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Polyhedral study of the capacitated vehicle routing problem

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The capacitated vehicle routing problem (CVRP) considered in this paper occurs when goods must be delivered from a central depot to clients with known demands, using k vehicles of fixed capacity. Each client must be assigned to exactly one of the vehicles. The set of clients assigned to each vehicle must satisfy the capacity constraint. The goal is to minimize the total distance traveled. When the capacity of the vehicles is large enough, this problem reduces to the famous traveling salesman problem (TSP). A variant of the problem in which each client is visited by at least one vehicle, called the graphical vehicle routing problem (GVRP), is also considered in this paper and used as a relaxation of CVRP. Our approach for CVRP and GVRP is to extend the polyhedral results known for TSP. For example, the subtour elimination constraints can be generalized to facets of both CVRP and GVRP. Interesting classes of facets arise as a generalization of the comb inequalities, depending on whether the depot is in a handle, a tooth, both or neither. We report on the optimal solution of two problem instances by a cutting plane algorithm that only uses inequalities from the above classes.

Key words: Vehicle routing, polyhedron, facet, branch and cut.

1. Introduction

The term Vehicle Routing refers to a wide class of managerial problems in which a fleet of vehicles located at a central depot is dispatched to visit a set of clients with known requirements for pick up or delivery. For example, it arises in situations of public concern such as school bus routing, mail delivery, and garbage collection. In the private sector as well, the importance of this problem has been demonstrated. A study [18] reports annual distribution costs of about $400 billion in the United States. As a basic example, consider the following problem: given (i) a road network and travel costs on each link, (ii) a fleet of identical vehicles with given capacity located at a central depot, and (iii) client demands and locations, construct routes

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for the vehicles in order to meet the client demands at minimum travel cost while satisfying the vehicle capacity constraints. Two versions of the problem arise depending on whether the client demands can be split among several vehicles or not. Traditionally, such problems have been addressed using heuristics, see, for instance, Toth [32], Magnanti [26], and Christofides [4]. Recently, however, some instances of significant size were solved to optimality. For example, Fisher [9] describes a branch-and-bound algorithm based on a Lagrangian relaxation. In [24], Laporte, Nobert and Desrochers give a cutting plane algorithm based on inequalities related to the so-called subtour elimination constraints. In spirit, the approach in [24] is similar to the groundbreaking work of Dantzig, Fulkerson and Johnson [8] on the Traveling Salesman Problem (TSP). First, a relaxed linear program is solved. Then the relaxation is strengthened by adding a set of violated inequalities to the formulation. Finally, a branch-and-bound algorithm is applied, based on the resulting linear programming bounds. Other exact approaches have been developed to solve vehicle routing problems and are discussed in the surveys of Christofides [4], Laporte and Nobert [23], Bodin, Golden, Assad and Ball [2].

For some time now, polyhedral combinatorics has established itself as a powerful tool for solving combinatorial optimization problems. In particular, impressive results have been obtained with the TSP. This problem, the most basic version of the vehicle routing problem (one vehicle with infinite capacity), has received special attention throughout the years. The book [25] provides an excellent review of the literature on the TSP. In the last 10 to 15 years, a better understanding of the underlying integer polyhedron has paved the way for the solution of large instances of the TSP. Instead of searching for violated inequalities among subtour elimination constraints only, other classes of valid inequalities have been considered, such as comb and clique tree inequalities, and have been found to be effective in strengthening the linear programming relaxation (Chvátal [6], Grötschel [15], Grötschel and Padberg [16], Padberg and Hong [30], Grötschel and Pulleyblank [17], Padberg and Rinaldi [31]). Despite the obvious analogy between the TSP and the vehicle routing problem, very little is known about the structure of the vehicle routing polyhedron. The motivation for this paper stemmed from the desire to narrow this gap by exploiting the knowledge about the facial structure of the traveling salesman polyhedron and the belief that, for some vehicle routing problems, a better understanding of the underlying integer polyhedron will also lead to the solution of larger instances. It turns out, as will be seen later, that there are striking structural similarities between these two combinatorial problems. Initial investigations along these lines were performed by Fleischmann [10], Laporte [20], Laporte and Nobert [22], Laporte and Bourjolly [21], Laporte, Nobert and Desrochers [24].

We briefly outline the notation used in this paper. Given an undirected graph $G=(V, E)$ and $W \subseteq V$, we denote by $E(W)$ the set of edges with both ends in W and by $\delta(W)$ the set of edges with exactly one endnode in W. For the sake of brevity, $\delta(v)$ denotes $\delta(\{v\})$ for $v \in V$. The graph $(W, E(W))$ induced by the node set W is denoted by $G(W)$. Given edges $e_1, \ldots, e_t \in E$, not necessarily distinct, and given the multiset $T=\{e_1,\ldots,e_t\}$, we denote by $V(T)$ the set of nodes incident with at least one edge of T. The graph $(V(T), T)$ induced by the multiset of edges T is denoted by $G(T)$. A graph is a *simple cycle* if it is connected and all its nodes have degree 2; it is a *cycle* if it is connected and all its nodes have even degrees. An edge with endnodes u and v is denoted by uv. For $U, W \subseteq V$ such that $U \cap W = \emptyset$, then (U, W) denotes the set of edges uw where $u \in U$ and $w \in W$. If J is a finite set, then $\mathbb{R}^J$ denotes the set of vectors $x=(x_j, j\in J)$ where x_j is a real number for each $j \in J$. Given elements $j_1,\ldots,j_t \in J$, not necessarily distinct, and the multiset $U=\{j_1,\ldots,j_t\}$, then x^U denotes the vector of $\mathbb{R}^J$ where x_j^U is the number of times that $j\in J$ appears in the multiset U. In the case where $U\subseteq J$ is a set, then x^U simply represents the characteristic vector of the set U. Given a multiset $U=\{j_1,\ldots,j_t\}$ and a vector $c\in\mathbb{R}^J$ then $c(U)$ denotes $\sum_{i=1}^{t} c_{j_i}$. In the case where $U\subseteq J$ is a set, then this notation states that $c(U)=\sum_{j\in U} c_j$.

Consider an undirected graph $G=(V, E)$, where $|V|=n+1$. A distinguished node $v_0 \in V$ is called the *depot*. The nodes $j\in V\backslash\{v_0\}$ are called the *clients*. A *cost* vector $c\in\mathbb{R}^E$ indexes the edges of G and a positive *demand* vector $d\in\mathbb{R}_+^{V\backslash\{v_0\}}$ indexes the clients. Finally, consider k *vehicles*, each with *capacity* $C>0$. A multiset of edges T is said to be a *simple k-tour* if it can be partitioned into k nonempty multisets $T_1,\ldots,T_k$ such that

(i) for $i=1,\ldots,k$ the graph $G(T_i)$ is a simple cycle and $v_0\in V(T_i)$;
(ii) each $j\in V\backslash\{v_0\}$ belongs to exactly one of the sets $V(T_i)$, $1\leqslant i\leqslant k$;
(iii) for $i=1,\ldots,k$, $d(V(T_i)\backslash\{v_0\})\leqslant C$.

The multiset T_i is called the ith *route* of the simple k-tour and the clients in $V(T_i)\backslash\{v_0\}$ are said to be *assigned* to the route T_i. Note that the only purpose of multisets in the definition of a simple k-tour is to allow for routes with a single client.

Conditions (i) and (ii) imply that v_0 is incident with exactly $2k$ edges of T and each $j\in V\backslash\{v_0\}$ is incident with exactly two edges of T. Checking whether there exists T satisfying conditions (i) and (ii) in a general graph G is NP-complete, even for $k=1$ (it is a Hamiltonian cycle problem). So it is usually assumed in the literature that the graph G is complete.

Condition (iii) above guarantees that the sum of the demands of the clients assigned to a route does not exceed the vehicle capacity. Checking whether this condition can be satisfied is a bin packing problem (see [14]) with "items" of "size" d_j and identical "bins" of capacity C. So finding a simple k-tour, even when G is a complete graph, is an NP-hard problem.

The *capacitated vehicle routing problem*, denoted by CVRP, is to construct a simple k-tour T of minimum cost $c(T)$, when the underlying graph G is complete. In the case where $k=1$ and $C\geqslant d(V\backslash\{v_0\})$, CVRP reduces to the TSP. When we need to refer to the case where G is a general graph, we will use the notation CVRP(G).

For a set $S\subseteq V\backslash\{v_0\}$ of clients, we denote by $r(S)$ the smallest number of vehicles needed to meet the demands of the clients in S, given the vehicle capacity C and the client demands d_j, $j\in S$. In other words, $r(S)$ is the solution of a bin packing problem with items of size d_j, for $j\in S$, and identical bins of capacity C. Note that the clients that are not in S are ignored when computing $r(S)$. A related quantity, denoted by $R(S)$, is the smallest integer t such that $S_1, \ldots, S_t, \ldots, S_k$ is a partition of $V \backslash \{v_0\}$ satisfying $d(S_i) \leqslant C$, for $1 \leqslant i \leqslant k$, and $S \subseteq \bigcup_{i=1}^{t} S_i$. Clearly

$$\lceil d(S)/C \rceil \leqslant r(S) \leqslant R(S)$$

where $\lceil a \rceil$ denotes the smallest integer at least as large as a. The inequalities above can be strict. For example, let $k=4$, $C=7$, $d_1=5$, $d_2=d_3=d_4=3$, $d_5=2$, $d_6=d_7=d_8=4$ and consider $S=\{1,2,3,4\}$. It is easy to check that $\lceil d(S)/C \rceil = 2$, $r(S)=3$ and $R(S)=4$.

An integer programming formulation of CVRP can be stated as follows.

$$\text{Min} \quad cx \tag{1}$$

$$\text{s.t.} \quad x(\delta(v_0)) = 2k, \tag{2}$$

$$x(\delta(v)) = 2 \quad \text{for } v \in V \backslash \{v_0\}, \tag{3}$$

$$x(\delta(S)) \geqslant 2r(S) \quad \text{for } S \subseteq V \backslash \{v_0\},\ S \neq \emptyset, \tag{4}$$

$$x_e \in \{0, 1\} \quad \text{for } e \in E(V \backslash \{v_0\}), \tag{5}$$

$$x_e \in \{0, 1, 2\} \quad \text{for } e \in \delta(v_0). \tag{6}$$

Expressions (2) and (3) are the *degree constraints*, which state that each route must start and end at the depot and that each client must be visited by exactly one vehicle. Expressions (4), called the *capacity constraints*, ensure the connectivity of the graph induced by each vehicle route and impose the capacity requirements. Finally, conditions (5) and (6) specify the integrality restrictions on the variables. Condition (6) allows for routes containing a single client.

Computing $r(S)$ in the capacity constraints (4) requires the solution of a bin packing problem. So one may ask whether a "simpler" integer programming formulation exists for this problem. In fact, a valid integer programming formulation of CVRP would still be obtained by replacing $r(S)$ in (4) by $\lceil d(S)/C \rceil$. The reason is that the edges with $x_e > 0$ still induce k simple cycles T_i. Let S_i be the set of clients assigned to T_i. The capacity constraints $x(\delta(S)) \geqslant 2\lceil d(S)/C \rceil$ applied to the sets $S=S_i$, imply $d(S_i) \leqslant C$ for $1 \leqslant i \leqslant k$. So it is valid to replace $r(S)$ by $\lceil d(S)/C \rceil$ in (4). At the opposite, one may argue that a more useful formulation will be obtained if the right-hand side of (4) is increased to its maximum value, namely $2R(S)$. The only reason for not using the value $R(S)$ in (4) is that it may be harder to compute than $r(S)$. Although $r(S)$ is also NP-hard to compute, it is often relatively easy to obtain for typical values of k such as $2 \leqslant k \leqslant 10$. When neither $R(S)$ not $r(S)$ can be computed within a reasonable amount of time, then $\lceil d(S)/C \rceil$ can be used instead.

It should be emphasized that although the above formulation is closely related to that of the TSP, the vehicle capacities introduce new complexities. In fact, just checking whether the above vehicle routing problem has a feasible solution is NP-complete, even under our assumption that the underlying graph G is complete.

This is because checking whether k vehicles suffice to meet the client demands is a bin packing problem, as pointed out earlier. It follows that finding the dimension of the capacitated vehicle routing polyhdron CVRP $\equiv$ conv$\{x \in \mathbb{R}^E$: x satisfies (2)-(6)$\}$, is NP-hard. Furthermore, due to constraints (2) and (3), the polyhedron CVRP is never full dimensional which renders the situation even more complicated. As usual when dealing with a non full dimensional polyhedron, it is convenient to embed it into a full dimensional one which contains it as a face. This approach has the merit of simplifying the analysis and of providing a useful structural insight. For instance, the study of the graphical relaxation of the traveling salesman polyhedron (Cornuejols, Fonlupt and Naddef [7], Fleischmann [11], Fonlupt and Naddef [12], Naddef and Rinaldi [27, 28]) has lead to the discovery of new valid inequalities and facets for the TSP, such as the path inequalities and the path-tree inequalities. The recent results of Naddef and Rinaldi bring out clearly some useful connections between the classical traveling salesman polyhedron and the graphical traveling salesman polyhedron. In addition to the convenience for analysis, there are practical reasons for relaxing the CVRP formulation.

Frequently, in applications, there is no reason to require that each client be on exactly one route. Instead, it is sufficient (and sometimes desirable) that each client be on *at least* one route. Two versions of the problem which are of interest occur depending on whether the client demands can be split among several vehicles or not. Both cases arise in practice. Here, we restrict our attention to the case where demand splitting is not allowed. Another practical consideration is that the graph $G=(V, E)$ usually represents a sparse network with some structure (for example a planar road network) and therefore it may not be desirable to replace it by a complete graph, as we have assumed in defining CVRP. Now we will define the graphical vehicle routing problem for a general graph $G=(V, E)$.

