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On cuts and matchings in planar graphs

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We study the max cut problem in graphs not contractible to K_5, and optimum perfect matchings in planar graphs. We prove that both problems can be formulated as polynomial size linear programs.

Key words: Cut polytope, matching, multicommodity flows.

1. Introduction

The convex hull of the incidence vectors of matchings in a graph has been characterized by Edmonds (1965). In his pioneering paper, Edmonds showed that exponentially many inequalities may be necessary. Several other polytopes related to combinatorial problems have been characterized by systems whose number of inequalities is exponential in the size of the problem. One example is the Cut Polytope for graphs not contractible to K_5, see Barahona and Mahjoub (1986a). In this paper we show that the maximum cut problem in graphs not contractible to K_5, and the optimum perfect matching problem in planar graphs, can be formulated as polynomial size linear programs. For this reason we say that we present compact systems for those problems. A compact system for optimum arborescences has been presented in Wong (1984) and in Maculan (1985). Ball, Liu and Pulleyblank (1987) gave a compact system for two terminal Steiner trees. In Barahona and Mahjoub (1986b, 1987) we presented compact systems for the following problems in series-parallel graphs: stable sets, acyclic induced subgraphs, and bipartite induced subgraphs. In Barahona and Mahjoub (1989) we gave a compact system for stable sets in graphs with no W_4 minor. For matching in a complete graph, Yannakakis (1988) proved that there is no *symmetric* compact system. In Barahona (1988) it was shown that optimum matching in a general graph reduces to a sequence of $O(m^2 \log n)$ minimum mean cycle problems and this last problem admits a compact formulation. We denote by n the number of nodes and by m the number of edges.

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A connected graph G is said to be contractible to a graph H if H can be obtained from G by a sequence of elementary contractions, in which a pair of adjacent vertices is identified and all other adjacencies between vertices are preserved (multiple edges arising from the identification being replaced by single edges). The complete graph on n nodes is denoted by K_n.

Given a graph $G=(V, E)$, and $U \subseteq V$, the set of edges with exactly one endnode in U is called a cut and denoted by $\delta(U)$. The empty set is also a cut. Given a cut C, the incidence vector of C, x^C, is defined by

$$x^C(e)=\begin{cases}1 & \text{if } e \in C,\\ 0 & \text{if } e \notin C.\end{cases}$$

We denote by $P_C(G)$ the convex hull of incidence vectors of cuts of G. Given two sets S and T, their symmetric difference is denoted by $S \triangle T$. For $x \in \mathbb{R}^E$ and $T \subseteq E$ we denote by $x(T)$ the sum $\sum_{e \in T} x(e)$.

In this paper a simple cycle of G will be just called a *cycle*. We call $S(G)$ the following system of inequalities:

$$x(F)-x(C \setminus F) \leqslant |F|-1 \quad \text{for each cycle, } F \subseteq C,\ |F| \text{ odd}, \tag{1.1}$$

$$0 \leqslant x(e) \leqslant 1 \quad \text{for } e \in E. \tag{1.2}$$

Since the intersection between a cut and a cycle has even cardinality, every incidence vector of a cut satisfies (1.1). Moreover, given a cut D its incidence vector satisfies (1.1) as equation only if $|F \cap D|=|F|-1$ and $(C \setminus F) \cap D=\emptyset$, or $|F \cap D|=|F|$ and $|(C \setminus F) \cap D|=1$. These constraints are valid for $P_C(G)$, and are called *cycle inequalities*. In Barahona and Mahjoub (1986a) we proved that G is not contractible to K_5 if and only if $P_C(G)$ is defined by $S(G)$.

In Section 2 we describe Wagner's characterization of graphs not contractible to K_5. In Section 3 we prove that if G is not contractible to K_5 then $P_C(G)$ is defined by $S(G)$, the proof that we had given in Barahona and Mahjoub (1986a) was based on the work of Seymour (1981b) on the matroids with the *sum of circuits property*, the proof we present here does not involve Matroid Theory. In Section 4 we study the integrality of the dual solution and we give an algorithm for multicommodity flows in graphs not contractible to K_5. In Section 5 we study the system of inequalities defined by the odd cycles of a graph. In Section 6 we give a compact system for the max cut problem in graphs not contractible to K_5. In Section 7 we give a compact system for perfect matching in planar graphs. The reader that is only interested in compact systems can skip Sections 2, 3, 4 and 5.

