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PERFECT ZERO–ONE MATRICES *

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A zero–one matrix is called perfect if the polytope of the associated set packing problem has integral vertices only. By this definition, all totally unimodular zero–one matrices are perfect. In this paper we give a characterization of perfect zero–one matrices in terms of *forbidden submatrices*. Perfect zero–one matrices are closely related to perfect graphs and constitute a generalization of balanced matrices as introduced by C. Berge. Furthermore, the results obtained here bear on an unsolved problem in graph theory, the strong perfect graph conjecture, also due to C. Berge.

1. Introduction

In this paper we consider the polytope defined by the constraints of the following set packing problem:

$$\text{(P)} \qquad \begin{aligned} &\max c\,x, \\ &\text{subject to } A\,x \leqslant \tilde{e}, \\ &x_j = 0 \text{ or } 1 \quad \text{for all } j \in N = \{1, ..., n\}, \end{aligned}$$

where A is a $m \times n$ matrix of zeros and ones, $\tilde{e}^{\mathrm{T}} = (1, ..., 1)$ is the vector having all m components equal to one, and c is an arbitrary vector of reals. This problem has recently obtained much attention, see e.g. [1, 2, 6, 15, 17]. By (LP) we denote the linear programming problem obtained from (P) by dropping the integrality reguirement on x.

Whereas in an earlier paper [15] we have been concerned with the identification of *facets* of the convex hull of solutions to (P), we address ourselves here to the question under what conditions on the constraint matrix A the *relaxed* linear programming problem (LP),

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max $\{c\,x\colon A\,x \leqslant \tilde{e},\, x \geqslant 0\}$, yields an optimal *integral* solution vector *irrespective* of the linear from $c\,x$ that is maximized. This is obviously the case if the matrix A involved in problem (P) is *totally unimodular* [10]. This property remains true if the matrix A in the definition of (P), while not totally unimodular, is *balanced* [5]. Generally, the matrix A encountered in (P) is neither totally unimodular nor balanced. Nevertheless, for certain matrices A, the property that all basic feasible solutions to (LP) are integral remains true (see Section 3 for relevant examples). Using some results from graph theory we give a complete characterization of such matrices A in terms of *forbidden submatrices.* To this end we summarize in Section 2 some relevant theorems and concepts from the literature and introduce the notion of a *critically imperfect* ("*fast-perfekt*") graph. In Section 3 we derive the stated characterization, see Theorem 3.16.

2. Perfect and P-critical graphs

Let G denote a finite undirected graph without loops and multiple edges. By $\overline{G}$ we denote the compliment of G. Following the notation of C. Berge [3] we denote by $\alpha(G)$ the maximum cardinality of a stable (independent) node set in G, by $\theta(G)$ the minimal number of cliques which cover G, by $\gamma(G)$ the chromatic number of G, and by $\omega(G)$ the maximum cardinality of a clique in G. (A *clique* in G is a *maximal* complete subgraph of G.) A graph G is called γ-*perfect* if $\gamma(G') = \omega(G')$ for every induced subgraph G' of G; a graph G is called α-*perfect* if $\alpha(G') = \theta(G')$ for every induced subgraph G' of G. A graph G is *perfect* if it is both α-perfect and γ-perfect.

In a recent paper [13], Lovász has shown that a graph G is α-perfect if and only if G is γ-perfect. This result, usually referred to as the *Perfect Graph Theorem,* implies that it is sufficient to define perfection of a graph solely in terms of α-perfection (or *alternatively,* in terms of γ-perfection). With this in mind, the Perfect Graph Theorem can then be restated as follows: A graph G is perfect if and only if the complement G is perfect.

C. Berge has formulated the following conjecture in connection with this concept which characterizes perfect graphs in terms of forbidden subgraphs:

Strong Perfect Graph Conjecture [3, 4]. A graph G is perfect if and only if G does not contain any odd cycle C_{2k+1} without chords nor its complement $\bar{C}_{2k+1}$ (where $k \geqslant 2$).

A weaker form of the conjecture has recently been proven by Lovász [14] and we shall make use of his characterization of perfect graphs.

Theorem 2.1 [14]. *A graph is perfect if and only if $\omega(G')\,\omega(\bar{G}') \geqslant |G'|$ for every induced subgraph G' of G. ($|G'|$ is the cardinality of the node set of G'.)*

A further result that is strongly related to our problem is due to Chvátal [6] and Fulkerson [7, 8]: Let A be the incidence matrix of *all* cliques of G (rows of A) versus the nodes of G. Suppose that G has n nodes and define the polytope P as follows:

$$P = \{x \in \mathbf{R}^n \colon A\, x \leqslant \tilde{e},\, x \geqslant 0\}\,.$$

Theorem 2.2 ([16, Theorem 1]). *Given any graph G, the following two conditions are equivalent:*

(i) *Every vertex of P is integral,*

(ii) *G is perfect.*

A graph G is called *critically imperfect* (or shortly, *p-critical*) if G is imperfect and every proper induced subgraph of G is perfect. A graph with these properties has been considered by Lovasz [14], see also [16].

