1$), then:

$$Q^{\dagger} = F\begin{bmatrix} E \\ 0 \end{bmatrix}, \qquad Q^0 = F\begin{bmatrix} -K \\ I_{n_2} \end{bmatrix}. \tag{6}$$

Lemma 2. *IF Q has full row rank, then the $(n_1+n_2)\times(n_1+n_2)$ matrix $[Q^{\dagger}, Q^0]$ is nonsingular.*

Proof. Define $s \equiv (y, x)$ and consider the linear system $Qs = w$. Let $N(Q) = \{s \in \mathbf{R}^{n_2+n_1} \mid Qs = 0\}$ be the null space of Q and $R(Q') = \{s \in \mathbf{R}^{n_2+n_1} \mid s = Q'w, w \in \mathbf{R}^{n_1}\}$ be the range set of the transpose Q'. The two subspaces are orthogonal to each other: $N(Q) = R(Q')^{\perp}$. (The following is a short proof to this well-known relation. Since any $s \in N(Q)$ can be written as $s = Q^0 z$, $z \in \mathbf{R}^{n_2}$ (see Remark 3.2 below), the inner product of any vector in $R(Q')$ and any vector in $N(Q)$ is $(Q'w)'(Q^0z) = w'QQ^0z = 0$ (since $QQ^0 \equiv 0$), which implies that $N(Q) = R(Q')^{\perp}$.) Since Q has full row rank, dim $R(Q') = n_1$, and dim $N(Q) = n_2$. Therefore, a basis for $R(Q')$ and a basis for $N(Q)$ collectively constitute (n_1+n_2) linearly independent vectors that span $\mathbf{R}^{n_2+n_1}$. Q^0 is a basis for $N(Q)$, since rank $Q^0 = n_2$ (Lemma 1) and dim $N(Q) = n_2$; similarly, $Q^{\dagger}$ is a basis for $R(Q')$, since rank $Q^{\dagger} = n_1$ (Lemma 1) and dim $R(Q') = n_1$. Hence, $[Q^{\dagger}, Q^0]$ is a basis for $\mathbf{R}^{n_2+n_1}$, which means that $[Q^{\dagger}, Q^0]$ is nonsingular.

Remark 3.1. The proof of Lemma 2 as given above is general in the sense that it is independent of the particular method used to compute $Q^{\dagger}$ and Q^0. The proof becomes especially simple if $Q^{\dagger}$ is computed by using the Hermite normal form of Q, in which case $Q^{\dagger}$ and Q^0 are given in (6). Since E and F are nonsingular matrices of order $n_1 \times n_1$ and $(n_1+n_2)\times(n_1+n_2)$, respectively, and K is an $n_1 \times n_2$-matrix, the columns of the $(n_1+n_2)\times(n_1+n_2)$-matrix

$$\begin{bmatrix} E & -K \\ 0 & I_{n_2} \end{bmatrix}$$

are linearly independent. Hence,

$$[Q^{\dagger}, Q^0]^{-1} = \left\{F\begin{bmatrix} E & -K \\ 0 & I_{n_2} \end{bmatrix}\right\}^{-1} = \begin{bmatrix} E^{-1} & E^{-1}K \\ 0 & I_{n_2} \end{bmatrix} F^{-1}.$$

In general, the type of generalized inverse used in this paper is employed primarily for solving linear systems, as stated in Theorem 1 [26].

Theorem 1 [26]. *Let $Q \equiv [Q_2, \frac{1}{2}Q_1]$ be an $n_1 \times (n_1+n_2)$-matrix, and w be an n_1+1-vector. Then, the linear system $Q_2 y + \frac{1}{2}Q_1 x = w$ is consistent iff there is some generalized inverse $Q^{\dagger}$ of Q such that $QQ^{\dagger}w = w$, in which case the general solution of $Q_2 y + \frac{1}{2}Q_1 x = w$ is $\begin{bmatrix} y \\ x \end{bmatrix} = Q^{\dagger}w + p$, $p \in N(Q)$.*

Remark 3.2. Since Q^0 is a matrix for the null space basis of Q, $N(Q)$ can be written as $N(Q) = \{p \in \mathbf{R}^{n_1+n_2} \mid p = Q^0 z,\ z \in \mathbf{R}^{n_2}\}$. Then, if $Q_2 y + \frac{1}{2}Q_1 x = w$ is consistent, the set of all its solutions $\begin{bmatrix} y \\ x \end{bmatrix}$ is

$$S(Q, w) = \left\{\begin{bmatrix} y \\ x \end{bmatrix} \,\middle|\, \begin{bmatrix} y \\ x \end{bmatrix} = Q^{\dagger}w + Q^0 z,\ z \in \mathbf{R}^{n_2}\right\}. \tag{7}$$