2. Wagner's characterization of graphs not contractible to K_5

Let $G=(V, E)$ be a connected graph, and let $Y \subseteq V$ be a minimal articulation set (that is, the deletion of Y produces a disconnected graph, but no proper subset of Y has this property). Choose nonemtpy subsets T_1, T_2 of V, such that (T_1, Y, T_2)

is a partition of V, and no edge joins a node in T_1 to a node in T_2. Add a set Z of new edges joining each pair of nonadjacent nodes in Y. Let $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$ be subgraphs so that $V_i=T_i\cup Y$, $E_i=E(V_i)\cup Z$, $i=1,2$. Then if $|Y|=k$, $1\leq k\leq 3$, G is called a *k-sum* of G_1 and G_2. Let us notice that this decomposition is not necessarily unique. If Z is empty, the k-sum is called *strict.*

Let us denote by $\mathbb{W}$ the class of connected graphs not contractible to K_5. Wagner (1964, 1970) has shown that any graph $G\in\mathbb{W}$ can be obtained by means of k-sums starting from planar graphs and copies of V_8, which is the graph of Figure 1. This decomposition can be found in polynomial time.

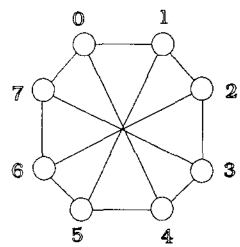

Fig. 1.

3. The cut polytope for graphs not contractible to K_5

In this section we assume that G is a graph not contractible to K_5. We shall prove that $P_C(G)$ is defined by $S(G)$.

Let us denote by $Ax\leq b$ the system $S(G)$, let $w: E\to\mathbb{Z}$ be a weight function. We have to prove that the problem

$$\begin{aligned}&\text{maximize} \quad wx\\ &\text{subject to} \quad Ax\leq b,\end{aligned} \tag{3.1}$$

has an integer optimal solution.

Suppose that for every node u the sum of the weights of the edges incident with u is even. In the next section we shall prove that, under this *evenness condition,* the dual problem has also an integer optimal solution.

Case 1. We first study planar graphs. We need a result of Edmonds and Johnson (1973) about the *Chinese postman problem.* This problem can be defined as follows. given a graph $H=(N,F)$, $T\subseteq N$, and a set of integer weights $d(e)\geq 0$, for $e\in F$,

$$\begin{aligned}&\text{minimize} \quad \sum d(e)x(e)\\ &\text{subject to} \quad \sum_{e\in\delta(v)} x(e)\equiv\begin{cases}1 \pmod 2 & \text{if } v\in T,\\ 0 \pmod 2 & \text{if } v\in V\setminus T,\end{cases}\\ &\qquad\qquad x(e)\in\{0,1\} \quad \text{for } e\in F.\end{aligned} \tag{3.2}$$

They proved that this problem is equivalent to the linear program

$$\begin{aligned} &\text{minimize} \quad \sum d(e)x(e) \\ &\text{subject to} \quad \sum_{e\in\delta(S)} x(e) \geq 1 \quad \text{for every set } S \subseteq N \text{ with } |S \cap T| \text{ odd}, \\ &\qquad\qquad x \geq 0. \end{aligned} \tag{3.3}$$

Assume now that we have a planar graph $G=(V, E)$ embedded in the plane, and a weight function $w: E \to \mathbb{Z}$. We are going to reduce (3.1) to a Chinese postman problem in the dual graph of G.

Define $E_1 = \{e\colon w(e) \leq 0\}$, $E_2 = \{e\colon w(e) > 0\}$, and $d(e) = |w(e)|$, for $e \in E$. Let H be the dual graph of G and T the set of faces D of G (i.e. nodes of H) with $|D \cap E_2|$ odd. Consider problem (3.3) associated to H. Notice that cuts in H correspond to unions of cyles in G. An inequality in (3.3) induced by a cut having a disconnected shore is redundant. Cuts of H with connected shores correspond to (simple) cycles of G, so problem (3.3) becomes

$$\begin{aligned} &\text{minimize} \quad dx \\ &\text{subject to} \quad \sum_{e\in C} x(e) \geq 1 \quad \text{for } C \in \mathbb{C}, \\ &\qquad\qquad x \geq 0. \end{aligned} \tag{3.4}$$

We denote by $\mathbb{C}$ the set of cycles C of G with $|C \cap E_2|$ odd.

The dual problem of (3.4) is

$$\begin{aligned} &\text{maximize} \quad \sum_{C\in\mathbb{C}} y_C \\ &\text{subject to} \quad \sum \{y_C : e \in C\} \leq d(e) \quad \text{for } e \in E, \\ &\qquad\qquad y \geq 0. \end{aligned} \tag{3.5}$$

Seymour (1981a) has proved that under the evenness condition (3.5) has an optimal solution that is integral. In Barahona (1990) we showed that a slight modification of the algorithm of Edmonds and Johnson produces this integer dual solution. From the solution of (3.4) we are going to derive a solution of (3.1).