Denote by $|G|$ the number of nodes of G and suppose that $|G| = n$. By G_i we denote the induced subgraph obtained from G by deleting the i^{th} node and all edges incident to it, $i = 1, ..., n$. P-critical graphs have the following known properties:

(P1) Every p-critical graph is connected and has at least five nodes. In fact, the "smallest" p-critical graph (smallest in terms of number of nodes and edges) is C_5, i.e. the cycle of length five having no chords.

(P2) A graph G is imperfect if and only if G contains an induced subgraph which is p-critical.

(P3) A graph G is p-critical if and only if its complement $\bar{G}$ is p-critical. This follows immediately from the Perfect Graph Theorem.

(P4) Let G be p-critical and $|G| = n$. Then $\alpha(G) = \alpha(G_i)$ for $i = 1, ..., n$ and $\alpha(G)\ \alpha(\bar{G}) = n - 1$. Indeed, since $\omega(H) = \alpha(H)$ is true for all graphs H, we have by Theorem 2.1 of Lovász [14], that $n-1 \leqslant \alpha(G_i)\,\alpha(G_i) \leqslant \alpha(G)\,\alpha(G) < n$ for all $i = 1, ..., n$. But $\alpha - 1 \leqslant \alpha(G_i) \leqslant \alpha = \alpha(G)$ for $i = 1, ..., n$. Consequently, (P4) is true.

(P5) If G is p-critical, then $2 \leqslant \alpha(G) \leqslant [\frac{1}{2}(n-1)]$ and $2 \leqslant \alpha(\overline{G}) \leqslant [\frac{1}{2}(n-1)]$. This is immediate from (P1) and (P4).

(P6) Let G be p-critical, $\alpha = \alpha(G)$ and $\bar{\alpha} = \alpha(\overline{G})$. Then every node of G is contained in at least one clique of maximum size, and $\theta(G) = \alpha+1$. Indeed, since G_i is perfect, we have by the definition of perfection and the Perfect Graph Theorem that $\theta(G_i) = \alpha$. Furthermore, $|G_i| = n-1$ for $i = 1, ..., n$. Hence for each $i \in \{1, ..., n\}$, G_i can be partitioned into α maximum cliques of G, each of which covers $\bar{\alpha} = \alpha(\overline{G})$ nodes. Consequently, $\theta(G) = \alpha + 1$.

(P7) Let G be p-critical and C any clique in G with $|C| \leqslant \bar{\alpha} = \alpha(\overline{G})$. Denote by $G \setminus C$ the induced subgraph obtained from G by deleting all nodes in C. Then there exists an independent node set F in $G \setminus C$ satisfying $|F| = \alpha = \alpha(G)$. Suppose not, then $\alpha(G \setminus C) \leqslant \alpha - 1$. Since $G \setminus C$ is perfect, we have $\theta(G \setminus C) \leqslant \alpha - 1$, which implies $\theta(G) \leqslant \alpha$, a contradiction to (P6).

Due to property (P2) of perfect graphs we can now reformulate C. Berge's Strong Perfect Graph Conjecture as follows (see also Sachs [16]): A graph G is p-critical if and only if it is an odd cycle C_{2k+1} without chords or its complement $\overline{C}_{2k+1}$, where $k \geqslant 2$.

3. Perfect zero–one matrices

Let A be any $m \times n$ matrix of zeros and ones, and define the polytopes P and P_I as follows:

$$P = \{x \in \mathbf{R}^n\colon A\,x \leqslant \tilde{e},\ x_j \geqslant 0, j = 1, ..., n\},$$
$$P_I = \operatorname{conv}\{x \in P\colon x_j = 0 \text{ or } 1, j = 1, ..., n\}, \tag{3.1}$$

where $\tilde{e}^{\mathrm{T}} = (1, ..., 1)$ has m components, all equal to one. The matrix A is called *perfect* if $P = P_I$, i.e., if the polytope P defined in (3.1) has only integral vertices. Denote by G the (intersection) graph associated with the matrix A, i.e., the nodes of G correspond to the columns of A and two nodes of G are linked by an edge if the associated columns a^i and a^j of A have at least one +1 entry in common. Consequently, G is a finite undirected graph without loops and multiple edges. Let C denote the node set of any clique in G. Then by [15, Theorem 2.4], the inequality

$$\sum_{j=1}^{n} a_j^C x_j \leqslant 1, \qquad a_j^C = \begin{cases} 1 & \text{if } j \in C, \\ 0 & \text{if } j \notin C \end{cases} \tag{3.2}$$

yields a *facet* of P_I, i.e., a face of dimension $n-1$ of P_I. Clearly, every facet of the polytope P_I is *essential* in defining P_I. Hence it is a necessary condition for A to be perfect that A contain the incidence (row-) vectors of all cliques of the associated graph G. Furthermore, if A is perfect, then every row of A that is *not* the incidence (row-) vector of a clique in G is dominated (in the usual vector sense) by some other row of A. Consequently, such a row is *non-essential* in defining the polytope P_I and can be deleted from A. In order to characterize perfect matrices we can thus restrict ourselves to considering only *clique-matrices*, i.e., matrices A which are such that every row vector of A is the incidence vector of a clique of the associated graph G and vice versa.