Let $Q_y^{\dagger}$ and Q_y^0 be the matrices made up from the first n_2 rows of $Q^{\dagger}$ and Q^0, respectively: $Q_y^{\dagger}$ is of order $n_2 \times n_1$, and Q_y^0 is of order $n_2 \times n_2$. Similarly, define $Q_x^{\dagger}$ and Q_x^0 to be the matrices made up from the remaining n_1 rows of $Q^{\dagger}$ and Q^0, respectively: they are of order $n_1 \times n_1$, and $n_1 \times n_2$ respectively. For a given $\bar{x} \in X$, let $\mathrm{P}'(\bar{x})$ be the following quadratic program:

$$\max \quad h(w, z) = d'\begin{bmatrix} w \\ z \end{bmatrix} + (w', z')D\begin{bmatrix} w \\ z \end{bmatrix}$$

$(\mathrm{P}'(\bar{x}))$

$$\begin{aligned} \text{s.t.} \quad & (1)\ A_2 Q_y^{\dagger} w + A_2 Q_y^0 z \le b - A_1 \bar{x}, \\ & (2)\ Q_y^{\dagger} w + Q_y^0 z \ge 0. \\ & (3)\ Q_x^{\dagger} w + Q_x^0 z = \bar{x}, \end{aligned}$$

where

$$d' = (q_1' Q_x^{\dagger} + q_2' Q_y^{\dagger},\ q_1' Q_x^0 + q_2' Q_y^0), \tag{8}$$

$$D = \begin{bmatrix} Q_x^{\dagger} + \frac{1}{2}Q_y^{\dagger\prime} Q_3 Q_y^{\dagger} & \frac{1}{2}[Q_x^{0\prime} + Q_y^{0\prime} Q_3 Q_y^{\dagger}]' \\ \frac{1}{2}[Q_x^{0\prime} + Q_y^{0\prime} Q_3 Q_y^{\dagger}] & \frac{1}{2}Q_y^{0\prime} Q_3 Q_y^0 \end{bmatrix} \tag{9}$$

are vector and symmetric matrix of order $1\times(n_1+n_2)$ and $(n_1+n_2)\times(n_1+n_2)$, respectively.

Theorem 2. *If Q has full row rank, then program* $\mathrm{P}'(\bar{x})$ *is equivalent to program* $\mathrm{P}(\bar{x})$, *and*:

(1) *If* $(\bar{w}, \bar{z})$ *solves* $\mathrm{P}'(x)$, *then the solution* $\bar{y}$ *to* $\mathrm{P}(\bar{x})$ *is given by* $\bar{y} = Q_y^{\dagger}\bar{w} + Q_y^0\bar{z}$.

(2) *If* $\bar{y}$ *solves* $\mathrm{P}(\bar{x})$, *then the solution* $(\bar{w}, \bar{z})$ *to* $\mathrm{P}'(\bar{x})$ *is the 'unique' solution to the linear system*

$$Q^{\dagger}w + Q^0 z = \begin{bmatrix}\bar{y}\\ \bar{x}\end{bmatrix}.$$

Furthermore, this solution is simply given by:

(i) $\bar{w} = Q_2\bar{y} + \frac{1}{2}Q_1\bar{x}$,

(ii) $\bar{z}_j = \left\{F'\begin{bmatrix}\bar{y}\\ \bar{x}\end{bmatrix}\right\}_{n_1+j}$, $j = 1, \dots, n_2$ *where* $\left\{F'\begin{bmatrix}\bar{y}\\ \bar{x}\end{bmatrix}\right\}_i$ *is the ith element of* $F'\begin{bmatrix}\bar{y}\\ \bar{x}\end{bmatrix}$.

Proof. Rank $Q = n_1$ implies $R(Q) = \mathbf{R}^{n_1}$ and that the linear system $Q_2 y + \frac{1}{2}Q_1 x = w$ is consistent for any $w \in \mathbf{R}^{n_1}$. It then follows from Theorem 1 that any $w \in \mathbf{R}^{n_1}$ can be written as

$$w = Q_2 y + \tfrac{1}{2}Q_1 x, \quad \text{where } \begin{bmatrix}y\\ x\end{bmatrix} = Q^{\dagger}w + p, \quad p \in N(A). \tag{10}$$

From Remark 3.2, $\begin{bmatrix}y\\ x\end{bmatrix}$ can be written as

$$\begin{bmatrix}y\\ x\end{bmatrix} = Q^{\dagger}w + Q^0 z, \quad z \in \mathbf{R}^{n_2}. \tag{11}$$

In other words, (10) and (11) constitute a one-to-one correspondence between all vectors $w \in \mathbf{R}^{n_1}$ and all translations $\begin{bmatrix}y\\ x\end{bmatrix}$ of $N(Q) = \left\{\begin{bmatrix}y\\ x\end{bmatrix} \in \mathbf{R}^{n_1+n_2} \mid Q_2 y + \frac{1}{2}Q_1 x = 0\right\}$ in $\mathbf{R}^{n_1+n_2}$. (The validity of (10) and (11) is clearly seen if one substitutes $\begin{bmatrix}y\\ x\end{bmatrix}$ as defined in (11) in $w = Q_2 y + \frac{1}{2}Q_1 x$, and notes that rank $Q = n_1$ implies $QQ^{\dagger} = I_{n_1}$ (Lemma 1), and that $QQ^0 \equiv 0$.)