Let $\bar{x}$ be an integer optimum of (3.4), and $S = \{e\colon \bar{x}(e) = 1\}$. Removing the edges in S breaks all the cycles in $\mathbb{C}$. We can assume that S is minimal with respect to this property because the objective function is nonnegative. Thus the node set V can be partitioned into V_1 and V_2 in such a way that every edge in $E_1 \backslash S$ has both endnodes in V_1 or both in V_2 and every edge in $E_2 \backslash S$ has one endnode in V_1 and the other in V_2. On the other hand every edge in $E_1 \cap S$ has exactly one endnode in V_1, and every edge in $E_2 \cap S$ has both endnodes in V_1 or both in V_2. Hence the vector $\hat{x}$ defined by

$$\hat{x}(e) = \begin{cases} \bar{x}(e) & \text{if } e \in E_1, \\ 1 - \bar{x} & \text{if } e \in E_2, \end{cases}$$

is the incidence vector of a cut. Now we have to show that it is an optimum of (3.1). To see this we shall construct an optimum of the dual problem.

Let us denote by $\beta(C, F)$ the dual variables associated with inequalities (1.1), and by $\gamma(e)$ the dual variable associated with

$$x(e) \leq 1.$$

Let $\bar{y}$ be an optimum of (3.5), let us define

$$\beta(C, F) = \begin{cases} \bar{y}_C & \text{if } F = |C \cap E_2|, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma(e) = \begin{cases} w(e) - \sum \{y_C : e \in C\} & \text{if } e \in E_2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\bar{y}$ satisfies the constraints of (3.5) we have that (β, γ) is a feasible vector for the dual of (3.1). Moreover, if $\bar{y}$ is integer valued then (β, γ) is also integer valued. Let α be the value of the optimum of (3.4), then $w\hat{x} = -\alpha + \sum_{e \in E_2} w(e)$ and (β, γ) has the value. Case 1 is complete. □

Case 2. Let G be the graph V_8.

As in Case 1, we consider the problem (3.4). We need the following lemma.

Lemma 3.1. *For any objective function w there are two nodes that cover all the cycles in* $\mathbb{C}$.

Proof. First, let us remark that if we switch all the signs of the weights of the edges incident with one node, the family $\mathbb{C}$ does not change. Hence we can assume that only edges ij, $j = i+1 \pmod 8$, may belong to E_2.

If the cycle whose nodes are 1, 2, 6, 5 is not in $\mathbb{C}$, we choose the nodes 0 and 3.

Otherwise, we study the cycle defined by 0, 1, 5, 6, 7. If it is not in $\mathbb{C}$ we choose 2 and 4.

Otherwise, we study the cycle defined by 1, 2, 3, 4, 5. If it is not in $\mathbb{C}$ we choose 0 and 6.

If the last three cycles are in $\mathbb{C}$ then the cycle defined by 2, 3, 4, 5, 6 is not in $\mathbb{C}$, we choose 1 and 7.

The proof of this lemma is complete. □

Let p and q be these two nodes, we can split them in such a way that the family $\mathbb{C}$ is a family of two-commodity paths as follows.

First partition the node set of $H = (N, F) = G \setminus \{p, q\}$ into N_1 and N_2 so that edges in $E_2 \cap F$ have exactly one endnode in N_1, and edges in $E_1 \cap F$ have both endnodes in N_1 or both in N_2. Now consider the edges incident with p, let p_1 and p_2 be its copies. If $pu \in E_1$ and $u \in N_1$ then we put the edge p_1u. If $pu \in E_1$ and $u \in N_2$ then we add p_2u. If $pu \in E_2$ and $u \in N_2$ then we put the edge p_1u. If $pu \in E_2$ and $u \in N_1$

then we add p_2u. The edges incident with q are treated in a similar way. This type of construction has been used in Barahona (1983b).

The dual problem (3.5) is a two-commodity flow problem. It follows from the work of Hu (1963) and the theory of blocking polyhedra of Fulkerson (1971) that (3.4) has an integer optimum. Rothschild and Whinston (1966) have proved that under the evenness condition the flow can be chosen to be integer, thus (3.5) has an integer optimum.

The remainder of the proof is the same as for Case 1. □

Case 3. G is a k-sum of two graphs G_1 and G_2, such that $P_C(G_1)$ and $P_C(G_2)$ are defined by $S(G_1)$ and $S(G_2)$ respectively.

We need a way to compose polyhedra. The following theorem appears in Barahona (1983a), the proof of it follows the arguments of Cornuéjols, Naddef and Pulleyblank (1985).

Theorem 3.2. *If G is a strict k-sum of G_1 and G_2, then a system of linear inequalities sufficient to define $P_C(G)$ is obtained from the union of the systems that define $P_C(G_1)$ and $P_C(G_2)$, and by identifying the variables associated with the edges in $G_1 \cap G_2$.*

Proof. Let Q be the polytope defined by the union of these systems. Clearly $P_C(G) \subseteq Q$, so we have to prove that every vector $x \in Q$ is a convex combination of vectors in $P_C(G)$.