A multiset of edges T is said to be a *k-tour* if it can be partitioned into nonempty multisets $T_1, \ldots, T_k$ so that

(i) for $i=1, \ldots, k$, the graph $G(T_i)$ is a cycle (not necessarily simple) and $v_0 \in V(T_i)$; T_i is called the ith *route*;

(ii) each client $j \in V\backslash\{v_0\}$ belongs to at least one of the sets $V(T_i)$, $1 \leq i \leq k$;

(iii) each $j \in V\backslash\{v_0\}$ is assigned to exactly one of the routes T_i such that $j \in V(T_i)$;

(iv) each route T_i has a nonempty set S_i of clients assigned to it, and for $i=1, \ldots, k$, $d(S_i) \leq C$.

The *graphical vehicle routing problem*, denoted by GVRP, is to construct a k-tour T of minimum cost $c(T)$. The vector $x^T \in \mathbb{R}^E$ where x_e^T denotes the number of times that the edge e appears in the multiset T is also called a k-tour. The convex hull of the k-tours will be denoted by GVRP(G). When G is a complete graph, this polyhedron will simply be denoted by GVRP. Since every simple k-tour is also a k-tour, it follows that CVRP $\subseteq$ GVRP. In fact, CVRP is a face of the polyhedron GVRP. It is worth noting that the knowledge of a k-tour T (or x^T) is usually not sufficient to uniquely determine which clients are assigned to each route, nor even what the routes are. In other words, for a k-tour T the partition $T_1, \ldots, T_k$ may not be unique. This is unlike the situation for CVRP where T is sufficient to define the k routes completely. It is not hard to see that if an integral vector x is a k-tour, then it satisfies

(i) $x_e \geqslant 0$ for $e \in E$,
(ii) $x(\delta(v_0)) \geqslant 2k$ and even,
(iii) $x(\delta(v)) \geqslant 2$ and even, for $v \in V \backslash \{v_0\}$,
(iv) $x(\delta(S)) \geqslant 2r(S)$ for $S \subseteq V \backslash \{v_0\}$, $S \neq \emptyset$.

However, the converse is not true as pointed out by Fleischmann [10]. Consider the following example (Figure 1.1), where $k=2$, $C=4$, and the number besides each client node j indicates its demand d_j. It is easily verified that the solution $x_e=1$ for $e \in E$ satisfies conditions (i)-(iv) but is not a 2-tour. Note that changing $r(S)$ into $R(S)$ in (iv) would not resolve this difficulty.

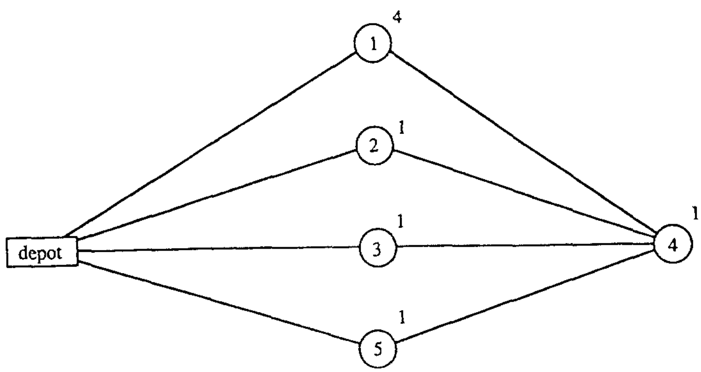

Fig. 1.1.

Given an inequality $fx \geqslant f_0$ defined on $\mathbb{R}^E$, the following sets will be useful throughout the paper.

$$\boldsymbol{H}^f = \{T\text{: } T \text{ is a simple } k\text{-tour and } fx^T = f_0\},$$

$$\boldsymbol{T}^f = \{T\text{: } T \text{ is a } k\text{-tour and } fx^T = f_0\}.$$

The basic polyhedral properties of CVRP and GVRP(G) are given in Section 2, including conditions under which the capacity constraints yield facets. Section 3 deals with path inequalities for GVRP(G). Section 4 describes the related comb inequalities for CVRP. Finally Section 5 illustrates the use of these inequalities as cutting planes. Although several of the conditions introduced in this paper to guarantee that an inequality defines a facet are hard to check, it must be stressed that these conditions need not be checked in a cutting plane algorithm since the inequalities we consider are always valid.

The reader not familiar with the basics of polyhedral theory is invited to consult [25] or [29].

2. Basic properties

In this section our main interest is to develop basic polyhedral properties of both GVRP(G) and CVRP. For that purpose, it is assumed in the remainder of this paper that the bin packing problem defined by demands d_j, $j=1,\ldots,n$, and bin capacity C has a feasible solution with k bins.

Theorem 2.1. *Let $G=(V,E)$ be a connected graph. Then* $\dim \mathrm{GVRP}(G)=|E|$. *If $fx \geqslant f_0$ is a valid inequality for* GVRP(G), *then $f_e \geqslant 0$ for $e \in E$.*

Proof. Our assumptions (G is connected and there exists a feasible solution to the bin packing problem) imply the existence of a k-tour, say $\bar{x}$. We produce $|E|$ additional affinely independent k-tours z^e as follows: $z^e=\bar{x}+2y^e$ where y^e is the unit vector such that $y^e_e=1$ and $y^e_j=0$ for $j \neq e$. This shows that $\dim \mathrm{GVRP}(G)=|E|$.

Since $fx \geqslant f_0$ is valid, $f\bar{x} \geqslant f_0$. Assume $f_e<0$ for some $e \in E$. Then $\bar{x}+2My^e$ is also a k-tour and $f(\bar{x}+2My^e)<f_0$ for a large enough positive integer M, a contradiction. □

Theorem 2.2. *Let $G=(V,E)$ be a connected graph. The nonnegativity constraint $x_e \geqslant 0$ defines a facet of* GVRP(G) *if and only if the graph $(V, E\backslash\{e\})$ is a connected graph.*

Proof. Suppose that $(V, E\backslash\{e\})$ is connected and let $\bar{x}$ be some k-tour of G such that $\bar{x}_e=0$. From $\bar{x}$ we construct $|E|-1$ k-tours z^j as follows: $z^j=\bar{x}+2y^j$, $j \neq e$, where y^j is the unit vector such that $y^j_j=1$ and $y^j_i=0$ for $i \neq j$. Hence we have produced $|E|$ affinely independent k-tours satisfying $x_e=0$.

Now, suppose that $(V, E\backslash\{e\})$ is disconnected. Let $S \subseteq V\backslash\{v_0\}$ be the node set of one of the connected components of $(V, E\backslash\{e\})$. Since $x_e \geqslant 2r(S)>0$ in every k-tour, the inequality $x_e \geqslant 0$ is dominated. □

Now we turn our attention to the capacity constraints. The inequality $x(\delta(W)) \geqslant 2R(W)$ where $W \subseteq V\backslash\{v_0\}$, $W \neq \emptyset$, always defines a nonempty face of GVRP but this may not be the case for GVRP(G) when G is not complete. For example, when G is a star graph with edges v_0v_i for $i=1,\ldots,n$, the inequality $x(\delta(W)) \geqslant 2|W|$ is valid and may strictly dominate $x(\delta(W)) \geqslant 2R(W)$. The next theorem gives simple conditions guaranteeing that $x(\delta(W)) \geqslant 2R(W)$ defines a facet of GVRP(G).

Theorem 2.3. *Let G be a connected graph and let $W \subseteq V\backslash\{v_0\}$, $W \neq \emptyset$.*

(a) *If $G(W)$ and $G(V\backslash W)$ are connected graphs, then the inequality $x(\delta(W)) \geqslant 2R(W)$ defines a facet of* GVRP(G).

(b) *If $G(V\backslash W)$ is a disconnected graph, then the inequality $x(\delta(W)) \geqslant 2R(W)$ does not define a facet of* GVRP(G).

(c) *If* $G(W)$ *is a disconnected graph, then the inequality* $x(\delta(W)) \geq 2r(W)$ *does not define a facet of* GVRP(G).

Proof. (a) Assume that both $G(W)$ and $G(V\backslash W)$ are connected. Then, it follows from the definition of $R(W)$ that a k-tour such that $x(\delta(W)) = 2R(W)$ exists. In fact, for each $j \in \delta(W)$, we can construct a k-tour x^j such that $x^j_j = 2R(W)$ and $x^j_h = 0$ for every $h \in \delta(W)\backslash\{j\}$. Let x' be one of the $|\delta(W)|$ k-tours just defined. We shall produce $|E\backslash\delta(W)|$ additional k-tours z^e, $e \in E\backslash\delta(W)$, from x' as follows: $z^e = x' + 2y^e$, for $e \in E\backslash\delta(W)$, where y^e is the unit vector such that $y^e_e = 1$ and $y^e_i = 0$ for $i \neq e$. Thus, we have $|E|$ affinely independent k-tours all satisfying $x(\delta(W)) = 2R(W)$, which completes the proof of (a).

(b) Assume that $G(V\backslash W)$ is disconnected. Let W_3 be the node set of a connected component of $G(V\backslash W)$ that does not contain v_0. Since G is connected, W_3 is connected to W. This implies that $\delta(W \cup W_3) \subset \delta(W)$ and the inclusion is strict. Therefore, $x(\delta(W \cup W_3)) \geq 2R(W \cup W_3)$ dominates $x(\delta(W)) \geq 2R(W)$, showing that this inequality is not facet inducing.

(c) Assume that $G(W)$ is disconnected, let $W_1 \subset W$ be a node set which induces a connected component of $G(W)$ and let $W_2 = W\backslash W_1$. Then, the edge cutset $\delta(W)$ is partitioned into $\delta(W_1) \cup \delta(W_2)$. Thus $x(\delta(W)) \geq 2r(W)$ is identical to or dominated by the sum of the valid inequalities $x(\delta(W_i)) \geq 2r(W_i)$ for $i = 1, 2$ since $r(W_1 \cup W_2) \leq r(W_1) + r(W_2)$. It follows that it is not a facet defining inequality. □

Note that, in the special cases where $|W| = 1$ or n, we get necessary and sufficient conditions for the degree constraints to be facet inducing for GVRP(G).

Naddef and Rinaldi [28] observed a fundamental relation between the traveling salesman polyhedron (TSP) and the graphical traveling salesman polyhedron (GTSP) on a complete graph. They showed that, under some conditions, facets of GTSP also define facets for TSP. As will be seen below, a similar relation persists to a large extent in our context. For that purpose, it is worthwhile to review some basic concepts which were essential in [28]. We begin by defining the tightly triangular inequalities in our setting.

In what follows, it is assumed that the graph G is complete.

Definition 2.4. An inequality $fx \geq f_0$ defined on $\mathbb{R}^E$ is said to be *tightly triangular* (henceforth abbreviated by TT) if:

(i) $f_{vw} \leq f_{uv} + f_{uw}$ for every triplet u, v, w of distinct nodes of V,

(ii) $\Delta_f(u) \equiv \{vw \in E: v, w \in V\backslash\{u\}$ and $f_{vw} = f_{uv} + f_{uw}\}$ is nonempty for all $u \in V$.

It is easy to check that the nonnegativity constraints $x_e \geq 0$, $e \in E$, and the degree constraints $x(\delta(v_0)) \geq 2k$ and $x(\delta(v)) \geq 2$, $v \in V\backslash\{v_0\}$, are not TT inequalities. The following analogue of Proposition 2.2 in [28] shows that all other facet inducing inequalities for GVRP are TT inequalities.

Proposition 2.5. *A facet inducing inequality* $fx \geq f_0$ *of* GVRP *is either an inequality* $x_e \geq 0$, *a degree inequality, or a TT inequality.*

Proof. Let $fx \geq f_0$ be a facet inducing inequality for GVRP. If condition (i) is not satisfied then there exists a triplet u, v, w of distinct nodes such that $f_{vw} > f_{uv} + f_{uw}$. Assume there exists a k-tour $T \in \boldsymbol{T}^f$ with $x^T_{vw} > 0$. Denote the routes of T by T_i, $i = 1, \ldots, k$, and assume w.l.o.g. that T_1 contains edge vw. Construct from T_1 another route $T'_1 = (T_1 \backslash \{vw\}) \cup \{uv, uw\}$ but do not reassign the demand of node u. It follows that the resulting k-tour T' with routes $T'_1, T_2, \ldots, T_k$ is feasible and $fx^{T'} < f_0$, a contradiction. Thus $x^T_{vw} = 0$ for every $T \in \boldsymbol{T}^f$. Therefore, the facet inducing inequality $fx \geq f_0$ must be the inequality $x_{vw} \geq 0$.

