Mathematical Programming 6 (1974) 48–61. North-Holland Publishing Company

# PROPERTIES OF VERTEX PACKING AND INDEPENDENCE SYSTEM POLYHEDRA*

G.L. NEMHAUSER and L.E. TROTTER, Jr.**
Cornell University, Ithaca, New York, U.S.A.

Received 8 May 1973
Revised manuscript received 10 December 1973

We consider two convex polyhedra related to the vertex packing problem for a finite, undirected, loopless graph $G$ with no multiple edges. A characterization is given for the extreme points of the polyhedron $\mathcal{L}_G = \{x \in \mathbf{R}^n\colon Ax \leq 1_m, x \geq 0\}$, where $A$ is the $m \times n$ edge-vertex incidence matrix of $G$ and $1_m$ is an $m$-vector of ones. A general class of facets of $\mathcal{B}_G$ = convex hull$\{x \in \mathbf{R}^n\colon Ax \leq 1_m, x \text{ binary}\}$ is described which subsumes a class examined by Padberg [13]. Some of the results for $\mathcal{B}_G$ are extended to a more general class of integer polyhedra obtained from independence systems.

## 1. Introduction

Consider a finite, undirected, loopless graph $G = (V, E)$ with no multiple edges; $V$ and $E$ are the vertex and edge sets of $G$, respectively. A *vertex packing* (*v-packing, anticlique, stable set, independent set*) in $G$ is a subset $P \subseteq V$ for which all $v_i, v_j \in P$ satisfy $(v_i, v_j) \notin E$. The family of all such v-packings in $G$ is denoted $\mathcal{P}_G = \{P \subseteq V\colon P \text{ a v-packing}\}$. A practical reason for studying v-packings is that packing and partitioning problems in a family of sets can be transformed into v-packing problems on the intersection graph defined by the family [5, 10, 13].

We will investigate two classes of convex polyhedra that arise naturally from the problem of determining a maximum weighted member of $\mathcal{P}_G$ (the *weighted vertex packing problem*). This problem can be formulated as an integer program by representing a packing $P$ by a binary vector $x$ such that $x_j = 1$ if and only if $v_j \in P$. It then can be formulated as a linear

* This research was supported by the National Science Foundation under Grant GK-32282X to Cornell University and by the United States Army under Contract No. DA-31-124-ARO-D-462 to the Mathematics Research Center, University of Wisconsin, Madison, Wisconsin, U.S.A.

** Presently at the Department of Administrative Sciences, Yale University, New Haven, Connecticut, U.S.A.

program by considering the convex hull of the binary vectors that correspond to packings. Thus the weighted vertex packing problem can be written as the linear program

$$\text{(VP)}\qquad \begin{aligned} &\max cx, \quad x \in \mathcal{B}_G \\ &\mathcal{B}_G = \text{convex hull}\,\{x \in \mathbf{R}^n \colon Ax \le 1_m,\ x \text{ binary}\} \end{aligned}$$

in which $n = |V|$,[1] $m = |E|$, $c$ is an arbitrary $n$-vector, $1_m = (1, \dots, 1)$ is an $m$-vector of 1's, and $A$ is the edge-vertex incidence matrix of $G$ ($a_{ij} = 1$ if and only if $v_j$ is an endpoint of $e_i$).

The related *fractional v-packing problem* is obtained from (VP) by deleting integrality restrictions. This yields

$$\text{(VLP)}\qquad \begin{aligned} &\max cx, \quad x \in \mathcal{L}_G \\ &\mathcal{L}_G = \{x \in \mathbf{R}^n \colon Ax \le 1_m,\ x \ge 0\}. \end{aligned}$$

In the following section we give a decomposition theorem which characterizes all extreme points of $\mathcal{L}_G$ in terms of certain elementary extreme points. In Section 3, a class of facets ($(n-1)$-dimensional faces) of $\mathcal{B}_G$ is characterized which subsumes the class investigated by Padberg [13]. Furthermore, our characterization of the extreme points of $\mathcal{L}_G$ leads to a simple proof that the subclass of facets of $\mathcal{B}_G$ developed in [13] removes all original non-integer extreme points of $\mathcal{L}_G$.