Suppose $k=3$ and that e, f and g are the edges in $G_1 \cap G_2$. The restriction x^1 of x to the component set E_1 belongs to $P_C(G_1)$ thus

component set E_1 belongs to $P_C(G_1)$ thus

$$x^1 = \sum_{i \in I} \lambda_i y^i, \quad \text{with } \sum_{i \in I} \lambda_i = 1, \ \lambda \geq 0,$$

and the vectors $\{y^i\}$ are extreme points of $P_C(G_1)$.

Let

$$l_{ef} = \sum \{\lambda_i : i \in I \text{ such that } y^i(e) = y^i(f) = 1\},$$

$$l_{fg} = \sum \{\lambda_i : i \in I \text{ such that } t^i(f) = y^i(g) = 1\},$$

$$l_{eg} = \sum \{\lambda_i : i \in I \text{ that } y^i(e) = y^i(g) = 1\},$$

$$l_0 = \sum \{\lambda_i : i \in I \text{ such that } y^i(e) = y^i(f) = y^i(e) = 0\}.$$

Note that

$$l_{ef} + l_{eg} = x(e), \qquad l_{ef} + l_{fg} = x(f), \qquad l_{eg} + l_{fg} = x(g),$$

$$l_{ef} + l_{eg} + l_{fg} + l_0 = 1.$$

This uniquely determines l_{ef}, l_{eg}, l_{fg} and l_0, given x.

Similarly, for the restriction x^2 of x to E_2, we have

$$x^2 = \sum_{j \in J} \mu_j z^j, \quad \text{with } \sum_{j \in J} \mu_j = 1, \ \mu \geqslant 0,$$

where the vectors $\{z^j\}$ represent cuts of G_2.

Let

$$m_{ef} = \sum \{\mu_j : j \in J \text{ such that } y^j(e) = y^j(f) = 1\},$$

$$m_{fg} = \sum \{\mu_j : j \in J \text{ such that } y^j(f) = y^j(g) = 1\},$$

$$m_{eg} = \sum \{\mu_j : j \in J \text{ such that } y^j(e) = y^j(g) = 1\},$$

$$m_0 = \sum \{\mu_j : j \in J \text{ such that } y^j(e) = y^j(f) = y^j(e) = 0\}.$$

Then

$$m_{ef} + m_{eg} = x(e), \qquad m_{ef} + m_{fg} = x(f), \qquad m_{eg} + m_{fg} = x(g),$$

$$m_{ef} + m_{eg} + m_{fg} + m_0 = 1.$$

This system of equations has a unique solution, then $l_{ef} = m_{ef}$, $l_{fg} = m_{fg}$, $l_{eg} = m_{eg}$ and $l_0 = m_0$. Thus we can match vectors y^i with vectors z^j to form incidence vectors of cuts of G, say $\{\chi^p\}$, and a family of coefficients $\{\beta_p\}$ such that

$$x = \sum \beta_p \chi^p, \qquad \sum \beta_p = 1 \quad \text{and} \quad \beta \geqslant 0.$$

If $k = 2$ the proof is similar, and if $k = 1$, it is obvious. $\square$

Now we have to study the case when the k-sum is not strict, i.e. we need to delete some artificial edges. This situation is considered in the lemma below.

Lemma 3.3. *Let G be a graph such that $P_C(G)$ is defined by $S(G)$ and let e be an edge of G. Then $P_C(G\backslash e)$ is defined by $S(G\backslash e)$.*

Proof. The polytope $P_C(G\backslash e)$ is the projection of $P_C(G)$ along the variable $x(e)$, i.e.

$$P_C(G\backslash e) = \left\{ y: \begin{bmatrix} y \\ x(e) \end{bmatrix} \in P_C(G) \text{ for some value of } x(e) \right\}.$$

In order to obtain a system of inequalites that defines this projection one can apply Fourier–Motzkin elimination to the system $S(G)$, see for instance Schrijver (1986).

Each inequality of the new system is obtained by adding an inequality $ax \leqslant \alpha$ that has the coefficient 1 for $x(e)$ with an inequality $bx \leqslant \beta$ that has the coefficient -1 for this variable. We have to prove that this new constraint is implied by the system $S(G\backslash e)$.

If we add $x(e) \leq 1$ to $x(F) - x(C \setminus F) \leq |F| - 1$, with $e \in C \setminus F$, we obtain

$$x(F) - x(C \setminus [F \cup \{e\}]) \leq |F|,$$

which is implied by $0 \leq x \leq 1$.

If we add $-x(e) \leq 0$ to $x(F) - x(C \setminus F) \leq |F| - 1$, with $e \in F$, we obtain

$$x(F \setminus \{e\}) - x(C \setminus F) \leq |F|,$$

which is also implied by $0 \leq x \leq 1$.