Recalling the definitions of $Q_y^{\dagger}$, $Q_x^{\dagger}$, Q_y^0 and Q_x^0, (11) can be written as

$$y = Q_y^{\dagger}w + Q_y^0 z, \qquad x = Q_x^{\dagger}w + Q_x^0 z. \tag{12}$$

By substituting (10) and (12) in program $\mathrm{P}(\bar{x})$, the equivalent program $\mathrm{P}'(\bar{x})$ is obtained.

Part (1) of the theorem follows directly from relations (10)–(12). As to part (2), suppose that given $\bar{x} \in X$, $\bar{y}$ is the solution to $\mathrm{P}(\bar{x})$. Then, the solution $(\bar{w}, \bar{z})$ to $\mathrm{P}'(\bar{x})$ is obtained by solving the linear system $Q^{\dagger}w + Q^0 z = \begin{bmatrix}\bar{y}\\ \bar{x}\end{bmatrix}$. This solution is unique, since the matrix $[Q^{\dagger}, Q^0]$ is nonsingular (Lemma 2). Furthermore, it is clear from (10) and (11) that $\bar{w}$ is simply given by $\bar{w} = Q_2\bar{y} + \frac{1}{2}Q_1\bar{x}$. Hence, $\bar{z}$ is the solution to the linear system

$$Q^0 z = \begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix} - Q^{\dagger}\bar{w}. \tag{13}$$

Using (6), (13) becomes

$$F\begin{bmatrix} -K \\ I_{n_2} \end{bmatrix} z = \begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix} - F\begin{bmatrix} E \\ 0 \end{bmatrix}\bar{w} \tag{14}$$

or

$$\begin{bmatrix} -K \\ I_{n_2} \end{bmatrix} z = F'\begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix} - \begin{bmatrix} E\bar{w} \\ 0 \end{bmatrix}, \tag{15}$$

since F is nonsingular, and $F^{-1} = F'$. Since z is an n_2-vector and the last n_2 elements of the vector $\begin{bmatrix} E\bar{w} \\ 0 \end{bmatrix}$ are zero, it follows that

$$\bar{z}_j = \left\{F'\begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix}\right\}_{n_1+j}, \quad j = 1, \dots, n_2.$$

Remark 3.3. Suppose that x in $P'(x)$ is a vector of variables rather than a fixed point, and denote by P' the resulting optimization program in the xwz-space. Clearly, P' and the original program P are equivalent, since $P'(x)$ and $P(x)$ are equivalent for *every* point $x \in X$. Furthermore, the equivalence between the original and the new formulation holds in the almost general case (i.e., Q needs to be of full row rank). Observe that under the new formulation the integer variables x are absent from the objective function of P', and the constraint set of P' is linear in x. In this regard it is worthwhile to observe that the new formulation is not weaker than the original, in the sense that *relaxing the integer restrictions does not destroy the equivalence between both formulations.*[1] Finally, note that the transformation of $P(\bar{x})$ into $P'(\bar{x})$ is accomplished, essentially, by a one-to-one mapping of the set $\{y \in \mathbf{R}^{n_2} \mid A_2 y \le b - A_1\bar{x},\ y \ge 0\}$ from the y-space onto the wz-space.

4. The decomposition algorithm

The generalized Benders decomposition method [12] is implemented in this section for solving program P. Based on the equivalent formulations developed in the last section, P is decomposed into a series of *integer linear master problems*, and standard quadratic subproblems.

[1] The last observation was pointed out to the author by an anonymous referee.

4.1. An outline of the decomposition strategy

Define

$$G = \begin{bmatrix} A_2 Q_y^\dagger & A_2 Q_y^0 \\ -Q_y^\dagger & -Q_y^0 \\ Q_x^\dagger & Q_x^0 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} b - A_1 x \\ 0 \\ x \end{bmatrix}. \tag{16}$$

G is an $(m+n_2+n_1)\times(n_1+n_2)$-matrix, and $g(x)$ is an $(m+n_2+n_1)\times 1$-vector. The dual $\mathrm{D}'(\bar{x})$ to program $\mathrm{P}'(\bar{x})$ is [9]:

$$(\mathrm{D}'(\bar{x})) \qquad \begin{array}{ll} \min & \phi(\bar{x}, u, v) = -u'Du + v'g(\bar{x}), \\ \text{s.t.} & G'v - 2Du = d; \quad v_i \geq 0, i = 1, \ldots, m+n_2. \end{array}$$

where u and v are vectors of dual variables of order $(n_1+n_2)\times 1$ and $(m+n_2+n_1)\times 1$, respectively. (Note that only the first $(m+n_2)$ elements of v are required to be nonnegative.)