In Section 4 we give a natural generalization of the technique for obtaining facets of $\mathcal{B}_G$ that characterizes certain facets for independence system polyhedra.

## 2. Extreme points of $\mathcal{L}_G$

**Proposition 2.1.**[2] *Let $x$ be an extreme point of $\mathcal{L}_G$. Then $x_j = 0, \frac{1}{2}$ or 1 for $1 \le j \le n$.*

[1] $|S|$ denotes the cardinality of the set $S$.

[2] This simple and direct proof of Proposition 2.1 was kindly provided by a referee. Alternatively, Proposition 2.1 could be proved, as indicated by Balinski and Spielberg [3] and spelled out by Trotter [14], using the facts that the edge matching problem in binary variables [6, 2] is asymptotically equivalent to (VP) in the sense of Gomory [11] and that the edge matching problem in real variables has extreme points all of whose components are (0, $\frac{1}{2}$, 1)-valued [1, p. 280].

**Proof.** Define

$$U_{-1} = \{j\colon 0 < x_j < \tfrac{1}{2},\ 1 \le j \le n\},$$

$$U_1 = \{j\colon \tfrac{1}{2} < x_j < 1,\ 1 \le j \le n\},$$

$$y_j = \begin{cases} x_j + k\epsilon & \text{if } j \in U_k\,, \\ x_j & \text{otherwise,} \end{cases} \qquad z_j = \begin{cases} x_j - k\epsilon & \text{if } j \in U_k\,, \\ x_j & \text{otherwise} \end{cases}$$

for $k = \pm 1$ and $1 \le j \le n$. Then $x = (y+z)/2$. If either $U_{-1}$ or $U_1$ is non-empty, $\epsilon > 0$ may be chosen small enough so that $y, z \in \mathcal{L}_G$ and $x$, $y$ and $z$ are distinct. Thus $x$ extreme implies $U_k = \emptyset$ for $k = \pm 1$.

In order to describe the fractional extreme points of $\mathcal{L}_G$ we need the following characterization of connected bipartite graphs.

**Proposition 2.2.** *Let $A$ denote the edge-vertex incidence matrix of a connected graph $G$. Then $A$ has full column rank if and only if $G$ is not bipartite.*

**Proof.** Only if: $G$ bipartite implies existence of a vertex partition into $V_1 \cup V_2 = V$ for which $\Sigma_{v_j \in V_1} a^j = \Sigma_{v_j \in V_2} a^j$, where $a^j$ denotes the $j^{\text{th}}$ column of $A$. Hence $A$ has linearly dependent columns.

If: Suppose $G$ is not bipartite and consider multipliers $\lambda_j$ for which $\Sigma_{v_j \in V} \lambda_j a^j = 0$. Let $C$ denote an odd cycle of $G$ and, without loss of generality, assume the vertices of $G$ are indexed so that $C = \{v_1, \ldots, v_{2k+1}\}$, $k \ge 1$. Now $\Sigma_{v_j \in V} \lambda_j a^j = 0$ implies $\lambda_1 = -\lambda_2 = \lambda_3 = \ldots = \lambda_{2k+1} = -\lambda_1$ so that $\lambda_j = 0$ for all $v_j \in C$. By the same reasoning we conclude that $\lambda_j = 0$ along any path from $v_j \in C$ to $v_k \in V \setminus C$. Since $G$ is connected, this implies $\lambda_j = 0$ for all $v_j \in V$. Hence the columns of $A$ are linearly independent.