Now consider the inequalities $x(F) - x(C \setminus F) \leq |F| - 1$, and $x(G) - x(D \setminus G) \leq |G| - 1$, with $e \in F \cap \{D \setminus G\}$. The symmetric difference $C \triangle D$ is a union of edge disjoint cycles $C_1, \ldots, C_p$, and the inequality

$$x(F \setminus \{e\}) + x(G) - x(C \setminus F) - x(D \setminus [G \cup \{e\}]) \leq |F| + |G| - 2$$

is implied by inequalities

$$x(F_i) - x(C_i \setminus F_i) \leq |F_i| - 1, \quad F_i \subseteq C_i, \quad F_i \subseteq C_i, \ |F_i| \text{ odd}, \ 1 \leq i \leq p,$$

and

$$0 \leq x(f) \leq 1 \quad \text{for } f \in C \cap D. \qquad \square$$

The proof of Case 3 is complete. □

Until now we have proved that if G is not contractible to K_5 then $P_C(G)$ is defined by $S(G)$. Now consider the graph $K_5 = (V, E)$, the inequality

$$\sum_{e \in E} x(e) \leq 6 \tag{3.6}$$

defines a facet of $P_C(K_5)$, see Barahona and Mahjoub (1986a). If G has a subgraph contractible to K_5, then G contains a subgraph H that has been obtained from K_5 by subdivision of edges and splitting of nodes. In Barahona and Mahjoub (1986a) we gave a construction that derives a facet defining inequality of $P_C(H)$ starting from (3.6) and applying these two operations. All coefficients of this inequality are nonzero, therefore $P_C(H)$ is not defined by $S(H)$. Lemma 3.3 implies that $S(G)$ is not sufficient to define $P_C(G)$.

Now we present a combinatorial algorithm to solve the max cut problem in graphs not contractible to K_5. This algorithm will be needed in the next section.

If the present graph is planar or V_8, we solve the problem as in Cases 1 and 2. Otherwise G is a k-sum of G_1 and G_2; where G_2 is a planar graph or V_8. We need a way to decompose the problem. Let us denote by $\lambda(S, T, H)$ the maximum weight of a cut of the graph H, containing the edge set S and having empty intersection with the edge set T. We write $\lambda(H)$ instead of $\lambda(\emptyset, \emptyset, H)$.

Suppose $k = 3$ and let e, f and g be the edges in $G_1 \cap G_2$.

The edge weights in G_2 are taken to be the same as for G. Then, the max cut problem is solved in G_1 where all the edge weights are taken to be the same as for

G, except for e, f, g, which are redefined as the solution of the following system of linear equations:

$$w'(e)+w'(f)=\lambda(\{e,f\},\emptyset,G_2)-\lambda(\emptyset,\{e,f,g\},G_2),$$

$$w'(f)+w'(g)=\lambda(\{f,g\},\emptyset,G_2)-\lambda(\emptyset,\{e,f,g\},G_2),$$

$$w'(e)+w'(g)=\lambda(\{e,g\},\emptyset,G_2)-\lambda(\emptyset,\{e,f,g\},G_2).$$

The above system reflects the fact that a cut contains 0 or 2 of these edges. Let us remark that if the original weights satisfy the evenness condition, then the new weights in G_1 also satisfy the evenness condition. We have that

$$\lambda(G)=\lambda(G_1)+\lambda(\emptyset,\{e,f,g\},G_2).$$

If $k=2$, let e be the edge in $g_1\cap G_2$. We take in G_2 the same weights as for G. In G_1 we redefine only the weight of e as

$$w'(e)=\lambda(\{e\},\emptyset,G_2)-\lambda(\emptyset,\{e\},G_2).$$

Then $\lambda(G)=\lambda(G_1)+\lambda(\emptyset,\{e\},G_2)$.

If $k=1$, the problem is solved independently in G_1 and G_2.

This algorithm appears in Barahona (1983a), it is an adaptation of an algorithm given by Cornuéjols, Naddef and Pulleyblank (1985), for the traveling salesman problem in graphs with 3-edge cutsets.

4. On the dual integrality of $S(G)$

In this section we shall prove that, under the evenness condition, the dual problem of (3.1) has an integer optimum. For reasons that will be clear later, we assume that the problem is a minimization problem.

We first study the case when the value of the optimum is zero, i.e. there is no cut of negative weight (a negative cut). Since the zero vector is an optimum, the only constraints from (1.1) whose dual variables can take a positive value are

$$x(C\setminus e)-x(e)\geq 0 \quad \text{for every cycle } C,\ e\in C. \tag{4.1}$$

These dual variables satisfy

$$yB\leq w, \qquad y\geq 0, \tag{4.2}$$

where $Bx\geq 0$ denotes the system (4.1).

Consider a multicommodity flow problem defined as follows. A negative value for $w(e)$ represents a demand between the endnodes of e, a nonnegative value for $w(e)$ represents the capacity of e. Let $\mathscr{C}$ be the family of cycles having exactly one edge e with a negative weight $w(e)$. A flow is a function $f:\mathscr{C}\to\mathbb{R}_+$, such that

$$\sum\{f(C)|e\in C\}\begin{cases}\leq w(e) & \text{if } w(e)\geq 0,\\ \geq -w(e) & \text{if } w(e)<0.\end{cases}$$

Lemma 4.1. *There is a flow if and only if there is a vector y that satisfies* (4.2),

Proof. If f is a flow then one can easily obtain from f a vector y that satisfies (4.2).