Remark 4.1. Letting u^i and v^i be fixed points and x a vector of variables it is clearly seen that $\phi(x, u^i, v^i)$ is linear in the integer variables x. Since $\phi(x, u^i, v^i)$, $i = 1, 2, \ldots$, are used to form Benders' cuts, this result makes it possible to overcome the difficulty that was encountered in Section 2. Furthermore, the dual constraint set is independent of x, and Geoffrion's 'Property P' holds [12]. This makes it possible to implement the generalized Benders decomposition method. (Similar results were obtained by Balas under his formulation.)

Let S_x' be the set of all admissible solutions x to $\mathrm{P}'(X)$. As noted in Section 2, one can use Farkas' Lemma and show that

$$S_x' = \{x \in \mathbf{R}^{n_1} \mid (r^j)'g(x) \geq 0, \quad j = 1, \ldots, n_r'; x \in X\}, \tag{17}$$

where $r^j j = 1, \ldots, n_r'$ are the generators of the convex polyhedron

$$R' = \{r \in \mathbf{R}^{m+n_2+n_1} \mid G'r = 0; r_i \geq 0, i = 1, \ldots, m+n_2\}.$$

Then, program P' (and, hence, the original program P) can be written as (see Section 2):

$$\underset{x \in S_x'}{\text{maximize}}\{\underset{(u,v)}{\text{minimize}}[\phi(x, u, v) \mid (u, v) \in \{U, V\}']\}, \tag{18}$$

where $\{U, V\}'$ is the feasible set of the dual $\mathrm{D}'(x)$. Finally, program (18) can be written as the following master problem MP':

$$(\mathrm{MP}') \qquad \begin{array}{ll} \max & \theta \\ \text{s.t.} & (1)\ \theta \leq -u'Du + v'g(x), \quad \text{all } (u, v) \in \{U, V\}' \\ & (2)\ (r^j)'g(x) \geq 0, \quad j = 1, \ldots, n_r', \\ & (3)\ x \in X. \end{array}$$

In the following *relaxed* master problem RMP′, only few of the constraints (1) and (2) of master problem MP′ are included:

$$\text{(RMP}'\text{)}\qquad \begin{array}{ll} \max & \theta, \\ \text{s.t.} & (1)\ \theta \le -(u^i)'Du^i + (v^i)'g(x), \quad i = 1, \ldots, k, \\ & (2)\ (r^i)'g(x) \ge 0, \quad i = 1, \ldots, t, \\ & (3)\ x \in X. \end{array}$$

An optimal solution $(\bar{\theta}, \bar{x})$ of RMP′ is also optimal for the *unrelaxed* master problem MP′ (and therefore, for the original program P) iff $(\bar{\theta}, \bar{x})$ is feasible for MP′. Furthermore, subproblem $P'(\bar{x})$ is used to test $(\bar{\theta}, \bar{x})$ for feasibility MP′:

(1) $\bar{x}$ satisfies the constraint set (2) of MP′ iff subproblem $P'(\bar{x})$ is feasible: the feasibility of $P'(\bar{x})$ implies that $\bar{x} \in S'_x$, which means that constraint set (2) of MP′ is satisfied.

(2) If $P'(\bar{x})$ is feasible, then $(\bar{\theta}, \bar{x})$ satisfies the constraint set (1) of MP′ iff $\bar{\theta} \le \phi(\bar{x}, \bar{u}, \bar{v})$.

Thus, $(\bar{\theta}, \bar{x})$ is feasible for MP′ iff (1) $P'(\bar{x})$ is feasible, and (2) $\theta \le \phi(\bar{x}, \bar{u}, \bar{v})$.

Suppose, first, that $P'(\bar{x})$ is feasible, but $\bar{\theta} > \phi(\bar{x}, \bar{u}, \bar{v})$: this implies that some of the constraints of type (1) of MP′ are violated. The 'most violated' constraint is created by solving $\min\{\phi(\bar{x}, u, v) \mid (u, v) \in \{U, V\}'\}$. This, however, is the dual $D'(\bar{x})$, whose optimal solution is $\phi(\bar{x}, \bar{u}, \bar{v})$. To satisfy the violated constraint, the following constraint is added to the relaxed master problem RMP′:

