

CLASSES OF FUNCTIONS AND FEASIBILITY CONDITIONS IN NONLINEAR COMPLEMENTARITY PROBLEMS*

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Given a mapping F from real Euclidean n -space into itself, we investigate the connection between various known classes of functions and the nonlinear complementarity problem: Find an x^* such that $Fx^* \geq 0$ and $x^* \geq 0$ are orthogonal. In particular, we study the extent to which the existence of a $u \geq 0$ with $Fu \geq 0$ (feasible point) implies the existence of a solution to the nonlinear complementarity problem, and extend, to nonlinear mappings, known results in the linear complementarity problem on P-matrices, diagonally dominant matrices with non-negative diagonal elements, matrices with off-diagonal non-positive entries, and positive semi-definite matrices.

1. Introduction

Let \mathbf{R}^n denote Euclidean n -space, (\cdot, \cdot) the usual inner product, and \mathbf{R}_+^n the set of x in \mathbf{R}^n with $x \geq 0$ in the componentwise ordering. Then given $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ defined on \mathbf{R}_+^n , the nonlinear complementarity problem consists of finding an $x^* \geq 0$ such that

$$Fx^* \geq 0, \quad (x^*, Fx^*) = 0. \quad (1)$$

This problem has received quite a lot of attention in both the linear case, where $Fx = Ax - z$ for some matrix A and vector z in \mathbf{R}^n , and in the nonlinear case [7,13,17]. In the nonlinear case, efforts to obtain existence results have mainly been based on growth conditions (coercivity conditions) on F . In the linear case, existence results have also been obtained by restricting the type of matrices under consideration, and by assuming the existence of a feasible point; that is, the existence of a $u \geq 0$ such that $Fu \geq 0$.

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For example, in 1964 Cottle [2] proved that if A is positive semi-definite, and $A u \geq z$ for some $u \geq 0$, then there is an $x^* \geq 0$ which satisfies (1) with $F x = A x - z$. The purpose of this paper is to investigate the extent to which results like these are carried over to nonlinear mappings.

In Section 2 we extend the result which guarantees a unique solution $x^* \geq 0$ to (1) if $F x = A x - z$ and A is a P-matrix. This is done by taking the P-functions, as defined by Moré and Rheinboldt [18], and proving that if F is a P-function on \mathbf{R}_+^n , then (1) has at most one solution in \mathbf{R}_+^n . An example shows that there may be no solutions.

In Section 3 we obtain some nonlinear feasibility results. Theorems analogous to those proved by Cottle [2] and Chandrasekaran [1] in the linear case are obtained by considering the monotone [15] and off-diagonally antitone [19] functions. These mappings are nonlinear versions of positive semidefinite matrices and matrices with non-positive off-diagonal entries, respectively. We also prove that if A is a diagonally dominant matrix with nonnegative diagonal entries, then A is a row adequate matrix, and thus a feasibility result for these matrices follows from a theorem of Eaves [7]. We then use the nonlinear version of diagonal dominance due to Moré [14] to extend this result.

One last point: We do not make any comments on the relationship between the various classes of functions; the interested reader should consult [18].

2. Uniqueness

Many researchers have contributed to one of the nicest theorems on the linear complementarity problem; for example, Cottle and Dantzig [4], and Ingleton [11]. Eaves [7, page 626] has a complete list of references.

Theorem 2.1. *The n by n real matrix A has positive principal minors if and only if for each z in \mathbf{R}^n there is a unique $x^* \geq 0$ with $A x \geq z$ and $(x^*, A x^* - z) = 0$.*

Previous to this result, Fiedler and Pták [9] had formally called all matrices with positive principal minors P-matrices and showed, among other things, that A is a P-matrix if and only if for each $x \neq 0$ there is

an index i for which $x_i y_i > 0$, where $y = Ax$. On the other hand, Moré and Rheinboldt [18], motivated by other problems, studied certain nonlinear mappings which were closely related to the P-matrices.

Definition 2.2. The mapping $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a P-function on the set D if for each $x \neq y$ in D there is an index $k = k(x, y)$ such that

$$(x_k - y_k) [f_k(x) - f_k(y)] > 0.$$

Here x_k and $f_k(x)$ are the k^{th} components of x and Fx , respectively.