Let A be any clique-matrix of size $m \times n$. By Theorem 2.2 we have that *A is perfect if and only if the associated graph G is perfect.* Consequently, by (P2), A is *imperfect* if and only if G contains an induced subgraph which is p-critical. Since induced subgraphs of G having k nodes correspond uniquely to $m \times k$ submatrices of A (and vice versa) we can make, without loss of generality, the temporary assumption that G itself is a p-critical graph. Let $\overline{G}$ denote the complement of G, and denote by B the clique-matrix of $\overline{G}$. Similarly to (3.1), define the polytopes Q and Q_I respectively as follows:

$$\begin{aligned} Q &= \{x \in \mathbf{R}^n\colon\ B\,x \leqslant \hat{e},\ x_j \geqslant 0, j = 1, \ldots, n\}, \\ Q_I &= \operatorname{conv}\{x \in Q\colon x_j = 0 \text{ or } 1, j = 1, \ldots, n\}, \end{aligned} \tag{3.3}$$

where $\hat{e}^T = (1, \ldots, 1)$ has all components equal to one and is dimensioned compatibly with B. The vertices of Q_I correspond to *complete* subgraphs of G and vice versa. Furthermore, every *maximal* independent node set in G defines a clique of $\overline{G}$ (and vice versa). Consequently, there exists an (incomplete) "duality" relation between the vertices of P_I and the facets of Q (and hence, between the vertices of Q_I and the facets of P), see e.g. [11]. Using the terminology of Fulkerson [7] we show that Q (P, respectively) is the *anti-blocker* of P_I (of Q_I, respectively), see Remark 3.4.

Let $j \in \{1, \ldots, n\}$ and let G_j denote the induced subgraph obtained from G by deleting the node j and all edges incident to it. Denote by P_j

the polytope defined with respect to the clique-matrix of G_j. Then we have that

$$P_j = P \cap \{x \in \mathbf{R}^n : x_j = 0\}. \tag{3.4}$$

To prove (3.4), we remark that by assumption, A contains *all* cliques of G. Consequently, since every clique of G not containing node j is a clique in G_j and every clique in G_j either is a clique in G (if node j is not contained in it) or becomes a clique in G after adjoining node j to it, relation (3.4) is true. Similarly, we define n polytopes Q_j with respect to $\overline{G}_j$ for $j = 1, ..., n$.

Remark 3.1. If G is p-critical, then P_j(Q_j, respectively) has integral vertices only for every $j \in \{1, ..., n\}$.

Remark 3.2. If G is p-critical, then P(Q, respectively) has at least one fractional vertex.

Remark 3.3. If G is p-critical, then every fractional vertex $\overline{x}$ of P (Q, respectively) satisfies $0 < \overline{x}_j < 1$ for all $j \in \{1, ..., n\}$.

Proof. Suppose that there exists $\overline{x}$, a fractional vertex of P having $\overline{x}_j = 0$. Then $\overline{x} \in P_j$ and $\overline{x}$ is a vertex of P_j. By Remark 3.1, this is impossible. Suppose that there exists $\overline{x}$, a fractional vertex of P, having $\overline{x}_j = 1$. Define $\overline{\overline{x}}$ by $\overline{\overline{x}}_k = \overline{x}_k$ for all $k \neq j$ and $\overline{\overline{x}}_k = 0$ for $k = j$. From the fact that A is a zero–one matrix and $\widetilde{e} = (1, ..., 1)$, it follows readily that $\overline{\overline{x}}$ is a vertex of P and since $\overline{\overline{x}}_j = 0$, $\overline{\overline{x}}$ must be a *fractional* vertex of P_j. Again by Remark 3.1, this is impossible.

Remark 3.4. If G is p-critical, then P (Q, respectively) is the *antiblocker* of Q_I (P_I, respectively).

Proof. Every vertex $\overline{x}$ of P either is an independent node set in G (if $\overline{x}$ is integral) or it satisfies $0 < \overline{x} < e$, where $e^{\mathrm{T}} = (1, ..., 1)$ has n components equal to +1. If $\overline{x}$ is integral, then $\overline{x}$ defines a maximal independent node set in G (or not, in which case it is contained in one or several maximal independent node sets). If $\overline{x}$ furnishes a maximal independent node set in G, this node set constitutes a clique of $\overline{G}$ and hence the constraint $\overline{x}^{\mathrm{T}} x \leqslant 1$ yields a facet for Q_I. (If $\overline{x}$ is not maximal, then $\overline{x}^{\mathrm{T}} x \leqslant 1$ yields a support of Q_I which is *non-essential* in defining Q_I). If $\overline{x}$ is a fractional vertex of P, then $0 < \overline{x} < e$. This implies that the matrix A is of size $m \times n$ with $m \geqslant n$, and that A is of full rank.

Furthermore, $A\,\bar{x} \leqslant \tilde{e}$ implies that $\bar{x}^{\mathrm{T}} x \leqslant 1$ is a support for Q_I which is satisfied by n linearly independent vertices of Q_I. (Take a $n \times n$ submatrix of A that defines $\bar{x}$.) Hence every vertex of P that either furnishes a maximal independent node set in G or that is fractional furnishes a facet of Q_I. On the other hand, let $\pi^{\mathrm{T}} x \leqslant \pi_0$ be any non-trivial facet of Q_I, i.e., $\pi^{\mathrm{T}} x \leqslant \pi_0$ is such that $\pi_0 > 0$. Then $\pi^{\mathrm{T}} x = \pi_0$ contains n linearly independent vertices of Q_I and $\pi^{\mathrm{T}} x \leqslant \pi_0$ for all $x \in Q_I$. Consequently, $(1/\pi_0)\,\pi$ is a vertex of P.