Now, suppose $\Delta_f(u)$ is empty for some $u \in V$ but condition (i) holds. Then for every pair of distinct nodes v, w in $V \backslash \{u\}$, $f_{uv} + f_{uw} > f_{vw}$. By condition (i), we have $f_{uv} + f_{wv} \geq f_{uw}$ and therefore we deduce $f_{uv} > 0$. Assume that u is distinct from the depot (u is the depot, respectively) and suppose there exists a k-tour $T \in \boldsymbol{T}^f$ such that $x^T(\delta(u)) \geq 4$ ($\geq 2k+2$, respectively). Let us first suppose that u has degree 4 or greater in one of the graphs $G(T_i)$, where T_i, $1 \leq i \leq k$, are the routes of T. $G(T_i)$ being a connected Eulerian graph, there is a cycle Γ that contains each edge of T_i exactly once, say $\Gamma = (\ldots, y, e_t, u, e_{t+1}, z, \ldots)$. Define $T'_i = (T_i \backslash \{e_t, e_{t+1}\}) \cup \{yz\}$ when $y \neq z$ and $T'_i = T_i \backslash \{e_t, e_{t+1}\}$ when $y = z$. In both cases, the degrees of the nodes in $G(T'_i)$ remain even. Furthermore, $G(T'_i)$ is still connected. So T'_i still induces a connected route. The new k-tour T' with routes T'_j, $j = 1, \ldots, k$ where $T'_j = T_j$ for $j \neq i$, satisfies $fx^{T'} < f_0$, contradiction. (When $y = z$ above, we use $f_{uy} > 0$ and when $y \neq z$, we use $f_{uy} + f_{uz} > f_{yz}$.) Next, suppose that $u \in V \backslash \{v_0\}$ belongs to two different routes T_i, T_j. Since no demand splitting occurs in problem GVRP, the demand d_u is assigned to only one route, say T_j. Then consider the other route T_i and proceed as before to find y, z, T'_i and finally T' such that $fx^{T'} < f_0$. Hence, in every k-tour of $\boldsymbol{T}^f$, node u has degree 2 ($2k$, respectively) and the facet inducing inequality must be $x(\delta(u)) \geq 2$ ($x(\delta(u)) \geq 2k$, respectively).

This shows that the degree constraints are the only facets of GVRP which satisfy (i) but not (ii) in the definition of a TT inequality. □

For the traveling salesman problem, dim TSP $= |E| - |V|$ and dim GTSP $= |E|$. For CVRP, the situation is slightly more complex. We still have dim GVRP $= |E|$ by Theorem 2.1, but we can only state that dim CVRP $\leq |E| - |V|$. This follows from the observation that, in a complete graph with more than two nodes, the degree constraints are affinely independent but, here, the capacity constraints may reduce further the dimension of the polyhedron CVRP. To guarantee that CVRP has the maximum possible dimension we generalize notions that appeared in [28].

Definition 2.6. A k-tour T is an *almost simple k-tour* if either

(i) node v_0 has degree $2k+2$ in the graph $G(T)$ and every node of $V \backslash \{v_0\}$ has degree 2, or

(ii) node v_0 has degree $2k$, some node $u \in V \backslash \{v_0\}$ has degree 4 and every node of $V \backslash \{v_0, u\}$ has degree 2.

Let $fx \geq f_0$ be a facet inducing inequality of GVRP. A *basis* of $fx \geq f_0$ is a set of $|E|$ affinely independent points in $\boldsymbol{T}^f$. A basis is *canonical* if it contains $|E|-|V|$ simple k-tours and $|V|$ almost simple k-tours.

The next result follows as a consequence of this definition.

Lemma 2.7. *Let $fx \geq f_0$ be a facet inducing inequality of* GVRP. *If $fx \geq f_0$ has a canonical basis and there exists at least one simple k-tour $\bar{x}$ such that $f\bar{x} > f_0$, then $fx \geq f_0$ also induces a facet of* CVRP. *Furthermore,* dim CVRP $= |E|-|V|$.

Proof. If $fx \geq f_0$ has a canonical basis, then by definition, there are $|E|-|V|$ affinely independent simple k-tours satisfying $fx = f_0$. Since CVRP is not entirely contained in the hyperplane $fx = f_0$, it follows that dim CVRP $\geq |E|-|V|$. Since dim CVRP $\leq |E|-|V|$ always holds, we conclude that dim CVRP $= |E|-|V|$ and that $fx \geq f_0$ induces a facet of CVRP. □

Definition 2.8. Let $e = vw$ and $h = yz$ be two distinct edges and u a node of $V \backslash \{v, w, y, z\}$. We say that e and h are *f-adjacent* in u if

(i) e and h belong to $\Delta_f(u)$,

(ii) there exists an almost simple k-tour, say $T \in \boldsymbol{T}^f$, that contains the edges uv, uw, uy and uz,

(iii) $(T \backslash \{uv, uw\}) \cup \{vw\}$ and $(T \backslash \{uy, uz\}) \cup \{yz\}$ are simple k-tours.

A set of edges $J_u \subseteq \Delta_f(u)$ is *f-connected* in u if $|J_u| \geq 2$ and for every pair of distinct edges $h_1, h_2 \in J_u$, there exists a sequence $e_1, \ldots, e_r$ of edges in J_u with $e_1 \equiv h_1$ and $e_r \equiv h_2$, such that e_i is f-adjacent in u to e_{i+1}, for $i = 1, \ldots, r-1$.

Lemma 2.9. *Let $fx \geq f_0$ be a facet inducing TT inequality for* GVRP, *and let $J_u \subseteq \Delta_f(u)$ be f-connected in u. Then, there exists an almost simple k-tour $T \in \boldsymbol{T}^f$ such that, for every edge $vw \in J_u$, the vector $s^{uvw} \in \mathbb{R}^E$ (called* shortcut*) defined by*

$$s_e^{uvw} = \begin{cases} 1 & \text{if } e = vw, \\ -1 & \text{if } e = uv \text{ or } uw, \\ 0 & \text{otherwise,} \end{cases}$$

can be expressed as a linear combination of x^T and points x^H for H in $\boldsymbol{H}^f$.

Proof. Let $yz \in J_u$. From Definition 2.8, it follows that the shortcut s^{uyz} can be expressed as

$$s^{uyz} = x^H - x^T,$$

where $T \in \boldsymbol{T}^f$ is an almost simple k-tour containing uy, uz and $H = (T \backslash \{uy, uz\}) \cup \{yz\}$ belongs to $\boldsymbol{H}^f$. So the lemma holds for edge yz.

Now let $vw = h_2$ be an edge of J_u distinct from $yz = h_1$. Note that h_1 and h_2 can have a common endpoint. Since h_1 and h_2 are f-connected in u, there exists a sequence $h_1 \equiv e_1, \ldots, e_r \equiv h_2$ such that e_i is f-adjacent in u to e_{i+1} for $i = 1, \ldots, r-1$. Let $e_i = v_i w_i$ and $e_{i+1} = v_{i+1} w_{i+1}$. Their f-adjacency in u implies the existence of an almost simple k-tour $T_i \in \boldsymbol{T}^f$ containing uv_i, uw_i, uv_{i+1}, uw_{i+1} and such that

$$H_i = (T_i \backslash \{uv_i, uw_i\}) \cup \{v_i w_i\} \quad \text{and} \quad H_i' = (T_i \backslash \{uv_{i+1}, uw_{i+1}\}) \cup \{v_{i+1} w_{i+1}\}$$

are two simple k-tours in $\boldsymbol{H}^f$. It follows that the shortcut $s^{uv_{i+1}w_{i+1}}$ can be expressed as

$$s^{uv_{i+1}w_{i+1}} = x^{H_i'} - x^{H_i} + s^{uv_i w_i}.$$

By induction, the shortcut s^{uvw} can be expressed as a linear combination of simple k-tours in $\boldsymbol{H}^f$ and s^{uyz}. □

Lemma 2.10. *Let $fx \geq f_0$ be a facet inducing* TT *inequality for* GVRP *and, for every $u \in V$, let $J_u \subseteq \Delta_f(u)$ be f-connected in u. If there exists a basis B of $fx \geq f_0$ such that every k-tour in B can be transformed into a simple k-tour in $\boldsymbol{H}^f$ by adding shortcuts s^{uvw} where $vw \in J_u$, then $fx \geq f_0$ has a canonical basis.*

Proof. By Lemma 2.9, for every $u \in V$, there exists an almost simple k-tour $T^u \in \boldsymbol{T}^f$ such that, for every $vw \in J_u$, s^{uvw} can be expressed as a linear combination of T^u and simple k-tours in $\boldsymbol{H}^f$, viewed as points in $\mathbb{R}^E$. Therefore our assumptions in Lemma 2.10 imply that every k-tour in B can be expressed as a linear combination of $\{T^u$: $u \in V\}$ and simple k-tours in $\boldsymbol{H}^f$. Consider the affine ranks of these sets. Since arank $B = |E|$, arank $\boldsymbol{H}^f \leq |E| - |V|$ and $|\{T^u$: $u \in V\}| = |V|$, it follows that arank $\boldsymbol{H}^f = |E| - |V|$ and $\{T^u$: $u \in V\}$ is a set of $|V|$ additional affinely independent points. Hence $fx \geq f_0$ has a canonical basis. □

A partition of $V \backslash \{v_0\}$ into k nonempty sets $S_1, \ldots, S_k$ is called a *tight k-partition relative to* $W \subseteq V \backslash \{v_0\}$ if $d(S_i) \leq C$ for $i = 1, \ldots, k$ and $\{t$: $W \cap S_t \neq \emptyset\}$ has cardinality $R(W)$.

Theorem 2.11. *Let $W \subseteq V \backslash \{v_0\}$, $2 \leq |W| \leq |V| - 2$, be such that, for every triplet of clients i, j, $l \in V \backslash \{v_0\}$, there exists a tight k-partition relative to W, say $S_1, \ldots, S_k$, where i, j, $l \in S_1$. Then $x(\delta(W)) \geq 2R(W)$ induces a facet of* CVRP.

Proof. Using Theorem 2.3(a) and the fact that G is a complete graph, it follows that $x(\delta(W)) \geq 2R(W)$ induces a facet of GVRP. It will be convenient to denote this inequality by $fx \geq f_0$ in this proof. We will show that there is a canonical basis for $fx \geq f_0$ and that a simple k-tour $\bar{x}$ satisfying $f\bar{x} > f_0$ exists. Applying Lemma 2.7, it then follows that $fx \geq f_0$ induces a facet of CVRP.

Claim 1. A simple k-tour $\bar{x}$ satisfying $f\bar{x} > f_0$ exists.

Proof. Every simple k-tour x satisfying $fx = f_0$ has the property that each route contains 0 or 2 edges of $\delta(W)$. Therefore, to prove Claim 1, it suffices to exhibit a

k-tour $\bar{x}$ with a route containing at least 4 edges of $\delta(W)$. Since $2 \leq |W| \leq |V|-2$, there exist three distinct clients i, j, l with $i, j \in W$ and $l \in V \backslash (W \cup \{v_0\})$. By assumption, there exists a tight k-partition relative to W such that $i, j, l \in S_1$. Construct a simple k-tour where route T_1 visits first the client i, then l and then j followed by the remaining clients in S_1 in any order. We have $|T_1 \cap \delta(W)| \geq 4$, proving the claim.

Claim 2. $fx \geq f_0$ has a canonical basis.

Proof. To prove this claim, we use Lemma 2.10. We will show that $\Delta_f(u)$ is f-connected in u for every $u \in V$ and that all k-tours used in the proof of Theorem 2.3(a) can be transformed into simple k-tours of $\boldsymbol{H}^f$ by only adding shortcuts s^{uvw} where $vw \in \Delta_f(u)$. These statements are proved in Claims 3 and 4 respectively.

Claim 3. $\Delta_f(u)$ is f-connected in $u \in V$.

Proof. We consider three cases according to whether u is in W, $V \backslash (W \cup \{v_0\})$ or $\{v_0\}$.

First assume $u \in W$. Then $\Delta_f(u)$ consists of all edges vw where $w \in W \backslash \{u\}$ and $v \in V \backslash \{u, w\}$. Consider $j \in W \backslash \{u\}$. If $|W| \geq 3$, let $i \in W \backslash \{u, j\}$. By hypothesis, there exists a tight k-partition relative to W where $i, j, u \in S_1$. Therefore there exists an almost simple k-tour $T \in \boldsymbol{T}^f$ where the route T_1 contains edges v_0u, uj, ju, ui.

This yields two simple k-tours in $\boldsymbol{H}^f$, namely $H_1 = (T \backslash \{v_0u, uj\}) \cup \{v_0j\}$ and $H_2 = (T \backslash \{ju, ui\}) \cup \{ji\}$. Therefore the edges v_0j and ji are f-adjacent in u. So all edges vw where $w \in W \backslash \{u\}$ and $v \in \{v_0\} \cup (W \backslash \{u, w\})$ are f-connected in u. Now, for $|W| \geq 2$, consider $j \in W \backslash \{u\}$ as before and $i \in V \backslash (W \cup \{v_0\})$. There exists a tight k-partition relative to W where $i, j, u \in S_1$. Therefore there exists an almost simple k-tour in $\boldsymbol{T}^f$ where the route T_1 contains edges iu, uj, ju, ul where, w.l.o.g., $l \in \{v_0\} \cup (W \backslash \{u, j\})$. It follows that the edges ij and jl are f-adjacent in u. Since the choices of $j \in W \backslash \{u\}$ and $i \in V \backslash (W \cup \{v_0\})$ are arbitrary, we conclude that $\Delta_f(u)$ is f-connected in u.

Now assume $u \in V \backslash (W \cup \{v_0\})$. Then $\Delta_f(u)$ consists of all edges vw where $v \in V \backslash (W \cup \{u\})$ and $w \in V \backslash \{u, v\}$. If $|W| \leq |V|-3$, let $j \in V \backslash (W \cup \{u, v_0\})$, and let $i \in V \backslash \{u, j, v_0\}$. By hypothesis there is a tight k-partition relative to W where $i, j, u \in S_1$. Therefore there exists an almost simple k-tour in $\boldsymbol{T}^f$ where route T_1 contains the edges v_0u, uj, ju, ui. It follows that v_0j and ji are f-adjacent in u. There also exists an almost simple k-tour in $\boldsymbol{T}^f$ where route T_1 contains the edges iu, uv_0, v_0u, uj. It follows that v_0i and v_0j are f-adjacent in u. Therefore all edges vw where $v \in V \backslash (W \cup \{u\})$ and $w \in V \backslash \{u, v\}$ are f-connected in u. Now, for $|W| = |V|-2$, consider $j \in V \backslash \{u, v_0\}$ and $i \in V \backslash \{u, j, v_0\}$. There exists an almost simple k-tour in $\boldsymbol{T}^f$ where route T_1 contains the edges ju, uv_0, v_0u, ui. It follows that v_0j and v_0i are f-adjacent in u. We conclude that $\Delta_f(u)$ is f-connected in u.