The following result is an immediate consequence of the above definition.

Theorem 2.3. Let $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a P-function on \mathbf{R}_+^n . Then there is at most one x^* in \mathbf{R}_+^n which satisfies (1).

Proof. If x^* and y^* in \mathbf{R}_+^n satisfy (1), then

$$(x_i^* - y_i^*) [f_i(x^*) - f_i(y^*)] = -y_i^* f_i(x^*) - x_i^* f_i(y^*) \leq 0$$

for each i , and since F is a P-function, $y^* = x^*$ as desired.

Moré and Rheinboldt [18, Theorem 5.2] proved that if $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is (Fréchet) differentiable in \mathbf{R}_+^n and $F'(x)$ is a P-matrix for each x in \mathbf{R}_+^n , then F is a P-function on \mathbf{R}_+^n . Therefore, if $F'(x)$ is a P-matrix for each x in \mathbf{R}_+^n , then there is at most one x^* in \mathbf{R}_+^n which satisfies (1). This observation applies, in particular, to the result of Cottle [3].

Although the uniqueness part of Theorem 2.1 extends to the non-linear P-functions, this is not the case for the existence part.

Example 2.4. Define $F : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ by $f_1(x) = \varphi(x_1) + x_2$ and $f_2(x) = x_2$, where $\varphi : \mathbf{R}^1 \rightarrow \mathbf{R}^1$ is any strictly increasing function with $\varphi(t) \neq 0$ for each t . This mapping is a P-function, but if x^* in \mathbf{R}_+^2 satisfies (1), then $x_2^* = 0$ and thus $x_1^* \varphi(x_1^*) = 0$, which implies that $x_1^* = 0$. However, $F(0) \notin \mathbf{R}_+^2$.

Efforts to extend the existence part of Theorem 2.1 have usually been based on conditions involving the growth of F (coercivity conditions). For example, in [17] it is shown that if either F is a convex P-function on \mathbf{R}_+^n or if there is a $c > 0$ such that

$$\max_{1 \leq i \leq n} \{(x_i - y_i) [f_i(x) - f_i(y)]\} \geq c \|x - y\|^2$$

for all x and y in \mathbf{R}_+^n , then there is precisely one x^* in \mathbf{R}_+^n which satisfies (1). The proof of these results is based on Theorem 2.5, and on the fact that in either case there are constants α and β with $\alpha > 0$ such that

$$\max_{1 \leq i \leq n} \{x_i f_i(x) / \|x\|_\infty\} \geq \alpha \|x\|_\infty + \beta \tag{2}$$

for all $x \neq 0$ in \mathbf{R}_+^n , where $\|x\|_\infty = \max |x_i|$.

Theorem 2.5. *Assume $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is continuous on \mathbf{R}_+^n . If there is a u in \mathbf{R}_+^n and a constant $r > \|u\|_\infty$ such that*

$$\max_{1 \leq i \leq n} \{(x_i - u_i) f_i(x)\} > 0$$

for all x in \mathbf{R}_+^n with $\|x\|_\infty = r$, then (1) has a solution $x^ \geq 0$ with $\|x^*\|_\infty \leq r$.*

The proof, as well as a discussion of the relationship of Theorem 2.5 to other results in the literature can be found in [17, Section 2]. In that paper, Theorem 2.5 was used to obtain existence results under coercivity conditions such as (2); in this paper it will be used to obtain existence results under feasibility conditions.

3. Classes of functions

There are several results which guarantee the existence of a solution to the linear complementarity problem with a feasibility assumption. Eaves [7] has unified some of these results, and we now investigate the extent to which his results carry over to nonlinear mappings.

Theorem 3.1. *Let A be an n by n real matrix and assume that there is a $u \geq 0$ such that $A u \geq z$. If, in addition, A satisfies the following two assumptions, then there is an $x^* \geq 0$ such that $A x^* \geq z$ and $(x^*, A x^* - z) = 0$.*

- (a) *For each non-zero $x \geq 0$ there is an index k such that $x_k > 0$ and $(A x)_k \geq 0$.*
- (b) *For each non-zero $x \geq 0$ such that $A x \geq 0$ and $(x, A x) = 0$, there*

are nonnegative diagonal matrices D_1 and D_2 such that $D_2x \neq 0$ and $(D_1A + A^T D_2)x = 0$.