$$\theta \le \phi(x, \bar{u}, \bar{v}) \tag{19}$$

Next, suppose that $P'(\bar{x})$ is not feasible. This means that $\bar{x} \notin S'_x$, which implies the existence of a vector $\bar{r}$ such that $\bar{r}'g(\bar{x}) < 0$ (see (17)). (An alternative way to obtain this result is as follows. The generators r^j, $j = 1, \ldots, n'_r$ of the convex polyhedron $R' = \{r \in \mathbf{R}^{m+n_2+n_1} \mid G'r = 0;\ r_i \ge 0,\ i = 1, \ldots, m + n_2\}$ are the extreme rays of $\{v \mid G'v = d;\ v_i \ge 0, i = 1, \ldots, m + n_2\}$. The infeasibility of $P'(\bar{x})$ implies that the dual $D'(\bar{x})$ is unbounded. Then, there is an extreme ray $\bar{r}$ of the convex polyhedron $\{v \mid G'v = d;\ v_i \ge 0, i = 1, \ldots, m + n_2\}$ such that the dual objective function $\phi(\bar{x}, u, v)$ decreases infinitely along direction $v = \bar{v} + \lambda\bar{r}$, $\lambda \ge 0$; this happens only if $\bar{r}'g(\bar{x}) < 0$.) Hence, $\bar{x}$ does not satisfy some of the constraints of type (2) of MP′. To eliminate this inadmissible point $\bar{x}$, the following constraint is added to the relaxed master problem RMP′:

$$\bar{r}'g(x) \ge 0. \tag{20}$$

Finally, observe that Benders' cuts $\{\theta \le -(u^i)'Du^i + (v^i)'g(x), i = 1, \ldots, k\}$ of the relaxed master problem RMP′ are *linear in the integer variables* x, since $g(x)$ is linear in x. This is the result of the new equivalent formulation (see Remarks 3.3 and 4.1). As a consequence, RMP′ is an *integer linear program*, rather than a program with constraints that are quadratic in x. This renders the original problem P tractable.

4.2. The algorithm

The detailed decomposition algorithm to solve P is stated below.

Step 1. (a) Let UB $= +\infty$ and LB $= -\infty$ be the initial upper and lower bounds, respectively, on the optimal value of the objective function $f(x, y)$ of program P. Set a tolerance value $\epsilon > 0$, and $k = t = 0$.

(b) Generate $\bar{x} \in X$, and go to Step 3.

Step 2. Solve the relaxed master problem RMP′.

(a) If RMP′ has no feasible solution, then there is no feasible solution to P.

(b) Let $(\bar{\theta}, \bar{x})$ be the optimal solution to RMP′. Put UB $= \bar{\theta}$. If UB $-$ LB $\leq \epsilon$, terminate; $(\bar{x}, \bar{y})$ is the optimal solution to P.

Step 3. Solve subproblem P$(\bar{x})$:

$$\begin{aligned} \max \quad & f(\bar{x}, y) = q_1'\bar{x} + \tfrac{1}{2}\bar{x}'Q_1\bar{x} + (q_2' + \bar{x}'Q_2)y + \tfrac{1}{2}y'Q_3y, \\ \text{s.t.} \quad & A_2y \leq b - A_1\bar{x}; \quad y \geq 0. \end{aligned}$$

(a) If P$(\bar{x})$ has no feasible solution, go to Step 6.

(b) Let $\bar{y}$ be the optimal solution to P$(\bar{x})$. If UB $- f(\bar{x}, \bar{y}) \leq \epsilon$ terminate: $(\bar{x}, \bar{y})$ is the optimal solution to P.

Step 4. (a) Determine the optimal solution $(\bar{w}, \bar{z})$ to program P′$(\bar{x})$ by setting (Theorem 2):

(i) $\bar{w} = Q_2\bar{y} + \tfrac{1}{2}Q_1\bar{x}$.

(ii) $\bar{z}_j = \{F' \begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix}\}_{n_1+j}$, $j = 1, \dots, n_2$.

(b) Find the optimal solution $(\bar{u}, \bar{v})$ to the dual D′$(\bar{x})$ by [9]:

(i) $\bar{u} \equiv (\bar{w}, \bar{z})$,

(ii) $\bar{v}$ is the optimal solution to the linear program:

$$(\text{LP}(\bar{x})) \quad \begin{aligned} \min \quad & g(\bar{x})'v, \\ \text{s.t.} \quad & G'v = 2D\bar{u} + d; \quad v_i \geq 0, i = 1, \dots, m + n_2. \end{aligned}$$

Step 5. If $f(\bar{x}, \bar{y}) >$ LB, set LB $= f(\bar{x}, \bar{y})$. Let $k = k + 1$ and $(u^k, v^k) = (\bar{u}, \bar{v})$ and go to Step 2.

Step 6. Let $t = t + 1$ and generate the extreme ray r^t of the polyhedron $G'v = d$ such that the dual functional $\phi(\bar{x}, u, v)$ decreases infinitely along the direction $v = \bar{v} + \lambda r^t$, $\lambda \geq 0$. Go to Step 2. (If the quadratic subproblem is solved by a simplex-based algorithm, r^t can be generated using standard linear programming techniques. See [10, 22].)