Finally, assume $u = v_0$. Then $\Delta_f(u)$ consists of all edges vw where $v \in V \backslash (W \cup \{v_0\})$ and $w \in V \backslash \{v, v_0\}$. Consider $j \in V \backslash (W \cup \{v_0\})$. If $|W| = |V|-2$, let i, l be two distinct nodes in W. There exists an almost simple k-tour in $\boldsymbol{T}^f$ where route T_1 contains the edges iv_0, v_0j, jv_0, v_0l and therefore ij and jl are f-adjacent in u. If $|W| \leq |V|-3$, let $i \in V \backslash (W \cup \{j, v_0\})$ and $l \in V \backslash \{i, j, v_0\}$. Again, there exists an almost simple k-tour in $\boldsymbol{T}^f$ where route T_1 contains the edges iv_0, v_0j, jv_0, v_0l. Therefore ij and jl are f-adjacent in u. This shows that $\Delta_f(u)$ is f-connected in u.

This completes the proof of Claim 3.

Claim 4. All k-tours used in the proof of Theorem 2.3(a) can be transformed into simple k-tours of $\boldsymbol{H}^f$ by only adding shortcuts s^{uvw} where $vw \in \Delta_f(u)$.

Proof. First we describe a recursive procedure for transforming a k-tour $T \in \boldsymbol{T}^f$ into a simple k-tour. Let T_i be any route of T, C_i an Eulerian cycle with edge set T_i, and S_i the set of clients assigned to T_i. If vu and uw are consecutive edges of C_i and $u \notin S_i \cup \{v_0\}$ then apply the shortcut s^{uvw}. Similarly, if vu and uw are consecutive edges and, when starting from v_0, node $u \in S_i \cup \{v_0\}$ has already been visited in C_i, then apply the shortcut s^{uvw}. Modify T appropriately and repeat the above procedure until no shortcut can be found. This yields a simple k-tour. Consider any shortcut s^{uvw} used in the above procedure. If $vw \notin \Delta_f(u)$, then the resulting k-tour T' (obtained by applying s^{uvw} on T) satisfies $f(T') < f_0$, because $x(\delta(W)) \geqslant 2R(W)$ is in TT form. Claim 4 follows.

This completes the proof of the theorem. □

Applied to the TSP ($k=1$, $C=\infty$), Theorem 2.11 shows that $x(\delta(W)) \geqslant 2$ induces a facet whenever both W and $V \setminus W$ have cardinality at least 2, i.e., for every TSP with four or more nodes.

The question of finding a necessary and sufficient condition for $x(\delta(W)) \geqslant 2R(W)$ to induce a facet of CVRP is open. A different proof technique would be required. The above results can be simplified when all the demands are equal and the capacity is expressed as the maximum number of clients in a subtour.

Corollary 2.12. *Assume that* $k \geqslant 2$, $C \geqslant 3$, $k+4 \leqslant n \leqslant kC$ *and* $d_i = 1$ *for* $i = 1, \ldots, n$. *Then*

(1) $\dim \text{CVRP} = |E| - |V|$,

(2) $x_e \geqslant 0$ *induces a facet of* CVRP,

(3) $x(\delta(W)) \geqslant 2\lceil |W|/C \rceil$ *induces a facet of* CVRP *for* $W \subseteq V \setminus \{v_0\}$, $2 \leqslant |W| \leqslant (C/(C-1))(n-k-1)$, $|W| \neq pC$ *and* $pC-1$, $p = 1, \ldots, k$.

Proof. (1) follows from Lemma 2.7 and the fact that at least one set W satisfies the conditions of Theorem 2.11. For example, we can choose $|W| = 4$ if $C = 3$ and $|W| = 2$ if $C \geqslant 4$. (2) can be proved directly as in [16] for the TSP. We prove (3) by showing that the conditions of Theorem 2.11 are satisfied. These conditions require the existence of a simple k-tour where $\lceil |W|/C \rceil$ routes intersect W, $k - \lceil |W|/C \rceil$ routes do not and, for any $\{i, j, l\} \subseteq V \setminus \{v_0\}$, some route contains $\{i, j, l\}$.

When $i, j, l \notin W$, this simple k-tour exists if there are at least $k - \lceil |W|/C \rceil + 2$ clients in $V \setminus W$, i.e., $n - |W| \geqslant k - \lceil |W|/C \rceil + 2$. This is implied by $|W| \leqslant (C/(C-1))(n-k-1)$ and $|W| \neq pC$. When $|\{i, j, l\} \cap W| = 1$ or 2, the simple k-tour exists as a consequence of $|W| \neq pC$, $|W| \neq pC-1$ and of the previous condition. Finally, in the case $\{i, j, l\} \subseteq W$, the conditions of Theorem 2.11 hold without further restriction on $|W|$. □

Corollary 2.12 is related to results of Campos, Corberan and Mota [3] and of Araque [1]. In [3], the variables x_e for e incident with the depot are split into two 0, 1 variables, whereas in [1] the number of vehicles is not fixed. In both cases, all clients have equal demand. Theorems 8 and 10 in [3] contain conditions similar to (3) of Corollary 2.12 above. It is interesting to note that, in both [1] and [3], the capacity inequalities are shown to induce facets even when $|W| = pC - 1$ (in [3], this result is stated under the restriction that $n \geqslant (k-1)C+3$). This case is still open for our version of CVRP. When $|W| = pC$, it is easy to check that the capacity inequalities do not induce facets.

Checking whether an inequality $x(\delta(W)) \geqslant a$ is valid for the polytope CVRP is NP-complete in general. In the case of equal demands, this is easy to check, but it is not known whether the "constraint identification problem" [29] for capacity constraints is polynomially solvable.

3. Path inequalities

The purpose of this section is to characterize families of facets of GVRP(G) related to the path, wheelbarrow and bicycle inequalities of the graphical traveling salesman polyhedron GTSP(G). We then relate facets of GVRP and CVRP using the ideas developed in the preceding section. We use the notation given in Cornuejols, Fonlupt and Naddef [7].

A *path configuration* is defined by an odd integer $s \geqslant 3$, integers $n_i \geqslant 2$ for $i = 1, \ldots, s$, and a partition of V into sets A, Z, B_j^i for $i = 1, \ldots, s$ and $j = 1, \ldots, n_i$ such that

(i) B_j^i is nonempty and $G(B_j^i)$ is a connected graph for $i = 1, \ldots, s$ and $j = 0, 1, \ldots, n_i + 1$, and

(ii) the edge set (B_j^i, B_{j+1}^i) is nonempty for $i = 1, \ldots, s$ and $j = 0, 1, \ldots, n_i$, where, for convenience, we use the convention $B_0^i \equiv A$ and $B_{n_i+1}^i \equiv Z$ for $i = 1, \ldots, s$. Then, the *path inequality* corresponding to this configuration is

$$\sum_{e \in E} f_e x_e \geqslant f_0, \tag{7}$$

where the right-hand side f_0 and the coefficients f_e, $e \in E$, are defined by

$$f_0 = 1 + \sum_{i=1}^{s} \frac{n_i + 1}{n_i - 1}$$

and

$$f_e = \begin{cases} 1 & \text{for } e \in (A, Z), \\ \dfrac{|j-p|}{n_i - 1} & \text{for } e \in (B_j^i, B_p^i),\ i = 1, \ldots, s, \\ & \quad \text{and } j \neq p \text{ such that } |j-p| \leqslant n_i, \\ \max\left\{\dfrac{p}{n_l - 1} - \dfrac{j-2}{n_i - 1}, \dfrac{j}{n_i - 1} - \dfrac{p-2}{n_l - 1}\right\} & \text{for } e \in (B_j^i, B_p^l),\ i \neq l,\ j = 1, \ldots, n_i, \\ & \quad \text{and } p = 1, \ldots, n_l, \\ 0 & \text{otherwise.} \end{cases}$$

Two related configurations are the wheelbarrow and bicycle configurations. *Wheelbarrows* correspond to the case where Z is empty. Then condition (i) above is required only for $j = 0, 1, \ldots, n_i$, condition (ii) only for $j = 0, 1, \ldots, n_i - 1$, and the following additional condition must hold:

(iii) $(B^i_{n_i}, B^{i+1}_{n_{i+1}})$ is nonempty for $i = 1, \ldots, s$,

where, by convention, $B^{s+t}_{n_{s+t}} \equiv B^t_{n_t}$ for $t = 1, \ldots, s$.

Bicycle configurations correspond to the case where both A and Z are empty. Condition (i) must hold for $j = 1, \ldots, n_i$, condition (ii) for $j = 1, \ldots, n_i - 1$ and, in addition, we must have conditions (iii) and

(iv) (B^i_1, B^{i+1}_1), (B^i_1, B^{i+2}_1) and $(B^i_{n_i}, B^{i+2}_{n_{i+2}})$ are nonempty for $i = 1, \ldots, s$.

The coefficients in the inequality $fx \geqslant f_0$ associated with wheelbarrow or bicycle configurations are the same as for path inequalities. The path, wheelbarrow and bicycle inequalities always define facets of GTSP(G).

Since GVRP(G) $\subseteq$ GTSP(G), it follows that the path, wheelbarrow and bicycle inequalities are valid for GVRP(G). It turns out that these inequalities preserve their facetial properties as stated below.

Theorem 3.1. *Consider a path, bicycle or wheelbarrow inequality $fx \geqslant f_0$. Let B^i_j be the set that contains the depot and $S = V \backslash B^i_j$. Assume that $R(S) = 1$. Then the inequality $fx \geqslant f_0$ defines a facet of* GVRP(G).

Proof. Follows directly from Theorems 3.2 and 3.5 in [7], by noting that only one vehicle is needed for the clients in S. $\square$

A natural question is the following: Are inequalities (7) still facet inducing when $R(S) > 1$ in Theorem 3.1? We will proceed to answer this question by distinguishing between two cases, depending on whether the depot is in some *inner* set B^i_j, for $i = 1, \ldots, s$, and $j = 1, \ldots, n_i$, or in a *pole* set (either A or Z). For simplicity of exposition, we assume that $A \neq \emptyset$ and $Z \neq \emptyset$. We start by the case where the depot is located in a pole set, say A.

Let $P_i = B^i_1 \cup \cdots \cup B^i_{n_i}$ for $i = 1, \ldots, s$. We define a *tight k-partition* relative to a path configuration to be a partition of $V \backslash \{v_0\}$ into k nonempty subsets $S_1, \ldots, S_k$ such that

(a) $d(S_t) \leqslant C$ for $t = 1, \ldots, k$;

(b) for every $i = 1, \ldots, s$, there exists $t \in \{1, \ldots, k\}$ such that $P_i \subseteq S_t$;

(c) the set S_1 contains an odd number ($\geqslant 3$) of P_i's (S_1 is called the *odd set* of the k-partition); for $t = 2, \ldots, k$, the set S_t contains an even number (possibly 0) of P_i's; finally the sets S_t which contain no P_i are contained in A.

The following condition will be used in the next theorem.

Condition (C). For every $i \in \{1, \ldots, s\}$, there exists a tight k-partition where P_i is contained in the odd set. Furthermore, if G has an edge joining B^i_j to B^h_q for $i \neq h$,

$1 \leqslant j \leqslant n_i$, $1 \leqslant q \leqslant n_h$, then there exists a tight k-partition where $P_i \cup P_h$ is contained in the odd set.

Theorem 3.2. *If a path inequality with the depot in set A satisfies Condition* (C), *then it defines a facet of* GVRP(G).

Proof. The proof follows the lines of that of Theorem 3.2 in [7]. Let $fx \geqslant f_0$ be a path inequality. Consider an inequality $cx \geqslant f_0$ which defines a facet of GVRP(G) and contains the face defined by $fx \geqslant f_0$. In the following steps, we prove that $c_e = f_e$ for all $e \in E$.

Let $z \in Z$ and $b_j^t \in B_j^t$ for $t = 1, \ldots, s$ and $j = 1, \ldots, n_t$. For convenience, let $b_0^t = v_0$ and $b_{n_t+1}^t = z$ for $t = 1, \ldots, s$. Define k-tours $x^i \in \mathbb{R}^E$ as follows, for $i = 1, \ldots, s$.

$$x_e^i = \begin{cases} 2 & \text{for } e = b_j^t u, \text{ all } t = 1, \ldots, s, \text{ all } j = 0, \ldots, n_t + 1 \text{ and all } u \in B_j^t \backslash \{b_j^t\}, \\ 2 & \text{for } e = b_j^i b_{j+1}^i, \text{ all } j = 1, \ldots, n_i, \\ 1 & \text{for } e = b_j^t b_{j+1}^t, \text{ all } t \neq i \text{ and all } j = 0, \ldots, n_i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $fx^i = f_0$ and, as a consequence of Condition (C), x^i is a feasible k-tour where P_i is contained in the odd set. (Recall that the routes can go through nodes that are not assigned to them. Here node z belongs to all k routes but is only assigned to one.)

Since $fx^i = f_0$, x^i also satisfies $cx^i = f_0$. Now consider an edge e with both ends in the same B_j^t for some $t = 1, \ldots, s$ and $j = 0, \ldots, n_t + 1$. Another k-tour is $x' = x^i + 2y^e$ where y^e is the unit vector such that $y_e^e = 1$ and $y_h^e = 0$ for $h \neq e$. Since $f_e = 0$, $fx' = f_0$ and therefore x' also satisfies $cx' = f_0$. This implies $c_e = 0$.