Although Theorem 3.1 has been extended by Garcia [10], we will not consider his result since even Theorem 3.1 cannot be generalized to nonlinear mappings in the sense that the P-matrices clearly satisfy the hypotheses of Theorem 3.1, but the mapping F defined in Example 2.4 is a P-function, there is a $u \geq 0$ such that $Fu \geq 0$, but (1) does not have any solution $x^* \geq 0$.

However, it is possible to extend parts of Theorem 3.1 to nonlinear mappings. For example, consider the monotone mappings [15]: A function $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is *monotone* on D if $(x - y, Fx - Fy) \geq 0$ for each x and y in D . If F is defined by $Fx = Ax - z$ for some matrix A and z in \mathbf{R}^n , then F is monotone on \mathbf{R}_+^n if and only if A is positive semidefinite, and if this is the case, then A satisfies the assumptions of Theorem 3.1 when D_1 and D_2 are the n by n identity matrices. Therefore, the following result extends Theorem 3.1 in the case that A is positive semidefinite.

Theorem 3.2. *Let $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a continuous monotone mapping on \mathbf{R}_+^n . If there is a $u \geq 0$ with $Fu > 0$, then (1) has a solution $x^* \geq 0$.*

Proof. Since F is monotone,

$$(x - u, Fx) \geq (x - u, Fu) = (x, Fu) - (u, Fu),$$

and since $Fu > 0$, it is clear that there is an $r > \|u\|_\infty$ such that $(x, Fu) > (u, Fu)$ for all $x \geq 0$ with $\|x\|_\infty = r$. Theorem 2.5 now gives the result.

It is not known whether this result holds if we only assume that $Fu \geq 0$. However, if F is *strictly monotone* on \mathbf{R}_+^n (if $x \neq y$ are in \mathbf{R}_+^n , then $(x - y, Fx - Fy) > 0$), then a result of Karamardian [12, Theorem 3] implies that for any u in \mathbf{R}_+^n there is a $v \geq u$ such that $Fv > Fu$. Thus, if F is strictly monotone on \mathbf{R}_+^n , Theorem 3.2 holds if we only assume that $Fu \geq 0$, and since now F is a P-function, (1) has precisely one solution $x^* \geq 0$.

To generalize Theorem 3.1 in another direction, consider a mapping $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that for each $x \neq y$ in \mathbf{R}^n and any index k with $|x_k - y_k| = \|x - y\|_\infty$ it follows that

$$(x_k - y_k) [f_k(x) - f_k(y)] \geq 0. \quad (3)$$

These mappings come up naturally in the work of Moré [14] on non-linear generalizations of matrix diagonal dominance; in the linear case, it is easy to prove that they coincide with the diagonally dominant matrices with nonnegative diagonal elements. In fact, if for some matrix A we have

$$a_{ii} \geq \sum_{j \neq i} |a_{ij}| \quad \text{for } i = 1, \dots, n,$$

then for any $x \neq 0$ and index k with $|x_k| = \|x\|_\infty$,

$$\begin{aligned} x_k \sum_{j=1}^n a_{kj} x_j &\geq a_{kk} |x_k|^2 - \sum_{j \neq k} (|a_{kj}| |x_j| |x_k|) \\ &\geq \left(a_{kk} - \sum_{j \neq k} |a_{kj}| \right) \|x\|_\infty^2 \geq 0. \end{aligned}$$

Conversely, let $x_j = -\text{sgn } a_{kj}$ for $j \neq k$ and $x_k = 1$. Then $x \neq 0$ and $|x_k| = \|x\|_\infty$. Hence

$$x_k \sum_{j \neq k} a_{kj} x_j = a_{kk} - \sum_{j \neq k} |a_{kj}| \geq 0$$

as desired.

However, it is not yet clear that the diagonally dominant matrices with nonnegative diagonal entries satisfy the hypotheses of Theorem 3.1. To do this, we will prove that any such matrix is row adequate in the following sense: A real n by n matrix A is row adequate if

(a) A is a P_0 -matrix (for each $x \neq 0$ in \mathbf{R}^n there is an index k with $x_k \neq 0$ and $x_k y_k \geq 0$, where $y = Ax$),

(b) If a principal submatrix B of A is singular, then the rows of A corresponding to those of B are dependent.