Lemma 3.5. *Let G be p-critical and $\bar{\alpha} = \alpha(G)$. Then $\Sigma_{j=1}^{n} x_j \leqslant \bar{\alpha}$ is a facet of Q_I, i.e., $\bar{x} = (1/\bar{\alpha})\,e$ is a (fractional) vertex of P.*

Proof. Denote by e^j the row vector with $n-1$ components equal to $+1$ and having a zero in the j^{th} component. Then we have that $\max_{x \in P} e^j x = \max_{x \in P_j} e^j x = \alpha$ for $j = 1, ..., n$.

Define H_j to be the halfspace given by

$$H_j = \{x \in \mathbf{R}^n : e^j x \leqslant \alpha\} \quad \text{for } j = 1, ..., n.$$

Consequently, $P \subseteq H_j$ for $j = 1, ..., n$ or equivalently, $P \subseteq H = \bigcap_{j=1}^{n} H_j$. By definition of e^j, $j = 1, ..., n$, H is a *pointed* polyhedral convex cone with its apex at $x = (\alpha/(n-1))\,e$. By property (P4) of p-critical graphs we have that $A\,\bar{x} \leqslant (\bar{\alpha}\,\alpha/(n-1))\tilde{e} = \tilde{e}$. Consequently, since H is pointed, $\bar{x} = (\alpha/(n-1))\,e = (1/\bar{\alpha})\,e$ is a *vertex* of P satisfying $0 < \bar{x} < e$. By Remark 3.4 we have that $\Sigma_{j=1}^{n} x_j \leqslant \bar{\alpha}$ is a facet of Q_I.

Remark 3.6. If G is p-critical, we have by property (P3) that $\bar{x} = (1/\alpha)\,e$ is a (fractional) vertex of Q and by Remark 3.4 that $\Sigma_{j=1}^{n} x_j \leqslant \alpha$ provides a facet for P_I.

Remark 3.7. Since $\bar{x} = (1/\bar{\alpha})\,e$ is a vertex of P, we can reorder the rows of the matrix A as follows:

$$A = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix},$$

where the row sums of $\tilde{A}_1$ all equal $\bar{\alpha}$ and the row sums of $\tilde{A}_2$ are all (strictly) less than $\bar{\alpha}$. Furthermore, $\tilde{A}_1$ is of size $\hat{m} \times n$ with $m \geqslant \hat{m} \geqslant n$, and $\tilde{A}_1$ contains an (at least one) nonsingular submatrix of size $n \times n$. By Remark 3.6 we have a similar partitioning of B into $\tilde{B}_1$ and $\tilde{B}_2$, where $\tilde{B}_1$ has $\tilde{m}$ rows, $\tilde{m} \geqslant n$, having row sums equal to α, and $\tilde{B}_1$ contains a $n \times n$ nonsingular submatrix.

Remark 3.8. In graphical notation, Remark 3.7 implies that every p-critical graph G contains at least $|G|$ maximum cliques of the cardinality $\bar{\alpha} = \alpha(\bar{G})$. This has been observed earlier by H. Sachs and can be found, though without proof, in [16].

Remark 3.9. The point $\bar{x} = (\alpha/(n-1))\, e$ *uniquely* maximizes the linear from $e^{\mathrm{T}} x = \sum_{j=1}^{n} x_j$ over the convex polyhedral cone H defined in the proof of Lemma 3.5. Since $P \subseteq H$ and $\bar{x} \in P$, $\bar{x}$ *uniquely* maximizes the (same) linear form $e^{\mathrm{T}} x = \sum_{j=1}^{n} x_j$ over P. Consequently, there exists an *optimal* basis A_* (in the linear programming sense) associated with $\bar{x}$ of the following form:

$$A_* = \begin{bmatrix} A_1 & 0 \\ A_2 & \mathrm{I} \end{bmatrix},$$

where A_1 is a nonsingular submatrix of $\widetilde{A}_1$ (as defined in Remark 3.7) satisfying $A_1 \bar{x} = e$ *and* $e^{\mathrm{T}} A_1^{-1} \geqslant 0$. The latter inequality follows from the fact that $\bar{x}$ *uniquely* maximizes $e^{\mathrm{T}} x$ over P.

Proof. The convex polyhedral cone H is given by the system of inequalities $e^j x \leqslant \alpha$ for $j = 1, ..., n$. Call F the matrix made up of the n vectors e^j. Then F^{-1} exists and $F^{-1}(\alpha e) = (\alpha/(n-1))\, e = \bar{x}$. Furthermore, $e^{\mathrm{T}} F^{-1} = (1/(n-1))\, e^{\mathrm{T}} > 0$. Consequently, $\bar{x}$ *uniquely* maximizes the linear form $e^{\mathrm{T}} x$ over H. The rest of Remark 3.9. then follows by a simple contradiction.

To state the next lemma we recall that a $n \times n$ matrix R of zeros and ones is called a *permutation* matrix if R contains exactly one +1 entry in every row and column.