Next, we show that c_e has the same value for all edges e in (B_j^t, B_{j+1}^t). Consider x^i, $i \neq t$. Let $e_1 = b_j^t b_{j+1}^t$. Modify x^i so that it contains another edge of (B_j^t, B_{j+1}^t), say edge e_2 instead of e_1. In order to still have a k-tour, we have to change x^i within B_j^t and B_{j+1}^t, but we can keep x^i unchanged anywhere else. Since both k-tours satisfy $fx = f_0$ (and therefore $cx = f_0$), we must have $c_{e_1} = c_{e_2}$. The k-tour x^i has the property that $x_e^i = 0$ for $e \in (A, B_1^i)$ and $x_e^i = 2$ for $e = b_j^i b_{j+1}^i$, $j \geqslant 1$. Another k-tour satisfying $fx = f_0$ is obtained from x^i by setting $x_e = 2$ for $e = v_0 b_1^i$ and $x_e = 0$ for $e = b_j^i b_{j+1}^i$ for some $j \geqslant 1$. Note that this can be achieved without reassigning the clients to different subtours so that feasibility of the k-tour is maintained. Therefore, for $e \in (B_j^i, B_{j+1}^i)$, the value of c_e does not depend on j, but only on $i = 1, \ldots, s$. These s values satisfy the system of s equations $cx^i = f_0$. Its unique solution is $c_e = 1/(n_i - 1)$ for $e \in (B_j^i, B_{j+1}^i)$.

So far, we have shown that $c_e = f_e$ for every edge e with both ends in B_j^t or for $e \in (B_j^t, B_{j+1}^t)$ for all t and j. These edges define the *skeleton* of the path configuration. Now consider an edge $e \in (B_j^i, B_q^i)$ where $q \geqslant j + 2$. There exists a k-tour x' such that $fx' = f_0$ which only contains the edge e and edges of the skeleton. (Such a k-tour can be obtained from x^i by modifying x^i_h for edges h with endpoints in $\bigcup_p B^i_p$ and without reassigning the clients.) Since $cx' = f_0$ we get $c_e = f_e$.

Finally, consider $e \in (B^i_j, B^h_q)$ for $i \neq h$, $1 \leq j \leq n_i$, $1 \leq q \leq n_h$. By Condition (C), there exists a tight k-partition where $P_i \cup P_h$ is included in the odd set. This implies that there exists a k-tour x'' satisfying $fx'' = f_0$ which only contains the edge e and edges of the skeleton. Again this implies $c_e = f_e$. □

Now we assume that G is a complete graph. In the next theorem, use will be made of the condition stated below.

Condition (C'). For every $i, j, l \in \{1, \ldots, s\}$ and $t, y, z \in Z \cup A \backslash \{v_0\}$, there exists a tight k-partition where $P_i \cup P_j \cup P_l \cup \{z\}$ is included in the odd set, and one where $P_i \cup P_j \cup \{t, y, z\}$ is included in the odd set.

Theorem 3.3. *Let $fx \geq f_0$ be a path inequality with the depot in the set A. If Condition* (C′) *holds, then $fx \geq f_0$ defines a facet of* CVRP.

Proof. Since the underlying graph G is complete, it is easy to construct a simple k-tour $\bar{x}$ such that $f\bar{x} > f_0$ (strictly). So, by Lemma 2.7 and Theorem 3.2, it suffices to show that $fx \geq f_0$ has a canonical basis. In order to do this, we use Lemma 2.10. This requires showing that, for each $u \in V$, some $J_u \subseteq \Delta_f(u)$ is f-connected in u (this will be done in Lemmas 3.4 and 3.5). Then we must show that every k-tour used in the proof of Theorem 3.2 can be transformed into a simple k-tour in $\boldsymbol{H}^f$ by adding shortcuts s^{uvw} where $vw \in J_u$ (this will be done in Lemma 3.6). □

Lemma 3.4. *Let $u \in A \cup Z$. If Condition* (C′) *holds, then $\Delta_f(u)$ is f-connected in u.*

Proof. First, let $u \in A$. It can be readily verified that any edge $e = vw$ of $\Delta_f(u)$ has at least one of its endpoints in $(A \backslash \{u\}) \cup (\bigcup_{i=1}^s B^i_1)$. Furthermore, if $v \in B^i_1$, then $w \in V \backslash (P_i \cup \{u\})$ and if $v \in A \backslash \{u\}$, then $w \in V \backslash \{u, v\}$. This follows from the definition of the coefficients f_e, $e \in E$. We consider two cases.

Case 1. $A = \{v_0\}$. Let $v \in B^i_1$, $w \in B^j_1$, $y \in B^l_1$ where $i \neq j \neq l$ and $z \in P_j \backslash \{w\}$ if $|B^j_1| \geq 2$, $z \in P_j$ if $|B^j_1| = 1$. By Condition (C′), there exists a tight k-partition where $P_i \cup P_j \cup P_l$ is included in the odd set. So one can construct an almost simple k-tour $T^u \in \boldsymbol{T}^f$ containing the edges uv, uw, uy and uz. Such a k-tour can be obtained from a k-tour of $\boldsymbol{T}^f$ that uses only edges of the skeleton by shortcuts that preserve the partition and the relation $fx = f_0$. Furthermore, T^u can be chosen so that $H_1 = (T^u \backslash \{uv, uw\}) \cup \{vw\}$ and $H_2 = (T^u \backslash \{uy, uz\}) \cup \{yz\}$ are simple k-tours of $\boldsymbol{H}^f$. It follows that the two edges vw and yz are f-adjacent in u. By induction, this shows that the edges of $\Delta_f(u)$ with no endpoint in Z are all f-connected in u. Now consider $z \in Z$ and let $v \in B^i_1$, $w \in B^j_1$ and $y \in B^l_1$. By Condition (C′), there exists a tight k-partition where $P_i \cup P_j \cup P_l \cup \{z\}$ is included in the odd set. The argument above shows that vw and yz are f-adjacent in u. It follows that all the edges of $\Delta_f(u)$ are f-connected in u.

Case 2. $|A| \geq 2$. Let $v \in B_1^i$, $w \in B_1^j$, $y \in A \backslash \{u\}$ and $z \in (A \cup P_j \cup Z) \backslash \{u, y, w\}$ if $|B_1^j| \geq 2$ and $z \in (A \cup P_j \cup Z) \backslash \{u, y\}$ if $|B_1^j| = 1$. By Condition (C′), there exists a tight k-partition where $P_i \cup P_j \cup \{u, y, z\} \backslash \{v_0\}$ is included in the odd set. This implies the existence of an almost simple k-tour, say $T^u \in \boldsymbol{T}^f$, containing the edges uv, uw, uy and uz such that $H_1 = (T^u \backslash \{uv, uw\}) \cup \{vw\}$ and $H_2 = (T^u \backslash \{uy, uz\}) \cup \{yz\}$ are simple k-tours of $\boldsymbol{H}^f$. This implies that the edges vw and yz are f-adjacent in u. By varying the choice of i, j, v, w, y, z, it follows that the edges of $\Delta_f(u)$ with at least one endpoint in $A \backslash \{u\}$ are all f-connected in u. Note that $H_1' = (T^u \backslash \{uv, uz\}) \cup \{vz\}$ and $H_2' = (T^u \backslash \{uw, uy\}) \cup \{yw\}$ are also simple k-tours of $\boldsymbol{H}^f$. This implies that vz and yw are f-adjacent in u. So the edges of $\Delta_f(u)$ with one endpoint in B_1^i are f-connected in u to those with one endpoint in $A \backslash \{u\}$. It follows that all edges of $\Delta_f(u)$ are f-connected in u.

Now suppose that $u \in Z$. It follows from the definition of f_e, $e \in E$, that $\Delta_f(u)$ consists of the edges vw with $v \in B_{n_i}^i$, $w \in V \backslash (P_i \cup \{u\})$ and those with $v \in Z \backslash \{u\}$, $w \in V \backslash \{u, v\}$. Again we consider two cases. In the case where $Z = \{u\}$, Condition (C′) implies that, for $i \neq j \neq l$, there exists a tight k-partition where $P_i \cup P_j \cup P_l \cup \{z\}$ is included in the odd set. By choosing $v \in B_{n_i}^i$, $w \in B_{n_j}^j$ and $y \in B_{n_l}^l$ and using the same argument as in Case 1 above, we get that the edges of $\Delta_f(u)$ with no endpoint in A are f-connected in u. Next, let $t \in A$. By Condition (C′) there exists a tight k-partition where $P_i \cup P_j \cup \{t, u\}$ is included in the odd set. Let P_l be a path included in the odd set which is distinct from P_i and P_j, and let $v \in B_{n_i}^i$, $y \in B_{n_j}^j$, $w \in B_{n_l}^l$. There exists an almost simple k-tour $T^u \in \boldsymbol{T}^f$ containing edges uv, uw, uy, ut and simple k-tours in $\boldsymbol{H}^f$, $H_1 = (T^u \backslash \{uv, uw\}) \cup \{vw\}$ and $H_2 = (T^u \backslash \{uy, ut\}) \cup \{yt\}$. Therefore vw and yt are f-adjacent in u. It follows that all edges of $\Delta_f(u)$ are f-connected in u. In the case where $|Z| \geq 2$, we let $v \in B_{n_i}^i$, $w \in B_{n_j}^j$, $y \in Z \backslash \{u\}$ and $t \in \{A \cup P_j \cup Z\} \backslash \{u, y, w\}$ if $|B_{n_j}^j| \geq 2$ and $t \in \{A \cup P_j \cup Z\} \backslash \{u, y\}$ if $|B_{n_j}^j| = 1$. The same argument as in Case 2 above shows that $\Delta_f(u)$ is f-connected in u. □

Lemma 3.5. *Let $u \in B_j^i$ where $1 \leq i \leq s$ and $1 \leq j \leq n_i$. Then $J_u = \{vw \in E(V \backslash \{u\})$: $v \in B_q^i$, $w \in B_r^i$ where $0 \leq q \leq j \leq r \leq n_i + 1$ and $r - q \leq n_i\}$ is f-connected in u.*

Proof. Let $z \in B_{j+1}^i$ and $y \in B_j^i$, $y \neq u$. (If $B_j^i = \{u\}$, then let $y \in B_{j-1}^i$.) Now, let $vw \in J_u$ where at least three of the nodes v, w, y, z are distinct. By Condition (C′) there exists an almost simple k-tour $T^u \in \boldsymbol{T}^f$ which contains the edges uv, uw, uy and uz. In addition the two simple k-tours $H_1 = (T^u \backslash \{uv, uw\}) \cup \{vw\}$ and $H_2 = (T^u \backslash \{uy, uz\}) \cup \{yz\}$ belong to $\boldsymbol{H}^f$. Therefore yz is f-adjacent in u to any $vw \in J_u$. This shows that J_u is f-connected in u. □

Lemma 3.6. *Every k-tour used in the proof of Theorem* 3.2 *can be transformed into a simple k-tour of $\boldsymbol{H}^f$ by adding only shortcuts s^{uvw} with $vw \in \Delta_f(u)$ for $u \in A \cup Z$ and $vw \in J_u$ as defined in Lemma* 3.5 *for $u \in V \backslash (A \cup Z)$.*

Proof. First we show that the k-tour x^i, defined in the proof of Theorem 3.2, can be transformed into a simple k-tour of $\boldsymbol{H}^f$ by only adding shortcuts s^{uvw} of the type claimed in Lemma 3.6 (referred to as valid shortcuts). If b_j^t has degree greater than two in x^i for some $1 \leqslant t \leqslant s$ and $1 \leqslant j \leqslant n_t$, the required shortcut is s^{uvw} where $u = b_j^t$ and v and w are the nodes just before and just after the second visit of node u in an Eulerian cycle traversing the k-tour x^i. It is easy to verify that the edge vw so defined belongs to J_u as defined in Lemma 3.5 and therefore is a valid shortcut. Such shortcuts are applied recursively until the only nodes of the tour with degree greater than two occur in $A \cup Z$. Then valid shortcuts s^{uvw} with $u \in A \cup Z$ and $vw \in \Delta_f(u)$ are performed to complete the transformation of x^i into a simple k-tour of $\boldsymbol{H}^f$. Figure 3.1 illustrates the transformation of a k-tour x^i into a simple k-tour of $\boldsymbol{H}^f$.

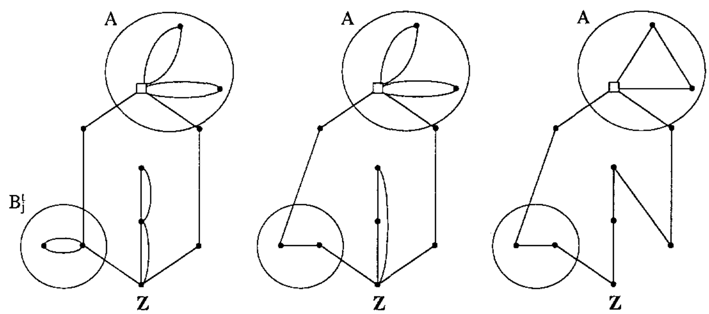

Fig. 3.1. Initial 2-tour x^i, 2-tour after shortcuts for $u \in V\backslash(A \cup Z)$, and final simple 2-tour.

The next family of k-tours used in the proof of Theorem 3.2 is $x^i + 2y^e$ where $e \in E(B_j^t)$. For these k-tours, one may apply two valid shortcuts s^{uvw} in order to obtain x^i and then the earlier argument can be used. (If $|B_j^t| \geqslant 3$, we can choose u, v, $w \in B_j^t$ and if $|B_j^t| = 2$, we can choose u, v, $\in B_j^t$ and $w \in B_j^{t+1}$ or B_j^{t-1}.)