Some comments are now in order regarding this procedure and various computational aspects related to it.

(a) The original mixed-integer quadratic problem P is decomposed into a series of integer linear programs and standard quadratic programs. These programs are significantly simpler and more tractable than P, and efficient procedures are available for solving both types of programs. Furthermore, the objective of rendering P tractable is achieved *without introducing new primary variables and/or new constraints into the subproblem* $P(\bar{x})$ *and the relaxed master problem* RMP′. However, the algorithm requires the solution of an additional standard linear program $LP(\bar{x})$ in each iteration, which has $(m+n_2+2n_1)$ variables and (n_1+n_2) constraints. Further discussion of this issue follows.

(b) The decomposition strategy of the algorithm is based on the new formulation P′ of P and, hence, on the primal $P'(\bar{x})$ and the dual $D'(\bar{x})$. Each iteration requires the solution $(\bar{u}, \bar{v})$ to the dual $D'(\bar{x})$ which is readily available upon solving $P'(\bar{x})$: $\bar{u} \equiv (\bar{w}, \bar{z})$, and $\bar{v}$ is given as a byproduct. $P'(\bar{x})$ is a quadratic program with $2(n_1+n_2)$ variables (the factor 2 is due to transforming the unrestricted variables (w, z) into nonnegative variables), and $(m+n_1+n_2)$ constraints. The original subproblem $P(\bar{x})$, however, is a smaller quadratic program: it has only n_2 variables and m constraints. Furthermore, Theorem 2 makes it possible to solve $P(\bar{x})$ instead of $P'(\bar{x})$: both programs are equivalent, and the solution $(\bar{w}, \bar{z})$ to $P'(\bar{x})$ is easily obtained upon solving $P(\bar{x})$. This, of course, has a clear computational significance. As to the dual solution $(\bar{u}, \bar{v})$ to $D'(\bar{x})$: $\bar{u} \equiv (\bar{w}, \bar{z})$, but $\bar{v}$ is not readily available as a byproduct of solving $P(\bar{x})$, and the linear program $LP(\bar{x})$ has to be solved in order to determine $\bar{v}$. Thus, in the tradeoff between solving the relatively big quadratic program $P'(\bar{x})$ or the much smaller quadratic program $P(\bar{x})$ *and* the linear program $LP(\bar{x})$, the latter is more attractive from a computational point of view. (Further research into the nature of the new formulation suggested in this study and the relationships between programs $P(\bar{x})$ and $D(\bar{x})$ and programs $P'(\bar{x})$ and $D'(\bar{x})$, may eliminate the need to solve linear program $LP(\bar{x})$. For instance, it may be possible to compute $\bar{v}$ directly from the primal solution to $P(\bar{x})$ and the dual solution to $D(\bar{x})$. It is also worthwhile to observe that the matrix of coefficients G' of $LP(\bar{x})$ is independent of $\bar{x}$, and remains unchanged from one iteration to another. Hence, the solution v^k to $LP(x^k)$ in iteration k can be used as an initial basic solution to $LP(x^{k+1})$ in iteration $(k+1)$; if v^k is not feasible for $LP(x^{k+1})$, the dual simplex algorithm can be used to restore primal feasibility.) As a result of solving $P(\bar{x})$ as a subproblem, the algorithm does not introduce new primary variables and/or new constraints: the variables of $P(\bar{x})$ are the n_2 original continuous variables $y_1, \dots, y_{n_2}$ and its constraints are the original m constraints; the variables of RMP′ are the n_1 original integer variables $x_1, \dots, x_{n_1}$ and its constraints are generated iteratively as needed.

(c) *Solving the subproblem and the relaxed master problem.* Quadratic programming techniques (e.g., [4, 23, 28]) can be used to solve subproblem $P(\bar{x})$. The relaxed master problem RMP′ can be solved using one of the available approaches for solving integer linear programs: branch-and-bound; cutting plane; implicit enumeration (see, e.g., [10]).

(d) *Computational experience with Benders' method.* As in any implementation of Benders' method, the computational performance of the procedure described here is highly dependent upon the structure of the relaxed master problem RMP′ and upon the all-integer algorithm used to solve it: a complete computational study of the procedure will be reported separately. Several researchers report favorable computational experience with Benders' method. Benders reports encouraging results in solving mixed-integer linear programs, and so are the results of Geoffrion and Marsten [14]. Geoffrion and Graves' [13] experience in solving multicommodity distribution flow problems with hundreds of 0–1 variables and thousands of rows and continuous variables suggests convergence with a relatively small number of Benders' cuts. Of a particular relevance to this study is the computational experience of Armstrong and Willis [1]. They applied Benders' method to solve a simple form of mixed-integer quadratic problems (namely, with an objective function that is *linear in the integer variables*). The sub and master problems in their procedure are of the same form as the corresponding problems in this study: quadratic subproblems, and integer linear master problems. Their experience suggests a rapid computational convergence of the algorithm: a maximum of five Benders' cuts for problems with 73 integer variables. However, other researchers had different experience with Benders' method. For instance, Bazaraa and Sherali's quadratic assignment problem [3] required a large number of Benders' cuts. (These researchers attempted the use of Gomory's [19] dual all-integer cuts to solve the 0–1 linear master problems, but report the experience of problems; they have also experienced convergence problems with Glover's [15] pseudo-primal–dual integer programming algorithm. Finally, they adopted an implicit enumeration scheme.)