Lemma 3.10. *Let A be the $m \times n$ clique-matrix of a p-critical graph G and let B be the $m \times n$ clique-matrix of $\bar{G}$. Then A and B, respectively, contain $n \times n$ submatrices A_1 and B_1 satisfying the matrix equation*

$$A_1 B_1^{\mathrm{T}} = E - R, \tag{3.5}$$

where E is the $n \times n$ matrix consisting entirely of ones and R is a $n \times n$ permutation matrix.

Proof. From Remark 3.7 we have that A has $\hat{m} \geqslant n$ rows having row sums equal to $\bar{\alpha}$, i.e., the polytope Q has $\hat{m}$ integer vertices satisfying

$$\sum_{j=1}^{n} x_j \leqslant \bar{\alpha} \tag{3.6}$$

with equality. The vector $\tilde{x} = (\bar{\alpha}/n)\, e$ satisfies $B\tilde{x} \leqslant (\alpha\,\bar{\alpha}/n)\, e < e$ and $\tilde{x} > 0$. Consequently, $\tilde{x} \in \operatorname{int} Q$. Furthermore, $\tilde{x}$ satisfies (3.6) with equality. We show next that $\tilde{x} \in Q_I$, i.e., more precisely, that

$$\tilde{x}^{\mathrm{T}} = \gamma^{\mathrm{T}} A_1 , \tag{3.7}$$

with $\gamma_i \geqslant 0$ for $i = 1, ..., n$ and $\Sigma_{j=1}^{n} \gamma_i = 1$, where A_1 is the nonsingular submatrix of A defined in Remark 3.9. Since by definition, $\tilde{x}^{\mathrm{T}} = (\bar{\alpha}/n)\, e^{\mathrm{T}}$, we have from the nonsingularity of A_1 and from Remark 3.9 that $(\bar{\alpha}/n)\, e^{\mathrm{T}} A_1^{-1} = \gamma^{\mathrm{T}} \geqslant 0$. Furthermore,

$$\gamma^{\mathrm{T}} e = \frac{\bar{\alpha}}{n}\, e^{\mathrm{T}} A_1^{-1} e = \frac{\bar{\alpha}}{n}\, e^{\mathrm{T}} \left(\frac{\alpha}{n-1}\, e \right) = 1 .$$

Consequently, (3.7) is true. Now let B_1 be any $n \times n$ nonsingular submatrix of the submatrix $\tilde{B}_1$ defined in Remark 3.7. Then we obtain from (3.7) that

$$\gamma^{\mathrm{T}} A_1 B_1^{\mathrm{T}} = \tilde{x}^{\mathrm{T}} B_1^{\mathrm{T}} = \frac{\bar{\alpha}}{n}\, e^{\mathrm{T}} B_1^{\mathrm{T}} = \frac{n-1}{n}\, e^{\mathrm{T}} .$$

Since both A_1 and B_1 are nonsingular, it follows that $D = A_1 B_1^{\mathrm{T}}$ is nonsingular. Furthermore, since the columns of B_1^{T} are (a subset of) vertices of P_I (satisfying $e^{\mathrm{T}} x \leqslant \alpha$ with equality), we have that $A_1 B_1^{\mathrm{T}} \leqslant E$, i.e., that D consists of zeros and ones only. Suppose now that D contains a row of +1 entries. Then there exists a row of A, say a^{T}, such that $a^{\mathrm{T}} B_1^{\mathrm{T}} = e^{\mathrm{T}}$. Since $B_1 e = \alpha e$, it follows that $a^{\mathrm{T}} = (1/\alpha)\, e^{\mathrm{T}}$. Since by property (P5) $\alpha \geqslant 2$, this contradicts the integrality of a^{T}. Consequently, D cannot contain a row of +1 entries. Hence we have that

$$\gamma^{\mathrm{T}} D = \frac{n-1}{n}\, e^{\mathrm{T}} , \tag{3.8}$$

with $\gamma_i \geqslant 0$ for $i = 1, ..., n$ and $\Sigma_{j=1}^{n} \gamma_i = 1$, and hence that $\gamma^{\mathrm{T}} D e = n-1$. Consequently, $\gamma_i > 0$ implies that the row sum of row i of D equals $n-1$, since $\Sigma_{i=1}^{n} \gamma_i = 1$. Suppose now that $\gamma_1 \leqslant ... \leqslant \gamma_k$, with $k < n$ satisfies $\gamma_1 > 0$ and $\gamma_{k+1} = ... = \gamma_n = 0$. Since D is nonsingular we can rearrange the rows and columns of D such that D has the form

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

with

$$D_1 = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & \cdot & & \vdots \\ \vdots & & \cdot & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix},$$

where D_1 is of size $k \times k$ and has zeros only in the main diagonal, D_2 is of size $k \times (n-k)$ and consists entirely of ones. If $k < n$, obviously (3.8) cannot have a solution satisfying $\gamma_i \geqslant 0$ and $\Sigma_{i=1}^n \gamma_i = 1$. On the other hand, $\Sigma_{i=1}^n \gamma_i = 1$ implies $k \geqslant 1$. Consequently, $k = n$ and D has the general form $E-R$, where E is the $n \times n$ matrix consisting entirely of ones and R is a $n \times n$ permutation matrix.