Now let us assume that y is a vector that satisfies (4.2). Denote by $y(C, e)$ the component of y associated with

$$x(C\backslash e) - x(e) \geqslant 0.$$

If $w(e) \geqslant 0$ and $y(C, e) > 0$ for some cycle C then e is said to be *wrong*. If $w(e) < 0$, $e \in C$, and $y(C, e') > 0$ for $e' \neq e$, then e is also said to be wrong. Otherwise e is called *right*.

Let e be a wrong edge, then $y(C, e) > 0$, $y(D, e') > 0$, and $e \in D$.

Set

$$\varepsilon = \min\{y(C, e), y(D, e')\}, \qquad y(C, e) \leftarrow y(C, e) - \varepsilon,$$

$$y(D, e') \leftarrow y(D, e') - \varepsilon.$$

Let F be the symmetric difference between C and D. The set F is a disjoint union of simple cycles. If e' belongs to one of them let us denote it by H. In this case set

$$y(H, e') \leftarrow y(H, e') + \varepsilon.$$

The new vector y also satisfies (4.2). We apply this procedure until the edge e is right. A right edge cannot become wrong. When every edge is right, we can derive a flow from the vector y. Note that this procedure does not increase the number of nonzero components of the vector y. □

This proves that for graphs not contractible to K_5, there is a flow if and only if there is no negative cut. This result has been proved by Seymour (1981b), he also proved that under the evenness condition the flow can be chosen to be integer. We give below a polynomial algorithm that constructs the flow.

To find those dual variables, we distinguish three cases as in Section 3.

If the graph is planar we solve a Chinese postman problem, see Case 1. If the graph is V_8 we solve a two commodity flow problem, see Case 2.

Now we assume that G is a k-sum of G_1 and G_2, where G_1 is planar or V_8. We apply the algorithm of Section 3 to redefine the capacities in G_1; we denote them by w'. Notice that they will satisfy the evenness condition and that G_1 does not have a negative cut. We can treat G_1 as in Case 1 or Case 2 of Section 3, therefore we can find an integer flow in G_1. Then we have to find a flow in G_2. We redefine the capacities of the edges in $G_1 \cap G_2$ as $w''(a) = w(a) - w'(a)$, for $a \in G_1 \cap G_2$. Notice that the new capacities in G_2 satisfy the evenness condition, and that G_2 will not have a negative cut. Once we have a flow for G_1 and G_2 we put together these two vectors to obtain a vector y that satisfies (4.2), then we apply the procedure of Lemma 4.1 to derive a flow in G. Notice that if the k-sum is not strict then we have some artificial edges of weight zero in $G_1 \cap G_2$, the procedure of Lemma 4.1 will produce a flow that does not use these artificial edges. We continue working recursively with G_2.

Thus finding a flow reduces to a sequence of Chinese postman problems in planar graphs and two-commodity flow problems in V_8. Given a planar graph with p nodes, the Chinese postman problem can be solved in $O(p^{3/2} \log p)$ time, cf. Barahona (1990). Thus if the original graph has n nodes and we have its decomposition, finding a flow takes $O(n^{3/2} \log n)$ time. The most efficient way we know to find the decomposition takes $O(n^2)$ time.

Now we study the case when the value of the optimum cut is negative. Let $\mathfrak{C}$ be a minimum cut with respect to the weights d. Let us define the weights w as follows:

$$w(e) = \begin{cases} -d(e) & \text{if } e \in \mathfrak{C}, \\ d(e) & \text{if } e \notin \mathfrak{C}. \end{cases}$$

It is easy to see that G has no negative cut with respect to w. Then the optimum value of the linear program below is 0.

$$\begin{aligned} &\text{Minimize} \quad wx \\ &\text{subject to} \quad x(C\backslash e) - x(e) \geqslant 0 \quad \text{for every cycle } C, \quad e \in C, \\ &\qquad\qquad\quad x(e) \geqslant 0 \quad \text{for } e \in E. \end{aligned} \tag{4.3}$$

Given a vector $\bar{y}$ that satisfies (4.2), let us denote by $\bar{y}(C, e)$ the variable associated with the inequality (4.1), and let $\bar{w} = w - \bar{y}B$.