Now consider the k-tours used in the proof of Theorem 3.2 to show that all edges $e \in (B_j^t, B_{j+1}^t)$ have the same coefficient value c_e. These k-tours are identical to x^i except within $B_j^t \cup B_{j+1}^t$. So these k-tours can also be transformed to simple k-tours of $\boldsymbol{H}^f$ by using only valid shortcuts.

The same argument can be made for the k-tours used in the proof of Theorem 3.2 to show that c_e for $e \in (B_j^i, B_{j+1}^i)$ only depends on i.

Now consider an edge $e \in (B_j^i, B_q^i)$ where $q \geqslant j+2$ and the k-tour x' used in the proof of Theorem 3.2 which contains only edge e and edges of the skeleton. We first apply shortcuts s^{uvw} where $u \in V\backslash(A \cup Z)$ and $vw \in J_u$ as defined in Lemma 3.5. This transforms the k-tour x' into a k-tour where the only nodes with degree greater than two occur in $A \cup Z$. Then shortcuts s^{uvw} with $u \in A \cup Z$ and $vw \in \Delta_f(u)$ are performed to complete the transformation of x' into a simple k-tour.

Finally, consider the k-tour x'' used in the proof of Theorem 3.2. Let $e = rt$ be the unique edge of this k-tour which is not an edge of the skeleton. We have

$rt \in (B_j^i, B_q^h)$ for $i \neq h$, $1 \leq j \leq n_i$ and $1 \leq q \leq n_h$. We can assume w.l.o.g. that each of the nodes r and t has degree 2 in the k-tour x''. So shortcuts of the form s^{rtw} or s^{trw} for $w \in V \backslash \{r, t\}$ are not needed. It follows that by first applying shortcuts s^{uvw} where $u \in V \backslash (A \cup Z)$, $vw \in J_u$ as defined in Lemma 3.5, and then shortcuts with $u \in A \cup Z$ and $vw \in \Delta_f(u)$, we can transform x'' into a simple k-tour of $\boldsymbol{H}^f$ which contains the edge rt. □

By analogous arguments, we may prove the following result for a path configuration where the depot is located in an inner set B_j^i for $1 \leq i \leq s$ and $1 \leq j \leq n_i$.

Theorem 3.7. *Consider a path inequality such that the depot belongs to B_j^i for $1 \leq i \leq s$ and $1 \leq j \leq n_i$. Assume that $R((V \backslash B_j^i) \cup \{t, y, z\}) = 1$ for any t, y, $z \in B_j^i \backslash \{v_0\}$. Then the path inequality defines a facet of* CVRP. □

In the remainder of this section, we consider a variation of path configurations which leads to a new class of inequalities for GVRP(G) and CVRP. In contrast with the situation for the previous path inequalities, these new inequalities are not valid for GTSP(G). Their validity for GVRP(G) depends on the capacity restrictions and therefore they can be expected to play a significant role in the solution of tightly capacitated vehicle routing problems. To simplify the exposition, we focus on the case where $n_i = 2$ for all $i = 1, \ldots, s$.

A *capacitated path configuration* is defined by an odd integer $s \geq 2k+1$ and a partition of V into sets A, Q, Z, B_1^i for $i = 2k, \ldots, s$ and B_2^i for $i = 1, \ldots, s$ such that, using the convention $A \equiv B_0^i$ for $i = 1, \ldots, s$, $Q \equiv B_1^i$ for $i = 1, \ldots, 2k-1$ and $Z \equiv B_3^i$ for $i = 1, \ldots, s$,

(i) the node set B_j^i is nonempty and $G(B_j^i)$ is a connected graph for $i = 1, \ldots, s$ and $j = 0, 1, 2, 3$;

(ii) the edge set (B_j^i, B_{j+1}^i) is nonempty for $i = 1, \ldots, s$ and $j = 0, 1, 2$;

(iii) $v_0 \in Q$, $R(V \backslash (Q \cup B_2^i)) = k$ for $i = 1, \ldots, 2k-1$.

See Figure 3.2.

The *capacitated path inequality* corresponding to this configuration is

$$\sum_{e \in E} f_e x_e \geq f_0, \tag{8}$$

where $f_0 = 3s - 2k + 3$ and

$$f_e = \begin{cases} 0 & \text{if } e \in E(B_j^i) \text{ for } i = 1, \ldots, s \text{ and } j = 0, 1, 2, 3, \\ 1 & \text{if } e \in (A, Z) \text{ or } e \in (B_j^i, B_{j+1}^i) \text{ for } i = 1, \ldots, s \text{ and } j = 0, 1, 2, \\ 2 & \text{if } e \in (B_j^i, B_{j+2}^i) \text{ for } i = 1, \ldots, s \text{ and } j = 0, 1 \text{ or} \\ & \text{if } e \in (B_j^i, B_j^h) \text{ for } i \neq h \text{ and } j = 1 \text{ or } 2, \\ 3 & \text{if } e \in (B_1^i, B_2^h) \text{ for } i \neq h. \end{cases}$$

When either $A = \emptyset$ or $Z = \emptyset$, we say that (8) is a *capacitated wheelbarrow inequality* and when $A = Z = \emptyset$, we say it is a *capacitated bicycle inequality*.

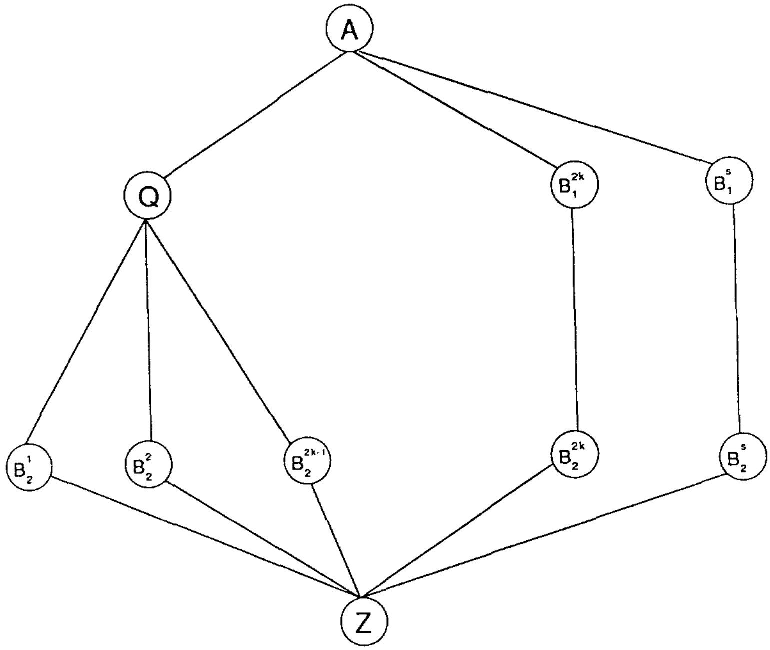

Fig. 3.2.

Theorem 3.8. *The capacitated path, wheelbarrow or bicycle inequality* (8) *is valid for* GVRP(G).

Proof. Let $\bar{x}$ be a k-tour such that $f\bar{x}$ achieves the smallest possible value. If $Z=\emptyset$, introduce a node in Z with 0 demand connected to B_2^j, $1 \leq j \leq s$. Similarly, if $A=\emptyset$, introduce a node in A with 0 demand connected to B_1^j, $1 \leq j \leq s$. Now consider any edge uv such that $f_{uv}=2$ (3 respectively) and $\bar{x}_{uv}>0$, if any. It follows from the definition of the coefficients f_e that there exists a path P_{uv} connecting u to v which contains 2 edges, say e_1 and e_2, such that $f_{e_1}=f_{e_2}=1$ (respectively 3 edges, say e_1, e_2, e_3, such that $f_{e_1}=f_{e_2}=f_{e_3}=1$) and all other edges of P_{uv} satisfy $f_e=0$. We get a new k-tour with the same value $f\bar{x}$ by reducing $\bar{x}_{uv}$ by 1 and increasing $\bar{x}_e$ by 1 for the edges e in the path P_{uv}. Therefore, we can assume w.l.o.g. that the k-tour $\bar{x}$ achieving the smallest value of $f\bar{x}$ contains only edges $e \in E$ such that $f_e=0$ or 1.

As a consequence of the subtour elimination constraints for B_1^j, B_2^j and $B_1^j \cup B_2^j$, we get

$$\bar{x}((A, B_1^j) \cup (B_1^j, B_2^j) \cup (B_2^j, Z)) \geq 3 \quad \text{for } j=2k, \ldots, s.$$

Furthermore, the subtour elimination constraints $\bar{x}(\delta(B_2^i)) \geq 2$ must hold for $i=1, \ldots, 2k-1$. Adding up these inequalities yields

$$f\bar{x} \geq 3s-2k+1+\bar{x}((A, Q))+\bar{x}((A, Z)).$$

Note that if $\bar{x}(\delta(B_2^i)) \geq 4$ for at least one index i, $1 \leq i \leq 2k-1$, then the inequality

$f\bar{x} \geqslant 3s-2k+3(=f_0)$ follows. To complete the proof, it only remains to consider the case where $\bar{x}(\delta(B_2^i))=2$ for every $i=1,\ldots,2k-1$, since $\bar{x}(\delta(B_2^i))$ is even.

First assume $\bar{x}((A,Q))=0$. Since $R(V\backslash(Q\cup B_2^i))=k$ for $i=1,\ldots,2k-1$, it follows from $\bar{x}(\delta(B_2^i))=2$ that each route of $\bar{x}$ visits at least two of the sets B_2^j, $1\leqslant j\leqslant 2k-1$. Since there are k such routes, we have a contradiction.

Now assume $\bar{x}((A,Q))=1$. The fact that $\bar{x}(\delta(A))$ is even implies that either $\bar{x}((A,Z))\geqslant 1$ or $\bar{x}((A,B_1^j)\cup(B_1^j,B_2^j)\cup(B_2^j,Z))\geqslant 4$ for some $j=2k,\ldots,s$. The inequality $f\bar{x}\geqslant 3s-2k+3$ follows. □

Define $t=s-2k+1$ and $P_j=B_1^{2k+j-1}\cup B_2^{2k+j-1}$ for $j=1,\ldots,t$. We define a *tight k-partition* relative to a capacitated path configuration to be a partition of $V\backslash\{v_0\}$ into k nonempty subsets $S_1,\ldots,S_k$ such that

(a) $d(S_i)\leqslant C$;

(b) for every $j=1,\ldots,t$, there exists $i\in\{1,\ldots,k\}$ such that $P_j\subseteq S_i$;

(c) S_1 contains exactly one of the sets B_2^j, $1\leqslant j\leqslant 2k-1$, and an even number ($\geqslant 2$) of sets P_j, $1\leqslant j\leqslant t$. S_1 is called the *odd set*;

(d) for $2\leqslant i\leqslant k$, the set S_i contains exactly two sets B_2^j, $1\leqslant j\leqslant 2k-1$ and an even number (possibly 0) of sets P_j, $1\leqslant j\leqslant t$. If S_i contains no P_j, then $S_i\cap A=\emptyset$. The sets S_i, $2\leqslant i\leqslant k$, are called the *even sets*.

We now introduce a sufficient condition for a capacitated path inequality to induce a facet of GVRP(G).

Condition (D). (i) For every B_2^i, $1\leqslant i\leqslant 2k-1$, and every P_j, $1\leqslant j\leqslant t$, there exists a tight k-partition where $B_2^i\cup P_j$ is included in the odd set.

(ii) For every P_i, P_j, $1\leqslant i,j\leqslant t$, there exists a tight k-partition where $P_i\cup P_j$ is included in the odd set.

(iii) For every B_2^i, B_2^j, $1\leqslant i,j\leqslant 2k-1$ and $i\neq j$, there exists a tight k-partition where $B_2^i\cup B_2^j$ is included in an even set which has an empty intersection with Z.

Theorem 3.9. *A capacitated path inequality which satisfies Condition* (D) *induces a facet of* GVRP(G).

Proof. It is very similar to the proof of Theorem 3.2. The k-tours x^i needed here are defined as follows. Let $y\in A$, $z\in Z$ and $b_1^j\in B_1^j$ for $j=2k,\ldots,s$ and $b_2^j\in B_2^j$ for $j=1,\ldots,s$. For convenience, let $b_1^j=v_0$ for $j=1,\ldots,2k-1$, $b_0^j=y$ and $b_3^j=z$ for $j=1,\ldots,s$.

$$x_e^i=\begin{cases}2 & \text{for } e=b_l^ju, \text{ all } j=1,\ldots,s, \text{ all } l=0,1,2,3 \text{ and all } u\in B_l^j\backslash\{b_l^j\},\\ 2 & \text{for } e=b_l^ib_{l+1}^i, \text{ all } l=1,2,\\ 1 & \text{for } e=b_l^jb_{l+1}^j, \text{ all } j\neq i \text{ and all } l=1,2,\\ 1 & \text{for } e=b_0^jb_1^j, \text{ all } j=2k,\ldots,s \text{ when } 1\leqslant i\leqslant 2k-1,\\ & \quad \text{all } j\neq i \text{ when } 2k\leqslant i\leqslant s,\\ 0 & \text{otherwise.}\end{cases}$$

The proof that $c_e=f_e$ for the edges e in $E(B_l^j)$, all $j=1,\ldots,s$, all $l=0,1,2,3$, and in (B_l^j, B_h^j), all $j=1,\dots,s$, all $0 \leq l < h \leq 3$, carries over from the proof of Theorem 3.2 using the k-tours x^i.

Now consider $e \in (Q, B_l^j)$ for $j=2k,\dots,s$ and $l=1, 2$. By Condition (D), there exists a tight k-partition where P_j is included in the odd set. This implies the existence of a k-tour in T^f containing edge e and edges of the skeleton. We deduce $c_e = f_e$.