Furthermore, we note that from (3.8) it follows that

$$\gamma^{\mathrm{T}} = \frac{n-1}{n} e^{\mathrm{T}} D^{-1} = \frac{1}{n} e^{\mathrm{T}}. \tag{3.9}$$

Theorem 3.11. *Let A be the $m \times n$ clique-matrix of a p-critical graph G. Then A contains a $n \times n$ nonsingular submatrix A_1 whose column and row sums are all equal to $\bar{\alpha} = \alpha(\bar{G})$. Furthermore, the row sums of the rows of A that are not contained in A_1 are all strictly less than $\bar{\alpha}$.*

Proof. Let A_1 be the $n \times n$ submatrix of A defined in Lemma 3.10. Then $A_1 e = \bar{\alpha} e$. Furthermore, from (3.9) we have that $\gamma^{\mathrm{T}} = (1/n) e^{\mathrm{T}}$; from (3.7) and the definition of $\bar{x}$ it follows that $e^{\mathrm{T}} A_1 = \bar{\alpha} e^{\mathrm{T}}$. Consequently, A contains a $n \times n$ nonsingular submatrix A_1 whose column and row sums are all equal to $\bar{\alpha}$. To complete the proof of Theorem 3.11 we note that the relation $A_1 B_1^{\mathrm{T}} = E-R$ of Lemma 3.10 implies that

$$A_1^{-1} = \frac{\alpha}{n-1} E - B_1^{\mathrm{T}} R^{\mathrm{T}}. \tag{3.10}$$

Suppose now that the matrix A_2 defined in Remark 3.9 contains a row a^{T} satisfying $a^{\mathrm{T}} e = \bar{\alpha}$. Then we have that $a^{\mathrm{T}} A_1^{-1} = e^{\mathrm{T}} - a^{\mathrm{T}} B_1^{\mathrm{T}} R^{\mathrm{T}}$. Since $0 \leqslant e^{\mathrm{T}} - a^{\mathrm{T}} B_1^{\mathrm{T}} R^{\mathrm{T}} \leqslant e^{\mathrm{T}}$, it follows that $0 \leqslant a^{\mathrm{T}} A_1^{-1} \leqslant e^{\mathrm{T}}$. Denoting by P^* the polytope obtained from P by dropping from the constraint set $A x \leqslant \tilde{e}$ the constraint $a^{\mathrm{T}} x \leqslant 1$, we conclude that

$$\max_{x \in P^*} a^{\mathrm{T}} x = \max_{x \in P} a^{\mathrm{T}} x = 1.$$

Consequently, we have that $P^* = P$, i.e., $a^T x \leqslant 1$ cannot be a facet of P and hence, in particular, it cannot be a facet of P_I. By assumption however, *all* rows of A define facets of P_I. Hence A_2 cannot contain such a row a^T. Consequently, the row sums of the rows of A that are not contained in A_1 are all strictly less than $\bar{\alpha}$.

Remark 3.12. An immediate consequence of Remark 3.4 and Theorem 3.11 is that the vertex $\bar{x} = (1/\bar{\alpha})e$ of P is *non-degenerate* and that P has *exactly n* (linearly independent) vertices x^i satisfying $\Sigma_{j=1}^n x_j \leqslant \alpha$ with equality, $i = 1, ..., n$. Furthermore, every vertex x^i is connected to $\bar{x}$ by an edge of P, i.e., x^i and $\bar{x}$ are *adjacent* vertices on P for $i = 1, ..., n$. An analogous statement holds for the polytope Q.

Remark 3.13. In graphical notation, Theorem 3.11 implies that every p-critical graph G has *exactly* $|G|$ maximum cliques of the cardinality $\bar{\alpha} = \alpha(\bar{G})$ and $|G|$ maximal independent node sets of the cardinality $\alpha = \alpha(G)$.

Theorem 3.11 suggests the following definition.

Definition 3.14. Let A be a zero–one matrix of size $m \times n$, $m \geqslant n$. A is said to have *property* $\pi_{\beta,n}$ if the following conditions are met:

(i) A contains a $n \times n$ *nonsingular* submatrix A_1 whose row and column sums are all equal to β.

(ii) If a^T is a row of A with row sum equal to β and a^T is not contained in the submatrix A_1 defined under (i), then there exists a row b^T of A_1 such that $a = b$ (equality is meant to hold componentwise).

(iii) All other rows of A have row sums strictly less than β.

Remark 3.15. Let A be a $m \times n$ matrix of zeros and ones and G its associated intersection graph. A is a clique-matrix, i.e., A contains as row vectors the incidence vectors of *all* cliques in G if and only if A does not contain any $m \times k$ submatrix A' having the property $\pi_{\beta,k}$ with $\beta = k-1$ and $\beta \geqslant 2$.

Proof (outline). The necessity of the condition is obvious. To prove sufficiency, let C be the node set of a clique in G such that the associated incidence vector a^T with $a_j = 1$ if $j \in C$, $a_j = 0$ otherwise, is *not* contained in A. Define

$$P' = P \cap \{x \in \mathbf{R}^n : x_j = 0 \text{ for all } j \in N \setminus C\},$$

$$P'_j = P' \cap \{x \in \mathbf{R}^n : x_j = 0\} \text{ for all } j \in C.$$

By assumption, $\max\{e^T x : x \in P', x_j = 0 \text{ or } 1, j = 1, ..., n\} = 1$, and furthermore, P' has at least one fractional vertex. We now distinguish two cases:

(i) P'_j has integral vertices only for all $j \in C$,

(ii) there exists a $j \in C$ such that P'_j has a fractional vertex.