For $S \subseteq V$ let $x^S \in \mathcal{L}_G$ be defined by

$$x_j^S = \begin{cases} \tfrac{1}{2} & \text{if } v_j \in S\,, \\ 0 & \text{else}\,, \end{cases} \tag{1}$$

and let $G_S$ denote the vertex generated subgraph of $G$ on $S$. If $x^S$ is extreme in $\mathcal{L}_G$ and $G_S$ is connected, then $x^S$ is called an *elementary fractional extreme point.*

**Proposition 2.3.** *Let $G_S$ denote the connected vertex generated subgraph of $G$ on $S \subseteq V$. Then $x^S$ defined by* (1) *is an elementary fractional extreme point of $\mathcal{L}_G$ if and only if $G_S$ contains an odd cycle.*

**Proof.** Since $x^S$ is extreme in $\mathcal{L}_G$ if and only if $\hat{x}^S = \frac{1}{2} \cdot 1_{|S|}$ is extreme in $\mathcal{L}_{G_S}$, we need only consider the case $S = V$ with $G$ a connected graph. Now, using the well-known correspondence between extreme points of $\mathcal{L}_G$ and basic feasible solutions to the system of inequalities which define this polyhedron, the desired result follows directly from Proposition 2.2.

Feasible integer solutions to (VLP) correspond to integer extreme points of $\mathcal{L}_G$. Two extreme points of $\mathcal{L}_G$ will be called *disjoint* if their sum is also in $\mathcal{L}_G$. All extreme points of $\mathcal{L}_G$ are described in terms of disjoint integer and elementary fractional extreme points by

**Theorem 2.4.** *A vector $x \in \mathbf{R}^n$ is an extreme point of $\mathcal{L}_G$ if and only if $x = x^0 + x^1 + \ldots + x^k$, where*
(i) *$x^0$ is an integer extreme point of $\mathcal{L}_G$,*
(ii) *$x^1, \ldots, x^k$ are elementary fractional extreme points of $\mathcal{L}_G$,*
(iii) *$x^0, x^1, \ldots, x^k$ are mutually disjoint.*

**Proof.** Only if: Let $S_1, \ldots, S_k$ denote the vertex sets of the components of the vertex generated subgraph of $G$ on $\{v_j \colon x_j = \frac{1}{2}\}$ and let $x^i = x^{S_i}$ be defined as in (1) for $1 \leq i \leq k$. Also let $S_0 = \{v_j \colon x_j = 1\}$ and define $x^0 = 2x^{S_0}$. From Proposition 2.1 we have that $x = x^0 + x^1 + \ldots + x^k$ and it is clear that (i) and (iii) hold. Furthermore, if any $x^i$ were not extreme, $1 \leq i \leq k$, then obviously $x$ could not be extreme, so (ii) holds also.

If: From $x^0, x^1, \ldots, x^k$ we define the vertex subsets $S_0, S_1, \ldots, S_k$ as above. Since $x^0, x^1, \ldots, x^k$ are mutually disjoint, $x \in \mathcal{L}_G$. If $x$ is not extreme, then a simple argument shows that the same is true for at least one of the vectors $x^1, \ldots, x^k$. Hence $x$ is extreme.

We call $x^{S_1}, \ldots, x^{S_k}$ the *elementary components* of $x$. Theorem 2.4 shows that an arbitrary extreme point of $\mathcal{L}_G$ can be represented *uniquely* as the sum of an integer extreme point and elementary fractional extreme points and, conversely, that any such sum of extreme points which yields a feasible solution to (VLP) produces an extreme point of $\mathcal{L}_G$. In contrast to the analogous situation in edge matching, where elementary fractional extreme points are in one-to-one correspondence with odd cycles in $G$, Theorem 2.4 shows that (VLP) generally produces many more elementary fractional extreme points – one, in fact, for each connected vertex generated subgraph of $G$ that contains an odd cycle. This characterization is used in Section 3 to show that certain simple facets of $\mathcal{B}_G$ remove all original fractional extreme points from $\mathcal{L}_G$.