For $e \in (B_2^h, B_l^j)$ where $h=1,\dots,2k-1$ and $j=2k,\dots,s$ and $l=1, 2$, the result follows from Condition (D)(i). For $e \in (B_q^h, B_l^j)$ where $2k \leq h < j \leq s$ and q, $l=1$ or 2, the result follows from Condition (D)(ii).

Finally consider $e \in (B_2^h, B_2^j)$ for $1 \leq h < j \leq 2k-1$. By Condition (D)(iii) there exists a k-tour in T^f containing edge e and edges of the skeleton. This shows $c_e = f_e$. □

One can find sufficient conditions in the spirit of Conditions (C′) under which the capacitated path inequalities (8) define facets of CVRP, but these conditions are more technical and we omit them here.

4. Comb inequalities

In the preceding section we derived sufficient conditions under which valid inequalities related to the path inequalities define facets of CVRP. In this section we demonstrate how the comb inequalities of the symmetric traveling salesman problem can be generalized to facets of CVRP depending on whether the depot is in a handle, a tooth, both or neither. Comb inequalities were introduced by Chvátal [6], and later generalized by Grötschel and Padberg [16]. For CVRP, the simplest but important case where the depot belongs neither to the handle nor to the teeth was addressed successively by Laporte and Nobert [22], and then Laporte and Bourjolly [21]. In what follows, the graph G is complete and $k \geq 2$. Let us now define the comb inequalities in our setting.

Formally, let $W_0, W_1, \dots, W_s \subseteq V$ satisfy

(i) $|W_i \setminus W_0| \geq 1$, $i=1,\dots,s$,

(ii) $|W_i \cap W_0| \geq 1$, $i=1,\dots,s$,

(iii) $|W_i \cap W_j| = 0$, $1 \leq i < j \leq s$,

(iv) s odd and ≥ 3.

The corresponding comb inequality is

$$\sum_{i=0}^{s} x(E(W_i)) \leq \left(\sum_{i=0}^{s} |W_i| \right) - \frac{3s+1}{2} + \alpha(k-1), \tag{9}$$

where

$$\alpha = \begin{cases} 0 & \text{if } v_0 \notin \bigcup_{i=0}^{s} W_i, \\ 1 & \text{if } v_0 \in W_0 \setminus \bigcup_{i=1}^{s} W_i \text{ or } v_0 \in W_j \setminus W_0 \text{ for some } j=1,\dots,s, \\ 2 & \text{if } v_0 \in W_j \cap W_0 \text{ for some } j=1,\dots,s. \end{cases}$$

Theorem 4.1. *The comb inequality* (9) *is valid for* CVRP *and induces the same face of* CVRP *as the path, wheelbarrow or bicycle inequality defined by* $n_i = 2$ *for* $i = 1, \ldots, s$ *and* $A = V \setminus \bigcup_{i=0}^{s} W_i$, $Z = W_0 \setminus \bigcup_{i=1}^{s} W_i$, $B_1^i = W_i \setminus W_0$, $B_2^i = W_i \cap W_0$ *for* $i = 1, \ldots, s$.

Proof. The degree constraints for CVRP imply

$$x(E(W)) = \begin{cases} |W| - \frac{1}{2}x(\delta(W)) & \text{if } v_0 \notin W, \\ |W| + (k-1) - \frac{1}{2}x(\delta(W)) & \text{if } v_0 \in W. \end{cases}$$

Therefore

$$\sum_{i=0}^{s} x(E(W_i)) = \sum_{i=0}^{s} (|W_i| - \tfrac{1}{2}x(\delta(W_i))) + \alpha(k-1),$$

where α is as defined in (9). Now the definition of f_e in (7), applied to the path, wheelbarrow or bicycle configuration A, Z, B_1^i, B_2^i, $1 \leqslant i \leqslant s$, as stated in Theorem 4.1, implies that

$$\sum_{e \in E} f_e x_e = \sum_{i=0}^{s} x(\delta(W_i)).$$

It follows that the path, wheelbarrow or bicycle inequality $\sum_{e \in E} f_e x_e \geqslant 3s + 1$ induces the same face of CVRP as the comb inequality (9). □

As a consequence of this result, Theorems 3.3 and 3.7 directly translate into facetial results for comb inequalities (9).

When $v_0 \in W_1 \setminus W_0$ and the comb satisfies

(v) $R(V \setminus W_1) = p$;

in addition to conditions (i)-(iv), then the comb inequality can be strengthened as follows.

$$\sum_{i=0}^{s} x(E(W_i)) \leqslant \left(\sum_{i=0}^{s} |W_i| \right) - \frac{3s+1}{2} + k - p. \tag{10}$$

Theorem 4.2. *If* $v_0 \in W_1 \setminus W_0$ *and* (v) *holds, then the comb inequality* (10) *is valid for* CVRP.

Proof. First, note that for $i = 2, \ldots, s$, $x(E(W_i)) \leqslant |W_i| - 1$, $x(E(W_i \setminus W_0)) \leqslant |W_i \setminus W_0| - 1$ and $x(E(W_i \cap W_0)) \leqslant |W_i \cap W_0| - 1$. Moreover, by condition (v), we have $x(E(W_1)) \leqslant |W_1| - 1 + k - p$, $x(E(W_1 \setminus W_0)) \leqslant |W_1 \setminus W_0| - 1 + k - p$ and $x(E(W_1 \cap W_0)) \leqslant |W_1 \cap W_0| - 1$. Adding these inequalities and the degree contraints for the nodes in W_0, applying a weight of $\frac{1}{2}$ and rounding up the resulting fractional coefficients yields the result. □

Checking condition (v) is an NP-hard problem but inequality (10) remains valid if p is replaced by $\lceil d(V\backslash W_1)/C\rceil$. This is the inequality that we recommend using in practice (instead of (9) or (10)), when $v_0\in W_1\backslash W_0$.

Now we introduce combs where several teeth can intersect. These combs are interesting because of their special form and intrinsic structure. The corresponding inequalities will be shown to be valid for CVRP but are not valid when the capacity constraints are dropped. For this reason, we call them *capacitated comb inequalities.* Let $W_0, W_1, \ldots, W_{2k}, \ldots, W_s \subseteq V$ satisfy (i), (ii), (iv) and the following conditions.

(iii') $W_i\cap W_j=\emptyset$, $1\leqslant i\leqslant s$, $2k\leqslant j\leqslant s$, $i\neq j$;

(vi) $W_i\cap W_j\cap W_0=\emptyset$, $1\leqslant i<j\leqslant 2k-1$;

(vii) $W_i\backslash W_0=Q$ for $i=1,\ldots,2k-1$;

(viii) $v_0\in Q$, $R(V\backslash W_i)=k$ for $i=1,\ldots,2k-1$.

Then the corresponding inequality is

$$\sum_{i=0}^{s} x(E(W_i))\leqslant\left(\sum_{i=0}^{s}|W_i|\right)-\left(\frac{3s-1}{2}\right). \tag{11}$$

This comb inequality can be derived from the capacitated path, wheelbarrow or bicycle inequality (8).

Theorem 4.3. *The capacitated comb inequality* (11) *is valid for* CVRP.

Proof. First, note that $x(E(W_i))=|W_i|-\frac{1}{2}x(\delta(W_i))$, for $i=0,\ 2k,\ldots,s$ and $x(E(W_i))=|W_i|-\frac{1}{2}x(\delta(W_i))+(k-1)$ for $i=1,\ldots,2k-1$. It follows that

$$\sum_{i=0}^{s} x(E(W_i))=\sum_{i=0}^{s}(|W_i|-\tfrac{1}{2}x(\delta(W_i)))+(2k-1)(k-1).$$

Now, it can be verified that $\sum_{i=0}^{s}x(\delta(W_i))=\sum_{e\in E}f_ex_e+(2k-2)x(\delta(Q))$. Since $\sum_{e\in E}f_ex_e\geqslant 3(s+1)-2k$ by Theorem 3.8, and $x(\delta(Q))\geqslant 2k$ by condition (viii), it follows that

$$\sum_{i=0}^{s} x(E(W_i))\leqslant\left(\sum_{i=0}^{s}|W_i|\right)-\left(\frac{3s+1}{2}\right)+1. \qquad \square$$

Finally, we note that the capacitated comb inequalities can be transformed into a different form, using the degree constraints. In this equivalent form, the depot belongs to the handle W_0 and exactly $2k-1$ teeth $W_1,\ldots,W_{2k-1}$. Formally, $W_0,\ldots,W_s$ satisfy conditions (i), (ii), (iv), (iii'),

(vi') $(W_i\backslash W_0)\cap(W_j\backslash W_0)=\emptyset$, $1\leqslant i<j\leqslant 2k-1$;

(vii') $Q=W_0\cap W_i$, $1\leqslant i\leqslant 2k-1$;

and (viii). Then the inequality

$$\sum_{i=0}^{s} x(E(W_i))\leqslant\left(\sum_{i=0}^{s}|W_i|\right)-\frac{3s+1}{2}+k \tag{12}$$

is valid for CVRP.

5. Two examples

In this section, we discuss the solution of two examples, using the results presented previously. It will also serve to highlight some of the difficulties inherent to vehicle routing problems. In the first example, we are given a set of 18 customers whose demands are all equal to one unit to be served from a depot. The vehicle capacity is ten units. Table 5.1 gives the symmetric distance matrix in which 0 denotes the depot.

Table 5.1

Distance matrix for the 18-customer problem

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0		126	76	51	39	63	65	107	38	61	79	70	73	110	16	23	97	80	115
1			98	148	94	118	146	52	164	133	206	173	198	20	121	133	65	50	38
2				125	42	121	53	52	100	127	138	146	141	94	86	98	115	54	113
3					83	37	115	145	62	23	87	25	67	128	39	28	96	112	124
4						79	60	68	72	85	114	106	111	81	45	57	85	44	93
5							127	126	90	16	122	60	104	98	46	46	60	92	90
6								105	66	127	95	129	105	137	81	88	145	97	152
7									140	138	181	170	179	55	109	122	94	34	79
8										82	42	67	41	148	49	44	134	116	154
9											111	44	91	113	45	40	77	104	106
10												79	24	190	88	80	173	158	194
11													56	154	61	48	121	137	149
12														181	77	67	159	153	183
13															103	115	46	40	23
14																13	85	78	106
15																	92	91	115
16																		64	30
17																			59

It is clear that at least two vehicles are necessary to visit all the customers. Here, we solve to optimality the 18-customer CVRP for three different values of k, i.e., $k = 2, 3, 4$. The method we use is a linear programming based cutting plane technique which proceeds in the following manner. First, a relaxed linear program which consists of the objective function, the degree constraints and the upper bound constraints is solved. If the optimal solution corresponds to a k-tour we are done. Else, we strengthen the relaxation by adding a set of facetial inequalities to the formulation. Following this, we solve the augmented linear program and proceed as before. We generate the facets by hand from the current optimal LP-solution. So far, no automatic techniques for identifying violated facets of CVRP are available to our knowledge. Of course, we would need to devise such procedures in order to solve large problems. The facet identification procedures would then be embedded in a cutting plane algorithm in the spirit of the branch-and-cut-algorithm of Padberg and Rinaldi [31].

Let us now present the solution of this example. In what follows, a violated capacity constraint will be designated by its client set W and right-hand side $\text{RHS} = |W| - r(W)$. It should be noted that we will use the capacity constraint $x(\delta(W)) \geq 2r(W)$ in the equivalent form $x(E(W)) \leq |W| - r(W)$. When $r(W) = 1$ for some W, we refer to the corresponding inequality as a subtour elimination constraint.

We begin by investigating the case when $k = 4$. The starting LP-solution, which is not a 4-tour, has value 767.0. The optimal 4-tour with length 807 was found after three iterations. It required the addition of one subtour elimination constraint, $W = \{1, 7, 13, 16, 17, 18\}$ with $\text{RHS} = 5$, and two capacity constraints, $W = \{1, 3, 5, 8, 9, 10, 11, 12, 13, 16, 18\}$ with $\text{RHS} = 9$ and $W = \{1, 3, 5, 7, 9, 10, 11, 12, 13, 16, 17, 18\}$ with $\text{RHS} = 10$.

When $k = 3$, the optimal 3-tour with length 767.0 was obtained after five iterations by imposing two subtour elimination and five capacity constraints. Now we turn to the tighter case $k = 2$. At the first iteration, the LP-solution value is 697.0. In subsequent iterations listed below, we added the following inequalities.

Iteration 2. Capacity constraint:

$$W = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18\}, \qquad \text{RHS} = 14.$$

LP-solution value: 719.5.

Iteration 3. Capacity constraint:

$$W = \{3, 5, 8, 9, 10, 11, 12, 14, 15, 16, 18\}, \qquad \text{RHS} = 9.$$

LP-solution value: 720.0.

Iteration 4. Subtour elimination constraint:

$$W = \{1, 13, 16, 17, 18\}, \qquad \text{RHS} = 4.$$

LP-solution value: 721.00.

Iteration 5. Subtour elimination constraint:

$$W = \{1, 7, 13, 16, 17, 18\}, \qquad \text{RHS} = 5.$$

LP-solution value: 726.5.

Iteration 6. Capacity constraint:

$$W = \{1, 2, 3, 5, 7, 9, 11, 13, 16, 17, 18\}, \qquad \text{RHS} = 9.$$

LP-solution value: 727.5.

Iteration 7. Capacity constraint:

$$W = \{1, 2, 4, 5, 6, 7, 9, 13, 16, 17, 18\}, \qquad \text{RHS} = 9.$$

LP-solution value: 738.66.