In case (i) we can show by an argument completely analogous to the one used in the proof of Lemma 3.5, that A contains a $m \times k$ submatrix A' having property $\pi_{\beta,k}$ with $\beta = k-1$, $\beta \geq 2$ and $k = |C|$. In case (ii) we can restrict attention to any P'_j having a fractional vertex for $j \in C$. Abusing slightly the notation we redefine C to be $C \setminus \{j\}$, redefine the polytopes P', P'_j with respect to the new set C and find again the two cases mentioned above. (Note that the new set C defines a complete subgraph in G, which is no longer a clique in G. This, however, does not affect the argument.) Clearly, case (ii) can happen only finitely many times, and finally we obtain a set C having *at least three* elements and which is such that case (i) prevails. This completes the outline of the proof of Remark 3.15.

We note for completeness that Remark 3.15 is equivalent to a theorem due to Gilmore [3, Ch. 17, Théorème 2]. The following theorem states a necessary and sufficient condition for an arbitrary zero–one matrix to be perfect.

Theorem 3.16. *Let A be any zero–one matrix of size $m \times n$. The following two conditions are equivalent:*

(i) *A is perfect.*

(ii) *For $\beta \geq 2$ and $3 \leq k \leq n$, A does not contain any $m \times k$ submatrix A' having the property $\pi_{\beta,k}$.*

Proof. Suppose that A is perfect and that (ii) is violated. Then there exists a $m \times k$ submatrix A' of A having property $\pi_{\beta,k}$ for some $\beta \geq 2$ and $k \geq 3$. Suppose the columns of A have been ordered such that A' coincides with the k first columns. Then $\bar{x}$, defined by $\bar{x}_j = (1/\beta)$ for $j = 1, ..., k$, $\bar{x}_j = 0$ for $j = k+1, ..., n$, is a *fractional* vertex of the polytope P defined in (3.1). Hence by definition, A cannot be perfect. On the other hand, suppose that A is such that (ii) holds. Then A must be perfect. For if not, then by Remark 3.15 and Theorem 2.2 the intersection graph G associated with A must be imperfect. By (P2), G contains an induced subgraph G' that is p-critical. Again by Remark 3.15, the clique-matrix of G' is a $m \times k$ submatrix A' of A, where $k = |G'|$. Let A'' denote the $m' \times k$ submatrix of A' whose rows correspond to the cliques of G'. By Theorem 3.11, A'' has the property $\pi_{\beta,k}$ with $\beta = \alpha(\overline{G}') \geqslant 2$. Since the $m-m'$ truncated rows of A not contained in A'' are dominated by some row in A'', the $m \times k$ submatrix A' of A must also have property $\pi_{\beta,k}$ with $\beta = \alpha(\overline{G}')$. Thus (ii) cannot be satisfied by A.

Remark 3.17. Let A be any $m \times n$ matrix of zeros and ones containing a $m \times k$ submatrix A' having property $\pi_{\beta,k}$ with $\beta \geqslant [\frac{1}{2}(k-1)]+1$. Let N' be the index set of the columns of A'. Then obviously, $\Sigma_{j \in N'} x_j \leqslant 1$ is a valid inequality for problem (P), and the nodes of the associated graph G that are contained in N' form a complete subgraph (possibly a clique) in G. Consequently, in view of Remark 3.15, condition (ii) of Theorem 3.16 can be written equivalently as folows:

(ii′) *A is a clique-matrix and A does not contain any $m \times k$ sub-matrix having property $\pi_{\beta,k}$ for $5 \leqslant k \leqslant n$ and $2 \leqslant \beta \leqslant [\frac{1}{2}(k-1)]$.*

Corollary 3.18. *Let A be a zero–one matrix of size $m \times n$. A is critically imperfect, i.e., A is the clique-matrix of a p-critical graph G, if and only if the following two conditions are met:*

(i) *A has the property $\pi_{\beta,k}$ for some β satisfying $2 \leqslant \beta \leqslant [\frac{1}{2}(n-1)]$ and $k = n$.*

(ii) *A does not contain any $m \times k$ submatrix having property $\pi_{\beta,k}$ for $\beta \geqslant 2$ and $3 \leqslant k \leqslant n-1$.*

Remark 3.19. The Strong Perfect Graph Conjecture [3, 4], if true, now is reduced to proving that the only zero–one matrix A of size $m \times n$ satisfying the conditions (i) and (ii) of Corollary 3.18 *and* $2 \leqslant \beta < [\frac{1}{2}(n-1)]$ is the circulant of odd size having exactly two positive entries in every row and column, i.e., A is the clique-matrix of an odd cycle without chords. A characterization of critically imperfect matrices in *graphical* terms appears advantageous if one wants to check the perfection of a zero–one matrix. For, similarly to the criterion for the total unimodularity of a matrix A, a direct check of the perfection of a zero–one matrix via the necessary and sufficient criterion of Theorem 3.16 is – computationally – an impossible task, whereas graphical criteria – at least in the context of total unimodularity – are relatively easily verified or known to be satisfied by the physical conditions of a problem under consideration.