## 3. Facets of $\mathcal{B}_G$

In [13], Padberg has shown that certain facets of $\mathcal{B}_G$ may be derived from maximal cliques (complete subgraphs) and odd holes (chordless cycles of odd length greater than 3) in $G$. The procedure used to obtain these facets applies much more generally and may be used to derive facets of $\mathcal{B}_G$ from arbitrary facets due to vertex generated subgraphs of $G$.

**Theorem 3.1.** *Suppose*

$$\sum_{v_j \in S} a_j x_j \leq \alpha_0 \tag{2}$$

*is a facet of $\mathcal{B}_{G_S}$, where $G_S$ is the subgraph of $G$ generated by $S \subseteq V$. Then there exists $\alpha_j$ for $v_j \in V \setminus S$ so that*

$$\sum_{v_j \in V} \alpha_j x_j \leq \alpha_0 \tag{3}$$

*is a facet of $\mathcal{B}_G$.*

**Proof.** It suffices to prove the theorem for the case $S = \{v_1, \ldots, v_{n-1}\}$. We define $\alpha_n = \max\{0, \alpha_0 - z^*\}$, where

$$\begin{aligned} z^* &= \max \sum_{v_j \in S} \alpha_j x_j, \\ &\sum_{v_j \in S} a^j x_j \leq 1_m - a^n, \qquad x_j = 0, 1, \qquad v_j \in S, \end{aligned} \tag{4}$$

and $a^j$ denotes the column of $A$ corresponding to $v_j$. We need only show that (3) is a valid inequality for (VP) that is satisfied at equality by $n = |V|$ affinely independent[3] feasible solutions to (VP).

For $P \in \mathcal{P}_G$, let $Q = P \cap S$ and let $x^P$ and $x^Q$ be the extreme points of $\mathcal{B}_G$ which correspond to $P$ and $Q$, respectively. Since $Q \subseteq S$, it is

[3] Here we are considering non-trivial facets (facets other than $x_j \geq 0$, $v_j \in V$) of $\mathcal{B}_G$, so $\alpha_0 > 0$ and consequently $(0, \ldots, 0)$ cannot satisfy (3) at equality. Hence, in this case, affine independence is equivalent to the more familiar concept of linear independence.

clear that $x^Q$ satisfies (3). If $P = Q$ or $\alpha_n = 0$, then $x^P$ also satisfies (3). If $\alpha_n > 0$, then $z^* < \alpha_0$ and since $x^Q$ is a feasible solution to (4),

$$\sum_{v_j \in S} \alpha_j \, x_j^Q \leq z^* .$$

Adding $\alpha_n = \alpha_0 - z^*$ to this inequality shows that $x^P$ satisfies (3). Thus (3) is a valid inequality for (VP).

Since (2) is a facet of $\mathcal{B}_{G_S}$, there are v-packings $P_1, \ldots, P_{n-1}$ in $S$ and corresponding affinely independent vectors $x^1, \ldots, x^{n-1}$ which satisfy (3) at equality. Now let $\hat{x}^n$ be an optimal solution to (4) with corresponding v-packing $\hat{P}_n$. Since $\hat{P}_n \in \mathcal{P}_G$ and the right-hand side of (4) is $1_m - a^n$, no edge joins $v_n$ to a vertex in $\hat{P}_n$, so that $P_n = \hat{P}_n \cup \{v_n\} \in \mathcal{P}_G$. The definition of $\alpha_n$ then forces the corresponding vector $x^n$ to satisfy (3) at equality. Furthermore, since $v_n \in P_n \setminus P_j$ for $1 \leq j \leq n-1$, we have that $x^n$ is affinely independent of $x^1, \ldots, x^{n-1}$, which completes the proof.