Iteration 8. Capacity constraint:

$$W = \{1, 3, 5, 7, 9, 10, 11, 12, 13, 16, 17, 18\}, \qquad \text{RHS} = 10.$$

LP-solution value: 740.5.

Iteration 9. Subtour elimination constraint:

$$W = \{8, 10, 12\}, \qquad \text{RHS} = 2.$$

LP-solution value: 741.

The current solution represents a 2-tour satisfying the capacity constraints and is therefore optimal. The routes are

route 1: 0 - 14 - 15 - 5 - 9 - 3 - 11 - 12 - 10 - 8 - 0,

route 2: 0 - 4 - 17 - 16 - 18 - 13 - 1 - 7 - 2 - 6 - 0.

The subtour elimination and capacity constraints were sufficient to solve all three instances of this 18 customer problem. Of course, in general, other inequalities, such as comb inequalities, are needed to solve vehicle routing problems to optimality. We encountered instances where the following type of inequalities had to be used. We say that $F \subseteq E$ is a *hypo-tour* if F does not contain a simple k-tour but $F \cup \{e\}$ does, for some $e \in E \setminus F$. Accordingly, we define

$$x(F) \leqslant n + k - 1, \tag{13}$$

the hypo-tour inequality. Clearly, this inequality is valid for CVRP. Moreover, by taking a maximal set F (with regard to inclusion), we get stronger hypo-tour inequalities. It is important to observe that the validity of the hypo-tour inequalities depends explicitly on the value of the vehicle capacity.

As it appears from the numerical example presented above, the addition of the subtour elimination and capacity constraints to the linear program has proven very powerful. Next, we attempt to investigate further the computational value of the capacity constraints by solving exactly a 50-customer benchmark test problem [5]. The solution procedure that we adopt is a two-phase method. The first phase, which can be viewed as a priori cut-generation, is used to rapidly generate potential capacity inequalities that might be necessary to cut off fractional points at low computational cost. This is achieved in the following manner.

Step 1. Construct a minimum spanning tree T on the set of all client nodes. Then, identify capacity inequalities from the tree T as follows. It is clear that the removal of any edge of the tree T will partition the node set of T into two nonempty subsets, say, S_1 and S_2. One can then associate a capacity inequality with either S_1 or S_2. Repeatedly removing single edges of the tree T will yield a number of capacity inequalities.

Step 2. Adjoin to the spanning tree T a set of $2k$ (cheapest) edges incident to the depot. Identify capacity inequalities that prevent the occurrence of illegal subtours (cycles). Observe simply that the addition of $2k$ edges (incident to the depot) to the spanning tree T may result in a number of subtours possibly infeasible.

The second phase is essentially identical to the solution procedure used to solve the 18-customer problem, except that the initial relaxed linear program is strengthened by adding all the valid inequalities found in the first phase. The purpose of this combined technique is to get as quickly and cheaply (in terms of computational effort) as possible a good lower bound on the integer optimum. Let us now illustrate this method by considering the 50-customer problem. To solve this problem, we considered the addition of four classes of cutting planes that include subtour elimination constraints (SEC), capacity constraints (CAPC), comb inequalities (TCOMB) without the depot, and comb inequalities (VCOMB) which contain the depot in some tooth only and are of the form (10). Let us note that the minimum number of vehicles needed for this problem is five. Moreover, the best heuristic solution that uses five vehicles has length 521.0, see [13]. Let it be remarked that the length of the relaxed linear program consisting only of the objective function, the degree constraints and the upper bound constraints is equal to 481.5. Therefore, the initial gap between the continuous and the integer (heuristic) solutions is 36.5. Now let us proceed to phase 1. In the first step, we have generated from the spanning tree 34 capacity inequalities out of which 11 are simple subtour elimination constraints. In the second step, we have derived 45 capacity inequalities. Therefore, the first phase has enabled us to identify 79 valid inequalities. Adding these inequalities to the relaxed linear program yields a solution of length 510.28. The gap is then substantially reduced to 10.72. Now we continue with the second phase. At the current fractional solution we have identified, using visual inspection, 23 violated capacity inequalities. The linear program is solved after adding these cuts and a lower bound of value 512.2 is obtained. Subsequently, we added a set of 71 violated cuts which includes 49 CAPC, 14 TCOMB and 8 VCOMB. The LP-solution is now 516.42. Note that the gap has been reduced to 4.58. At this point, we resort to branch-and-cut [31] using the GAMS system [19]. The optimal solution found at node 10 of the search-tree has value 521 (identical to the heuristic) and required the enumeration of 21 additional nodes to prove its optimality. During the enumeration of the 31 node search-tree, a total number of 45 cuts were generated consisting of 3 SEC, 37 CAPC, 3 TCOMB and 2 VCOMB. Overall, a set of 218 inequalities were generated out of which 177 are capacity inequalities. A summary of the search-tree is shown in Figure 5.1. Here the number besides and inside each node of the tree represents the lower bound obtained and the number of cuts added, respectively. This example clearly illustrates the computational value of the cuts considered, especially the capacity inequalities. These partial findings which are quite encouraging indicate that larger instances of CVRP may be solved using the

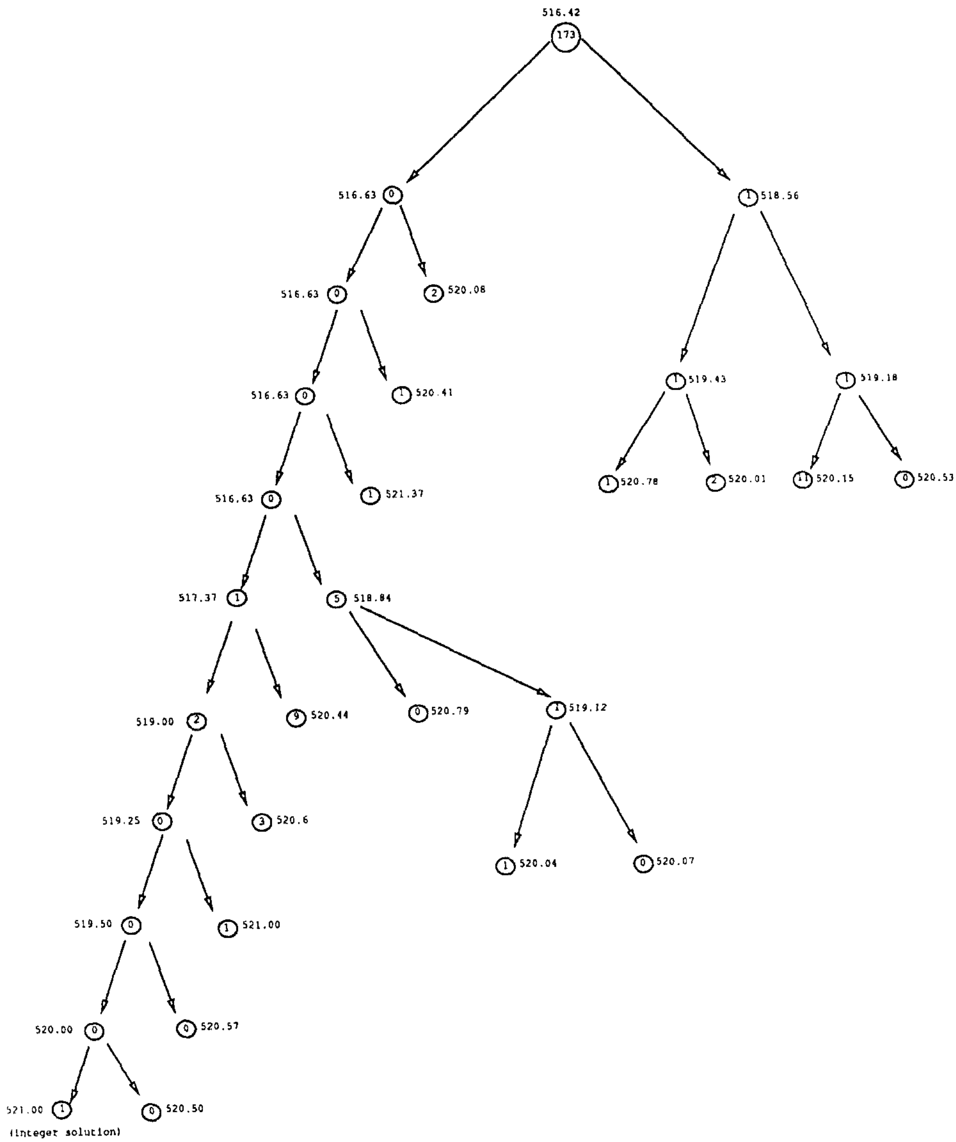

Fig. 5.1. Branch-and-cut search tree.

proposed method. Future research work will be focused on devising an automatic facet identification procedure.

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References

[1] J.R. Araque, "Solution of a 48-city vehicle routing problem by branch-and-cut," Working paper, department of Applied Mathematics and Statistics, SUNY (Stony Brook, NY, 1989).
[2] L.D. Bodin, B.L. Golden, A. Assad and M. Ball, "Routing and scheduling of vehicles and crews, the state of the art," *Computers and Operations Research* 10 (1983) 69–211.
[3] V. Campos, A. Corberan and E. Mota, "Polyhedral results for a vehicle routing problem," *European Journal of Operational Research* 52 (1991) 75-85.
[4] N. Christofides, "Vehicle routing," in: E.L. Lawler, J.K. Lenstra, A.H.G. Rinnoy Kan and D.B. Shmoys, eds., *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization* (Wiley, New York, 1985).
[5] N. Christofides and S. Eilon, "An algorithm for the vehicle dispatching problem," *Operations Research Quarterly* 20 (1969) 309-318.
[6] V. Chvátal, "Edmonds polytopes and weakly Hamiltonian graphs," *Mathematical Programing* 5 (1973) 29-40.
[7] G. Cornuejols, J. Fonlupt and D. Naddef, "The traveling salesman problem on a graph and some related integer polyhedra," *Mathematical Programming* 33 (1985) 1–27.
[8] G.B. Dantzig, D.R. Fulkerson and S.M. Johnson, "Solution of a large-scale traveling salesman problem," *Operations Research* 2 (1954) 393-410.
[9] M.L. Fisher, "Optimal solution of vehicle routing problems using minimum K-trees," Working paper 89-12-13, Department of Decision Sciences, The Wharton School, University of Pennsylvania (Philadelphia, PA, 1990).
[10] B. Fleischmann, "Linear programming approaches to traveling salesman and vehicle scheduling problems"' presented at the *XIth International Symposium on Mathematical Programming, Bonn,* 1982.
[11] B. Fleischmann, "A new class of cutting planes for the symmetric traveling salesman problem," *Mathematical Programming* 40 (1988) 225–246.
[12] J. Fonlupt and D. Naddef, "The traveling salesman problem with some excluded minors," Research Report 557, Université Scientifique et Médicale (Grenoble, France, 1985).
[13] B. Foster and D. Ryan, "An integer programming approach to the vehicle scheduling problem," *Operational Research Quarterly* 27 (1976) 367-385.
[14] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (Freeman, New York, 1979).
[15] M. Grötschel, "On the symmetric traveling salesman problem: Solution of a 120-city problem," *Mathematical Programming Studies* 12 (1980) 61–77.
[16] M. Grötschel and M.W. Padberg, "On the symmetric traveling salesman problem: I–II," *Mathematical Programming* 16 (1979) 265-280 and 281-302.
[17] M. Grötschel and W. Pulleyblank, "Clique tree inequalities and the traveling salesman problem," *Mathematics of Operations Research* 11 (1986) 537-569.
[18] A. Kearny, "Improving productivity in physical distribution," Report Undertaken for CPDM (London, 1980).
[19] D. Kendrick and A. Meerans, "GAMS - An introduction, user's manual for GAMS," Development and Research Department of the World Bank (1985).
[20] G. Laporte, "An integer programming approach to the vehicle scheduling problem," Cahiers du GERAD, G-82-10, Ecole des Hautes Etudes Commerciales de Montreal (1982).
[21] G. Laporte and J.M. Bourjolly, "Some further results on k-star constraints and comb inequalities," Cahiers du GERAD, G-84-17, Ecole des Hautes Etudes Commerciales (Montreal, Que., 1984).
[22] G. Laporte and Y. Nobert, "Comb inequalities for the vehicle routing problem," *Methods of Operations Research* 51 (1984) 271-276.

[23] G. Laporte and Y. Nobert, "Exact algorithms for the vehicle routing problem," *Annals of Discrete Mathematics* 31 (1987) 147-184.

[24] G. Laporte, Y. Nobert and M. Desrochers, 'Optimal routing under capacity and distance restrictions," *Operations Research* 33 (1985) 1050-1073.

[25] E.L. Lawler, J.K. Lenstra, A.H.G. Rinnoy Kan and D.B. Shmoys, eds., *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization* (Wiley, New York, 1985).

[26] T.L. Magnanti, "Combinatorial optimization and vehicle fleet planning: Perspectives and prospects," *Networks* 11 (1981) 179-214.

[27] D. Naddef and G. Rinaldi, "The traveling salesman problem: A composition of valid inequalities given by the graphical relaxation," to appear in *Mathematical Programming*.

[28] D. Naddef and G. Rinaldi, "The symmetric traveling salesman polytope: New facets from the graphical relaxation," forthcoming.

[29] G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization* (Wiley, New York, 1988).

[30] M.W. Padberg and S. Hong, "On the symmetric traveling salesman problem: A computational study," *Mathematical Programming Studies* 12 (1980) 78-107.

[31] M.W. Padberg and G. Rinaldi, "Optimization of a 532-city symmetric traveling salesman problem," *Operations Research Letters* 6 (1987) 1-8.

[32] P. Toth, "Heuristic algorithms for the vehicle routing problem," presented at *the Workshop on Routing Problems*, Hamburg, 1984.