Remark 3.20. By the definition of perfect zero–one matrices we have that every totally unimodular zero–one matrix is perfect. In a recent paper, Berge [5] has extended the notion of totally unimodular zero–one matrices by means of the theory of balanced hypergraphs [3]. A zero–one matrix A is called balanced if it is the incidence matrix of a balanced hypergraph. In order to show that every balanced zero–one matrix is perfect we use [5, Theorem 6], which states the following necessary and sufficient condition for a zero–one matrix to be balanced: A $m \times n$ zero–one matrix A is balanced if and only if for every zero–one vector $w \in \mathbf{R}^m$ and for every zero–one vector $b \in \mathbf{R}^n$ the linear programming problem

$$(\mathrm{LP}_w^b) \qquad \min \left\{ \sum_{j=1}^{m} y_j : y A \geqslant b,\ 0 \leqslant y \leqslant w \right\}$$

provides an integral solution. Consequently, if A is balanced, then the linear program

$$(\mathrm{LP}^b) \qquad \min \left\{ \sum_{j=1}^{m} y_j : y A \geqslant b,\ y \geqslant 0 \right\}$$

provides an integral solution for every zero–one vector $b \in \mathbf{R}^n$. Consequently, by [12, Theorem 5], the linear programs

$$(\mathrm{D}^b) \qquad \max \left\{ \sum_{i=1}^{n} b_i\, x_i : A\, x \leqslant \widetilde{e},\ x \geqslant 0 \right\}$$

provide integral solutions for *all* vectors b with integral components, i.e., A is perfect. Consequently, every balanced matrix is perfect. To prove that the reverse statement does not hold we use a criterion due to Hoffman and Oppenheim [12]: A is balanced if and only if A does not contain any submatrix of odd size having row and column sums equal to two. Consequently, Example 3.22 is an instance of a perfect zero–one matrix that is not balanced. In general, we have the statements: Every totally unimodular zero–one matrix is balanced; every balanced zero–one matrix is perfect; but none of the two preceding statements holds in the reverse direction.

Example 3.21. Since a zero–one matrix A is perfect if it is the clique-matrix of a perfect graph (and vice versa), the clique-matrices of perfect graphs furnish examples of zero–one matrices satisfying the condition (ii) of Theorem 3.11. Among the graphs known to be perfect are the rigid circuit graphs [7] (or "triangulated" graphs [2]), the comparability graphs, and the "i-triangulated" graphs, see e.g. [2]. The example of rigid circuit graphs provide examples of zero–one matrices which are perfect, but *not* totally unimodular. (I am indebted to D.R. Fulkerson for this example).

Example 3.22. Consider the graph G in Fig. 1 and its associated clique-matrix A in Fig. 2. The submatrix A' made up of the columns 1, 2, 3 and 4 and the rows 1, 6, 11 and 16 has a determinant of 3. Consequently, A is not totally unimodular. Furthermore, the submatrix A'' made up of the columns 1, 2 and 4 and the rows 1, 6 and 11 has row and column sums equal to two. Consequently, A is not balanced. Due to the simple structure of A we find by inspection that A is perfect. (Checking condition (ii) of Theorem 3.11 amounts to proving that G does not contain an odd cycle without chords.) Furthermore, the matrix A provides us with an example where A^T, the transpose of a perfect matrix, is *not* perfect. This is interesting since it is different from a property that both totally unimodular and balanced matrices have. The above example can be generalized to prove that given any natural number k, there exists a *perfect* matrix A having a minor whose determinant in absolute value equals k. (Replace K_4 by K_{k+1} and the G_i, $i = 1, ..., 4$, by $k + 1$ copies of G_1, say, each of which is connected to K_{k+1} in a similar fashion as done above.) This indicates why a characterization of perfect matrices in terms of *forbidden matrices* is appropriate rather than a characterization in terms of forbidden subdeterminants (which is not possible here, but possible for totally unimodular matrices).

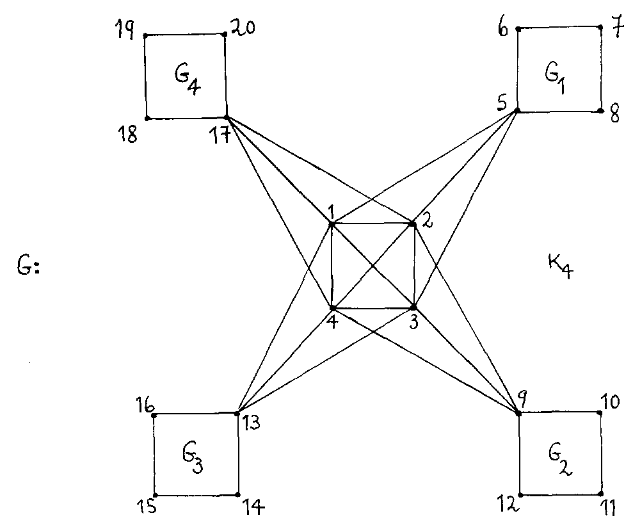

Fig. 1.

$$
A = \begin{vmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix}
$$

Fig. 2.

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