It is clear from the proof that the constraint (2), assumed to be a facet of $\mathcal{B}_{G_S}$, is both a support of $\mathcal{B}_G$ and the projection of the facet (3) of $\mathcal{B}_G$. The crux of the procedure described for "lifting" this projection up to a facet of $\mathcal{B}_G$ lies in the recursive definition of the $\alpha_j$'s for $v_j \in V \setminus S$ via (4). This procedure is simplified considerably by noticing that $\alpha_j = 0$ for $v_j \in V \setminus (S \cup N(S))$, where

$$N(S) = \{v_j \in V \setminus S\colon (v_i, v_j) \in E \text{ for some } v_i \in S\}.$$

Unfortunately, it may still be of little practical value, since (4) is itself a weighted v-packing problem which must be solved for each $v_j \in N(S)$. Furthermore, for a particular $S$, different orderings of the vertices of

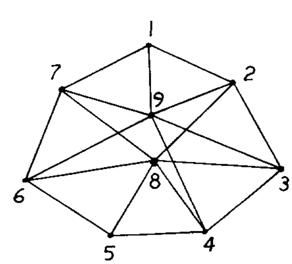


Fig. 1.

$N(S)$ may yield several different facets[4] of $\mathcal{P}_G$ from (3) as shown in

**Example** 3.2. In the graph $G$ of Fig. 1, $S = \{1, 2, 3, 4, 5, 6, 7\}$ is an odd hole and $\Sigma_{j=1}^{7} x_j \leq 3$ is a facet for $\mathcal{P}_{G_S}$ (see Corollary 3.6). We have $N(S) = \{8, 9\}$ and by first considering vertex 8 then vertex 9, we obtain the facet of $\mathcal{P}_G$,

$$x_1 + \ldots + x_7 + 2x_8 + x_9 \leq 3 .$$

If we consider first vertex 9 then vertex 8, we obtain a different facet,

$$x_1 + \ldots + x_7 + x_8 + 2x_9 \leq 3 .$$

Theorem 3.1 partitions the facets of $\mathcal{P}_G$ into two distinct categories – namely, those due to subgraphs of $G$ which can be lifted to $\mathcal{P}_G$, and those which are uniquely due to $G$. Thus the task of characterizing all the facets of $\mathcal{P}_G$ for an arbitrary graph may be viewed as a problem of determining those graphs which produce facets. We now outline several such classes of graphs. Obvious proofs are omitted.

**Proposition 3.3.** *Suppose $\alpha_0$ is the cardinality of a maximum v-packing in $G$ and suppose $G$ contains $n$ maximum v-packings $P_1, \ldots, P_n$ with corresponding affinely independent vectors $x^1, \ldots, x^n$. Then* (3) *is a facet of $\mathcal{P}_G$ with $\alpha_j = 1$ for all $v_j \in V$.*

**Proof.** Since $P_1, \ldots, P_n$ are maximum, each of the vectors $x^1, \ldots, x^n$ satisfies (3), with $\alpha_j = 1$ for all $v_j \in V$, at equality. This fact and the hypothesis that $x^1, \ldots, x^n$ are affinely independent imply that (3) is a facet of $\mathcal{P}_G$.

By considering certain maximal packings in $G$, it is possible to generate facets that cannot be obtained from Proposition 3.3. In particular,

**Proposition 3.4.** *Suppose $\alpha_0$ is the cardinality of a maximum v-packing in $G$ and suppose $G$ contains exactly $n$ maximal v-packings $P_1, \ldots, P_n$ with corresponding affinely independent vectors $x^1, \ldots, x^n$. Define $\alpha_j$ for $v_j \in V$ to be the unique solution to the system of equations*

$$\sum_{v_j \in V} \alpha_j x_j^i = \alpha_0, \quad i = 1, \ldots, n . \tag{5}$$

[4] This fact has been observed in [13] for the class of facets to be given in Corollaries 3.5 and 3.6.

*If* $\alpha_j \geq 0$ *for all* $v_j \in V$, *then* (3) *is a facet of* $\mathcal{B}_G$.