Let us define

$$\beta(C, F) = \begin{cases} \bar{y}(C, e) & \text{if } F = (C \cap \mathfrak{C}) \Delta \{e\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma(e) = \begin{cases} \bar{w}(e) & \text{if } e \in \mathfrak{C}, \\ 0 & \text{otherwise,} \end{cases} \qquad \delta(e) = \begin{cases} \bar{w}(e) & \text{if } e \notin \mathfrak{C}, \\ 0 & \text{otherwise.} \end{cases}$$

The vector (β, γ, δ) is a feasible solution of the dual of

$$\begin{aligned} &\text{minimize} \quad dx \\ &\text{subject to} \quad -x(F) + x(C\backslash F) \geqslant 1 - |F| \quad \text{for each cycle } C, \ F \subseteq C, \ |F| \text{ odd}, \\ &\qquad\qquad\quad -x(e) \geqslant -1, \quad x(e) \geqslant 0, \qquad \text{for } e \in E. \end{aligned}$$

This construction has the following interpretation. Let $ax \geqslant 0$ be one inequality of (4.3) which has a positive dual variable. We associate this dual value to the inequality $a'x \geqslant \alpha$, where

$$a'(e) = \begin{cases} -a(e) & \text{if } e \in \mathfrak{C}, \\ a(e) & \text{if } e \notin \mathfrak{C}, \end{cases} \quad \text{and} \quad \alpha = -\sum_{e \in \mathfrak{C}} a(e).$$

If w is a linear combination of the rows of (4.3), involving nonnegative coefficients, then we can use the same coefficients to write d as a linear combination of the rows given by the above procedure.

The incidence vector of $\mathfrak{C}$ together with (β, γ, δ) satisfy the complementary slackness conditions of linear programming.

5. Covering odd cycles

Given a graph $G=(V, E)$, consider the system

$$\begin{aligned} &x(C)\geqslant 1 \quad \text{for every odd cycle } C,\\ &x\geqslant 0. \end{aligned} \tag{5.1}$$

We shall see that if G is not contractible to K_5 then (5.1) defines an integral polyhedron. This has been proved by Fonlupt, Mahjoub and Uhry (1984) using composition techniques for the *Bipartite Subgraph Polytope.* It is an open problem to characterize the graphs that have this property. Graphs having two nodes that cover all the odd cycles have this property, cf. Barahona (1983b).

Suppose that we have weights $d(e)>0$ for every edge e. We have to prove that the linear program

$$\begin{aligned} &\text{minimize} \quad dx\\ &\text{subject to} \quad (5.1) \end{aligned} \tag{5.2}$$

has an integer optimum. Let $\bar{x}$ be an optimum with the additional constraint $x\in\{0,1\}^E$. The set $S=\{e\colon \bar{x}(e)=0\}$ is a maximum cut. Define w as follows:

$$w(e)=\begin{cases} -d(e) & \text{if } \bar{x}(e)=1,\\ d(e) & \text{if } \bar{x}(e)=0. \end{cases}$$

Consider the multicommodity flow problem of Section 4. The graph does not have a negative cut with respect to w, otherwise S would not be maximum. Therefore there is a flow f such that

$$\textstyle\sum f(C)=\sum\{w(e)\,|\,\bar{x}(e)=1\}.$$

If $f(C)>0$ then C contains exactly one edge e with $\bar{x}(e)=1$, every other edge in C belongs to S, therefore C is an odd cycle. This shows that f is a solution of the dual of (5.2) that proves the optimality of $\bar{x}$. In a similar way one can prove that for graphs not contractible to K_5, the system (3.4) defines an integral polyhedron.

6. A compact system for max-cut in graphs not contractible to K_5

Now we shall derive a compact system for the max cut problem in graphs not contractible to K_5. Consider the system $S(G)$, we just need the following three observations:

Remark 6.1. *Let G be an arbitrary graph and e an edge of G. Let us suppose that we eliminate $x(e)$ by applying Fourier–Motzkin elimination to $S(G)$. The system we obtain after deleting some redundant constraints is $S(G\setminus e)$.*

Proof. This is proved in Lemma 3.3. □

Remark 6.2. *If a cycle C has a chord e, then any inequality in S(G) associated with C is a sum of two other inequalities associated with the two new cycles obtained by adding e to C. This proves that the inequalities associated with C are redundant.*

Proof. Consider

$$x(F)-x(C\setminus F)\leqslant|F|-1, \quad F\subseteq C,\ |F| \text{ odd}.$$

Let C_1 and C_2 be the new cycles obtained by adding the chord e to C. Let $F_i=C_i\cap F$, $i=1,2$. Suppose that $|F_1|$ is odd, then the last constraint is the sum of the next two:

$$x(F_1)-x(C_1\setminus F_1)\leqslant|F_1|-1,$$

$$x(e)+x(F_2)-x(C_2\setminus[F_2\cup\{e\}])\leqslant|F_2|. \quad \square$$

Remark 6.3. *If the edge e belongs to a triangle (a cycle of cardinality three), then the inequalities $0\leqslant x(e)\leqslant 1$ are implied by the cycle inequalitiees associated with the triangle.*