That any diagonally dominant matrix with nonnegative diagonal entries is a P_0 -matrix is clear from (3), while (b) is proved below. We can now invoke a result of Eaves [7, Theorem 9.22] which implies that any row adequate matrix satisfies the hypotheses of Theorem 3.1.

We now prove that the diagonally dominant matrices satisfy (b).

Lemma 3.3. *If the n by n matrix A is diagonally dominant, and some*

principal submatrix B of A is singular, then the rows of A corresponding to those of B are linearly dependent.

Proof. By permuting the rows and columns of A we can assume that B is the principal submatrix of order $m < n$ in the upper left-hand corner of A . Since B is singular,

$$\sum_{j=1}^m a_{ij}x_j = 0, \quad 1 \leq i \leq m, \tag{4}$$

for some $x \neq 0$ in \mathbf{R}^m . Without loss of generality assume that

$$|x_i| = \max \{|x_j| : 1 \leq j \leq m\} \equiv \alpha$$

for $1 \leq i \leq r$, where $r \leq m$. We now show that $a_{ij} = 0$ for $1 \leq i \leq r$ and $j > r$. To see this, note that (4) implies that

$$|a_{ii}| |x_i| \leq \sum_{\substack{j=1 \\ j \neq i}}^m |a_{ij}| |x_j| \leq \alpha \sum_{\substack{j=1 \\ j \neq i}}^m |a_{ij}|, \tag{5}$$

and since $|x_i| = \alpha > 0$ for $1 \leq i \leq r$,

$$|a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^m |a_{ij}|.$$

This implies that $a_{ij} = 0$ for $j > m$ and $1 \leq i \leq r$. But if $a_{ij} \neq 0$ for $r < j \leq m$, then the second inequality in (5) would be strict, and hence the above inequality would also be strict. This contradicts the diagonal dominance of A .

Thus, it follows that $a_{ij} = 0$ for $1 \leq i \leq r$ and $j > r$, and that the principal submatrix of order r in the upper left-hand corner of A is singular. This implies that the first r rows of A are dependent and thus concludes the proof.

We have now proved that the diagonally dominant matrices with nonnegative diagonal elements form a subclass of those matrices defined by Theorem 3.1. Therefore, the following result is an extension to nonlinear functions of part (b) of Theorem 3.1.

Theorem 3.4. Assume that $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a continuous function on \mathbf{R}_+^n such that for each $x \neq y$ in \mathbf{R}_+^n and any index k with $|x_k - y_k| = \|x - y\|_\infty$

it follows that (3) holds. If $Fu \geq 0$ for some $u \geq 0$, then (1) has a solution $x^* \geq 0$.

Proof. Assume at first that $Fu > 0$. Now choose $r > 2\|u\|_\infty$ and take any $x \geq 0$ with $\|x\|_\infty = r$. Then $x \neq u$ and $|x_k - u_k| = \|x - u\|_\infty$ implies that $x_k > u_k$. Thus,

$$\begin{aligned} \max_{1 \leq i \leq n} \{(x_i - u_i) f_i(x)\} &\geq (x_k - u_k) f_k(x) \\ &\geq (x_k - u_k) f_k(u) > 0, \end{aligned}$$

and the existence of x^* follows from Theorem 2.5. Now assume that $Fu \geq 0$ and consider the mapping G defined by $g_i(x) = f_i(x) + \beta$ for any $\beta > 0$. Since $Gu > 0$, there is an $x^*(\beta) \geq 0$ with $\|x^*(\beta)\|_\infty \leq r$ for which

$$G[x^*(\beta)] \geq 0 \quad \text{and} \quad (x^*(\beta), G[x^*(\beta)]) = 0. \tag{6}$$

But since $\|x^*(\beta)\|_\infty \leq r$, there is a sequence $\{\beta_k\}$, converging to zero such that $\{x^*(\beta_k)\}$ converges to some $x^* \geq 0$. It follows from (6) that x^* solves (1).