4.3. *Upper and lower bounds*

Upper and lower bounds on the optimal objective value $f(x^*, y^*)$ of P can be computed at every iteration of the search. This allows a premature termination of the search, in which case the bounds provide estimates regarding the distance of the present solution from the optimal one. Let (x^*, y^*) be the optimal solution of P, and (w^*, z^*) be the solution to the linear system

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = Q^{\dagger}w + Q^0 z.$$

(w^*, z^*) is the solution of subproblem $P'(x^*)$ (Theorem 2). Thus, there is a solution (u^*, v^*) to the dual $D'(x^*)$ such that $u^* \equiv (w^*, z^*)$ and $\phi(x^*, u^*, v^*) = h(w^*, z^*)$ (duality). At some iteration, assume that $\bar{x} \in S'_x$ and let $(\bar{w}, \bar{z})$ and $(\bar{u}, \bar{v})$, respectively, be the solutions to the primal $P'(\bar{x})$ and the dual $D'(\bar{x})$. Then,

$$h(\bar{w}, \bar{z}) = \phi(\bar{x}, \bar{u}, \bar{v}) \leq \phi(x^*, u^*, v^*) \leq \phi(x^*, \bar{u}, \bar{v}). \tag{21}$$

Therefore, the lower bound on $\phi(x^*, u^*, v^*)$ is obtained by fixing $x = \bar{x}$ and solving either $P'(x)$ or $D'(\bar{x})$, and the upper bound is obtained by maximizing

$\phi(x, \bar{u}, \bar{v})$ over $x \in X$. Furthermore, $\max_{x\in X}\{\phi(x, \bar{u}, \bar{v})\} = \bar{\theta}$, where $\bar{\theta}$ is the solution of the relaxed master problem RMP′ constructed after solving $P'(\bar{x})$ and $D'(\bar{x})$. Finally, note that $\phi(\bar{x}, \bar{u}, \bar{v})$ and $\bar{\theta}$ are the lower and upper bounds, respectively, on the optimal objective value $\phi(x^*, u^*, v^*)$, whereas we are interested in the bounds on $f(x^*, y^*)$. But since $h(w^*, z^*) = \phi(x^*, u^*, v^*)$ (duality), and $h(w, z) \equiv f(x, y)$, it is clear that $\phi(\bar{x}, \bar{u}, \bar{v})$ and $\bar{\theta}$ are, respectively, the lower and upper bounds on $f(x^*, y^*)$ as well.

5. Numerical example

Let P be the following mixed-integer quadratic program:

$$\begin{aligned}\max\quad f(x, y) = {} & 80x_1 + 56x_2 + 105y_1 + 12y_2 + 150y_3 \\ & - 3x_1^2 - 18x_2^2 - 14x_1x_2 - 10x_1y_1 - 4x_1y_2 - 14x_1y_3 - 30x_2y_1 \\ & - 6x_2y_2 - 36x_2y_3 - 15y_1^2 - 3y_2^2 - 18y_3^2 - 30y_1y_3 - 6y_2y_3,\end{aligned}$$

$$\begin{aligned}\text{s.t.}\quad & (1)\ 2x_1 + x_2 + 4y_1 + 3y_2 + y_3 \le 60, \\ & (2)\ 3x_1 + 4x_2 + y_1 + 2y_2 + 2y_2 \le 60, \\ & \quad x_i = 0 \text{ or } 1, \quad i = 1, 2; \\ & \quad y_j \ge 0, \quad j = 1, 2, 3.\end{aligned}$$

Using the notation of this paper, we have

$$Q = [Q_2, \tfrac{1}{2}Q_1] = \begin{bmatrix} -10 & -4 & -14 & -3 & -7 \\ -30 & -6 & -36 & -7 & -18 \end{bmatrix},$$

and rank $Q = 2$. (Note that $n_1 = 2$, $n_2 = 3$, $m = 2$).