Proof. If $\{e, f, g\}$ is a triangle we have the constraints

$$x(e)+x(f)+x(g)\leqslant 2,$$

$$x(e)-x(f)-x(g)\leqslant 0,$$

$$-x(e)+x(f)-x(g)\leqslant 0,$$

$$-x(e)-x(f)+x(g)\leqslant 0.$$

By adding the first two we obtain $2x(e)\leqslant 2$, and adding the last two gives $-2x(e)\leqslant 0$. □

Let K_n be a complete graph, $n\geqslant 3$, we denote by $T(K_n)$ the following system:

$$x(e)+x(f)+x(g)\leqslant 2,$$

$$x(e)-x(f)-x(g)\geqslant 0,$$

$$-x(e)+x(f)-x(g)\leqslant 0,$$

$$-x(e)-x(f)+x(g)\leqslant 0,$$

for every trinagle $\{e, f, g\}$.

It follows from Remarks 6.2 and 6.3 that $S(K_n)$ and $T(K_n)$ define the same polytope.

Let G be subgraph of K_n that is not contractible to K_5. If we apply Fourier-Motzkin elimination to $S(K_n)$, to project the variables associated with the edges in $K_n \backslash G$, we obtain $S(G)$. This follows from Remark 6.1. Therefore, if we apply Fourier-Motzkin elimination to $T(K_n)$, to project the same variables, the system we obtain also defines $P_C(G)$. Let w be a weight function for the edges of G, then the value of

$$\begin{aligned} &\text{maximize} \quad \sum_{e \in G} w(e)x(e) \\ &\text{subject to} \quad x \text{ satisfies } T(K_n), \end{aligned} \tag{6.1}$$

is the value of a max cut of G.

We now state our main result.

Theorem 6.4. *Let G be a subgraph of K_n. The value of the optimum of* (6.1) *is the value of a max cut of G, for every weight function w, if and only if G is not contractible to K_5.* □

7. A compact system for perfect matching in planar graphs

In this section we shall use the results of Section 6 and planar duality to derive a compact system for perfect matching in planar graphs. In all the earlier sections we assumed that the graph has no loops and no parallel edges. In this section the graph may have loops and parallel edges. Parallelism is an equivalence relation. Its equivalence classes will be called parallel classes. Given a graph G, we shall denote by $\bar{G}$ the graph obtained from G by deleting loops and keeping only one representative of each parallel class.

Given a graph G, we call $U(G)$ the system

$$x(e_0) - x(e_i) = 0, \quad i = 1, \ldots, p, \quad \text{for each parallel class } \{e_0, \ldots, e_p\} \text{ of } G,$$

$$\begin{aligned} &x(e) + x(f) + x(g) \leq 2, \\ &x(e) - x(f) - x(g) \leq 0, \\ &-x(e) + x(f) - x(g) \leq 0, \\ &-x(e) - x(f) + x(g) \leq 0, \end{aligned}$$

for every triangle $\{e, f, g\}$ of $\bar{G}$,

$$0 \leq x(e) \leq 1$$

for every edge e of $\bar{G}$ that does not belong to a triangle.

Let $H = (V, E)$ be a planar graph, and G a dual graph of H. Cuts of G correspond to disjoint unions of cycles in H. Let $T \subseteq V$, with $|T|$ even. A T-join is a subgraph of H, such that the nodes in T (resp. not in T) have odd (resp. even) degree. The symmetric difference between two T-joints is a union of cycles. The symmetric difference between a T-join and a disjoint union of cycles is also a T-join. Let us add edges to G until $\bar{G}$ is complete. Denote by x the variables associated with the original edges and by z the variables for the new edges. The system $U(G)$ is a compact system for unions of cycles (cuts of the dual graph) of H. Let F be a (fixed) T-join of H. Let us define

$$y(e)=\begin{cases} x(e) & \text{if } e \notin F, \\ 1-x(e) & \text{if } e \in F. \end{cases}$$

This transformation will be denoted by $y = Dx + h$. The matrix D is nonsingular and h is a 0–1 vector.

Let w be a weight function for the edges of H. We obtain the value of a minimum weighted T-join by solving

$$\begin{aligned} &\text{minimize} && \sum_{e\in H} w(e)y(e) \\ &\text{subject to} && y = Dx + h \\ &&& (x, z) \text{ satisfies } U(G). \end{aligned}$$

If $T = V$ then the perfect matching polytope is the face of the T-join polytope defined by

$$\sum_{e\in H} y(e) = \tfrac{1}{2}|V|.$$

Thus if w is a weight function for the edges of H, the minimum weight of a perfect matching in H can be obtained by setting $T = V$, and solving

$$\begin{aligned} &\text{minimize} && \sum_{e\in H} w(e)y(e) \\ &\text{subject to} && y = Dx + h, \\ &&& (x, z) \text{ satisfies } U(G), \\ &&& \sum_{e\in H} y(e) = \tfrac{1}{2}|V|. \end{aligned}$$

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