We have not been able to generalize Theorem 3.1 any further. We now go outside these matrices and generalize a result of Chandrasekaran [1]: If the real n by n matrix A has non-positive off-diagonal elements, and if for some z in \mathbf{R}^n there is a $u \geq 0$ with $Au \geq z$, then there is an $x^* \geq 0$ with $Ax^* \geq z$ and $(x^*, Ax^* - z) = 0$. To state and prove the corresponding result, we consider nonlinear mappings which are off-diagonally antitone in the sense of [19]; for our purposes, a function $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is *off-diagonally antitone* on D if for each $u \geq v$ in D it follows from $u_k = v_k$ that $f_k(u) \leq f_k(v)$.

Theorem 3.5. *Let $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a continuous, off-diagonally antitone function on \mathbf{R}_+^n . If there is a $u \geq 0$ with $Fu \geq 0$, then there is an $0 \leq x^* \leq u$ which satisfies (1). Moreover, if $F0 \leq 0$, then $Fx^* = 0$.*

Proof. We construct a sequence $\{x^k\}$ such that $Fx^k \geq 0$, where $0 \leq x^{k+1} \leq x^k$ and

$$x_i^{k+1} f_i(x_1^k, \dots, x_{i-1}^k, x_i^{k+1}, x_{i+1}^k, \dots, x_n^k) = 0 \tag{7}$$

for $i = 1, \dots, n$, and $k = 0, 1, \dots$. To do this, we only need to show that

if $x^k \geq 0$ and $F x^k \geq 0$, then there is an $x^{k+1} \geq 0$ which satisfies the properties above. For any index i , set $x_i^{k+1} = 0$ if

$$f_i(x_1^k, \dots, x_{i-1}^k, 0, x_{i+1}^k, \dots, x_n^k) \geq 0 ;$$

otherwise, since $f_i(x^k) \geq 0$, there is an $x_i^{k+1} \in [0, x_i^k]$ which satisfies (7). In either case,

$$f_i(x_1^k, \dots, x_{i-1}^k, x_i^{k+1}, x_{i+1}^k, \dots, x_n^k) \geq 0 . \tag{8}$$

Thus, $0 \leq x^{k+1} \leq x^k$, and since F is off-diagonally antitone, $F x^{k+1} \geq 0$.

Now that the sequence is constructed, note that since $0 \leq x^{k+1} \leq x^k$, the iterates $\{x^k\}$ converge to some $x^* \geq 0$, and by (8) we have that $F x^* \geq 0$, while (7) shows that x^* satisfies (1). Finally, if $F 0 \leq 0$ but $f_i(x^*) > 0$ for some i , then (1) implies that $x_i^* = 0$, and therefore the off-diagonal antitonicity of F implies that

$$0 < f_i(x^*) \leq f_i(0) \leq 0 ,$$

which is a contradiction.

Theorem 3.5 was proved independently by A. Tamir in his Ph. D. thesis [20]. In his proof, Tamir generalizes the method of Chandrasekaran [1] to nonlinear mappings and actually shows that under the conditions of Theorem 3.5 there is a minimal solution x^* to (1) in the sense that $x^* \leq y^*$ for any other solution y^* of (1).

The sequence defined in the proof of Theorem 3.5 is a modification of the nonlinear Jacobi iteration used to solve certain nonlinear equation as in [16] and [19]; we could also have used an iterative scheme similar to the nonlinear (under-relaxed) Gauss–Seidel iteration, where (7) would have been replaced by

$$s_i^k [f_i(x_i^{k+1}, \dots, x_{i-1}^{k+1}, s_i^k, x_{i+1}^k, \dots, x_n^k)] = 0 ,$$

and $x_i^{k+1} = (1 - \omega) x_i^k + \omega s_i^k$, with $0 < \omega \leq 1$, but this would have lengthened the proof. However, iterative methods, and in particular successive over-relaxation, have long been used to solve linear complementarity problems. Cryer [5] has a brief historical survey and some results (see also [6]).