To compute $Q^\dagger$, Q is transformed into its Hermite normal form by a finite sequence of elementary row operations:

$$[Q \mid I_2] = \left[\begin{array}{ccccc|cc} -10 & -4 & -14 & -3 & -7 & 1 & 0 \\ -30 & -6 & -36 & -7 & -8 & 0 & 1 \end{array}\right] \to \cdots$$

$$\to \left[\begin{array}{ccccc|cc} 1 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{10} & -\frac{1}{15} \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{array}\right] = [EQ \mid E],$$

Clearly, the 5×5 permutation matrix F such that $EQF = [I_2, K]$ is the identity matrix I_5. Hence,

$$K = \begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{2} \\ 1 & \frac{1}{3} & \frac{1}{2} \end{bmatrix}.$$

Thus,

$$Q^{\dagger}=F\begin{bmatrix}I_2\\0\end{bmatrix}E=\begin{bmatrix}\frac{1}{10} & -\frac{1}{15}\\ -\frac{1}{2} & \frac{1}{6}\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix},\qquad Q^{0}=F\begin{bmatrix}-K\\ I_3\end{bmatrix}=\begin{bmatrix}-1 & -\frac{1}{6} & -\frac{1}{2}\\ -1 & -\frac{1}{3} & -\frac{1}{2}\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Also, the matrices and vectors D, d, G and $g(x)$ have to be constructed (see definitions (8), (9) and (16)).

Iteration 1. Set $\mathrm{UB}=+\infty$, $\mathrm{LB}=-\infty$, $\epsilon=10^{-6}$, $k=t=0$. Starting arbitrarily with $\bar{x}=(0,0)$, the quadratic subproblem $\mathrm{P}(\bar{x})$ is solved. Its solution is $\bar{y}=(0,0,4.167)$ and $f(\bar{x},\bar{y})=312.5$.

The solution $(\bar{w},\bar{z})$ to $\mathrm{P}'(\bar{x})$ is (Theorem 2) $\bar{w}=(-58.33,-150)$, $\bar{z}=(4.167,0,0)$, and $h(\bar{w},\bar{z})\equiv f(\bar{x},\bar{y})$. The solution $(\bar{u},\bar{v})$ to the dual $\mathrm{D}'(\bar{x})$ is $\bar{u}\equiv(\bar{w},\bar{z})$ and $\bar{v}=(0,0,20,13,0,21.667,-94)$.

Since $h(\bar{w},\bar{z})>\mathrm{LB}$, set $\mathrm{LB}\equiv h(\bar{w},\bar{z})=312.5$. Let $k=k+1=1$, $(u^1,v^1)=(\bar{u},\bar{v})$, and proceed to the next iteration.

Iteration 2. $\phi(x,u^1,v^1)=-u^{1\prime}Du^1+v^{1\prime}g(x)=312.5+21.5+21.667x_1-94x_2$ is used to construct the relaxed master problem RMP′:

$$\begin{aligned}\max\quad & \theta,\\ \text{s.t.}\quad & \theta\leqslant 312.5+21.667x_1-94x_2,\\ & x_1,x_2=0\quad\text{or}\quad 1\end{aligned}$$

Its optimal solution is $\bar{x}=(1,0)$ and $\bar{\theta}=334.167$.

Set $\mathrm{UB}\equiv\bar{\theta}$. Since $\mathrm{UB}-\mathrm{LB}=334.167-312.5>\epsilon$, subproblem $\mathrm{P}(\bar{x})$ is solved (for $\bar{x}=(1,0)$). Its optimal solution is $\bar{y}=(0,0,3.778)$ and $f(\bar{x},\bar{y})=333.889$. Hence, $\bar{w}=(-55.889,-143)$, $\bar{z}=(3.778,1,0)$, and $h(\bar{w},\bar{z})=333.889$ is the solution to $\mathrm{P}'(\bar{x})$. The solution $(\bar{u},\bar{v})$ to the dual $\mathrm{D}'(\bar{x})$ is $\bar{u}\equiv(\bar{w},\bar{z})$ and $\bar{v}=(0,0,18.333,14.667,0,21.111,-94)$.

Since $h(\bar{w},\bar{z})>\mathrm{LB}$, set $\mathrm{LB}\equiv h(\bar{w},\bar{z})=333.889$. Let $k=k+1=2$ and $(u^2,v^2)=(\bar{u},\bar{v})$.

Iteration 3. $\phi(x,u^2,v^2)=312.778+21.111x_1-94x_2$. The revised master problem is

$$\begin{aligned}\max\quad & \theta,\\ \text{s.t.}\quad & (1)\ \theta\leq 312.5+21.667x_1-94x_2,\\ & (2)\ \theta\leq 312.778+21.111x_1-94x_2,\\ & x_1,x_2=0\quad\text{or}\quad 1.\end{aligned}$$

Its optimal solution is $\bar{x}=(1,0)$ and $\bar{\theta}=333.889$.

Set $\text{UB} \equiv \bar{\theta}$. Since $\text{UB} - \text{LB} = 333.889 - 333.889 = 0$, the search is terminated. The optimal solution to the mixed-integer quadratic program P is $x^* = (1, 0)$, $y^* = (0, 0, 3.778)$, and $f(x^*, y^*) = 333.889$.

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