Another point of interest is that we were not able to prove Theorem 3.5 by means of Theorem 2.4. On the other hand, a constructive proof of Theorem 3.2 does not seem to be available unless, for example, F is

a gradient mapping. If this is the case and $Fx = \nabla g(x)$, then the iteration

$$x^{k+1} = [x^k - t_k \nabla g(x^k)]_+$$

where $[y]_+$ is obtained by replacing the negative coordinates of y by zero, and t_k is chosen to minimize $g([x^k - t_k \nabla g(x^k)]_+)$ for $t \geq 0$, is such that every limit point x^* of $\{x^k\}$ (which must necessarily lie in \mathbf{R}_+^n) will satisfy (1). See [14] for details. To show that there is at least one limit point, note that since F is monotone, g is convex on \mathbf{R}_+^n and thus

$$g(x) - g(u) \geq (\nabla g(u), x - u)$$

for $x \geq 0$. Hence, if $Fu = \nabla g(u) > 0$, it follows that $g(x) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$ and $x \geq 0$. Since $g(x^{k+1}) \leq g(x^k)$, this guarantees the existence of at least one limit point. Also note that if F is strictly monotone, then the whole sequence converges to the unique solution $x^* \geq 0$ of (1).

Similarly, with additional conditions on F , we can show that the iteration defined by (7) can be used to give a constructive proof of Theorem 3.4.

Theorem 3.6. *Let $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be continuous on \mathbf{R}_+^n , and assume that for each $x \neq y$ in \mathbf{R}_+^n ,*

$$\|x_k - y_k\| = \|x - y\|_\infty \quad \text{implies} \quad (x_k - y_k) [f_k(x) - f_k(y)] > 0. \quad (9)$$

If $Fu \geq 0$ for some $u \geq 0$, then for any $x^0 \geq 0$ a sequence $\{x^k\}$ is well-defined by (7). Moreover, $\{x^k\}$ converges to the unique $x^ \geq 0$ which satisfies (1).*

Proof. Clearly, (9) implies that F is a P-function on \mathbf{R}_+^n and hence, (1) has at most one solution $x^* \geq 0$. To prove the convergence of $\{x^k\}$, we first show that $\{x^k\}$ is given by $x^{k+1} = Hx^k$ for some mapping $H : \mathbf{R}^n \rightarrow \mathbf{R}^n$ with $\|Hx - Hy\|_\infty < \|x - y\|_\infty$ for any $x \neq y$ in \mathbf{R}_+^n . For this, let

$$\psi(t) = f_i(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n),$$

and note that (9) implies that ψ is a strictly increasing function on \mathbf{R}_+^1 . In addition, if $r = u_i + \|x - u\|_\infty$, then (9) implies that $\psi(r) \geq f_i(u) \geq 0$. Thus, if $\psi(0) < 0$, choose $h_i(x) \in (0, r)$ such that $\psi(h_i(x)) = 0$; otherwise, set $h_i(x) = 0$. In either case, it is clear that (7) is given by $x^{k+1} =$

Hx^k . To show that $\|Hx - Hy\|_\infty < \|x - y\|_\infty$ for $x \neq y$, note that the construction of H shows that

$$[h_i(x) - h_i(y)] [\psi(h_i(x)) - \psi(h_i(y))] \leq 0$$

for each index i . Thus (9) implies that $|h_i(x) - h_i(y)| < \|x - y\|_\infty$ and therefore, $\|Hx - Hy\|_\infty < \|x - y\|_\infty$.

To conclude the proof, we only need to show that $\{x^k\}$ has a convergent subsequence, for then, a theorem of Edelstein [8] shows that $\{x^k\}$ converges to a point which necessarily satisfies (1). Now, for any $x \geq 0$ we have $0 \leq h_i(x) \leq u_i + \|x - u\|_\infty$, since $h_i(x) \in [0, r]$. Thus

$$-\|u\|_\infty \leq h_i(x) - u_i \leq \|x - u\|_\infty$$

for each i , and consequently, either $\|Hx - u\|_\infty \leq \|x - u\|_\infty$, or $\|Hx - u\|_\infty \leq \|u\|_\infty$. This shows that $\{x^k\}$ is bounded for any x^0 in \mathbf{R}_+^n and therefore concludes the proof.

Condition (9) is too restrictive; in the linear case it is equivalent to assuming that the matrix is strictly diagonally dominant with positive diagonal entries. However, it is possible, using the techniques of [16], to extend Theorem 3.6 so as to cover the irreducibly diagonally dominant matrices with positive diagonal entries.

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