The following results can be obtained easily from either Proposition 3.3 or Proposition 3.4. If $S$ is a *maximal* clique in $G$, the procedure described in Theorem 3.1 forces $\alpha_j = 0$ for $v_j \in V \setminus S$. Consequently we have

**Corollary 3.5** ([8, 9, 13, 4]). *Let* $S \subseteq V$ *be a maximal clique in* $G$. *Then* $\Sigma_{v_j \in S}\, x_j \leq 1$ *is a facet of* $\mathcal{B}_G$.

As a consequence of Fulkerson's theory of anti-blocking polyhedra [8, 9] and the perfect graph theorem [12], it can be shown that the clique constraints of Corollary 3.5 are *all* the non-trivial facets of $\mathcal{B}_G$ if and only if $G$ is perfect.[5] Notice that the clique facet of Corollary 3.5 can be lifted from the facet $x_j \leq 1$ for the subgraph $(\{v_j\}, \phi)$, for any $v_j \in S$. Thus we can characterize perfect graphs as those for which the only (non-trivial) facet producing subgraphs are singleton vertices.

**Corollary 3.6** ([13]). *Let* $G$ *be an odd hole. Then* $\Sigma_{v_j \in V}\, x_j \leq \frac{1}{2}(|V|-1)$ *is a facet of* $\mathcal{B}_G$.

**Corollary 3.7**. *Let* $G$ *be an odd anti-hole* (*edge complement of an odd hole*). *Then* $\Sigma_{v_j \in V}\, x_j \leq 2$ *is a facet of* $\mathcal{B}_G$.

The facets of Corollaries 3.5, 3.6 and 3.7 all have the property that $\alpha_j = 1$ for all $v_j \in S$ in (2). However, it is not the case that all facets of this type can be derived from these corollaries, as shown in

**Example 3.8.** In Fig. 2, $G$ contains no 3-cliques, no 7-holes and no 7-anti-holes. There are, however, 8 5-holes of the form

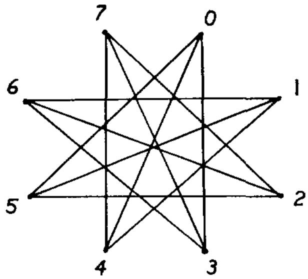


Fig. 2.

[5] Private communication, Fulkerson (1972). See also [4, 14].

$$\{i, i+3, i+6, i+1, i+4, i\}, \quad i = 0, 1, \ldots, 7,$$

where the indices are taken modulo 8. The constraint of the form (3) due to any one of the 8 5-holes, $S_k$, is $\Sigma_{j \in S_k} x_j \leq 2$. However, for $S = V$, we obtain 8 independent solutions of the form $\{i, i+1, i+2\}$ for $i = 0, 1, \ldots, 7$. These 8 solutions determine by Proposition 3.3 the facet $\Sigma_{j \in V} x_j \leq 3$ which is not implied by the 5-hole constraints, since the solution $x_0 = \frac{1}{2}, x_1 = \ldots = x_7 = \frac{3}{8}$ satisfies each 5-hole constraint but violates $\Sigma_{j \in V} x_j \leq 3$.

The graph of Fig. 2 and the cliques, odd holes and odd anti-holes of Corollaries 3.5–3.7 are all special instances of a class of facet producing graphs investigated in [15].

A facet that cannot be derived by lifting facets obtained from Corollary 3.6 has been provided by Chvátal [4]. More generally, Chvátal's example provides a facet that cannot be obtained from Proposition 3.3.

**Example 3.9.** In the graph of Fig. 3, the facet

$$2x_1 + \sum_{j=2}^{7} x_j \leq 3 \tag{6}$$

can be derived from Proposition 3.4 since $G$ has exactly 7 maximal v-packings and corresponding affinely independent vectors. Although we are not aware of any general results on lifting supports to facets, it happens that (6) can also be derived by lifting the support

$$x_2 + x_3 + x_4 \leq 3$$

of $\mathcal{B}_{G_S}$, where $S = \{2, 3, 4\}$, in the manner described in the proof of Theorem 3.1. However, no $S \subseteq V$ satisfying the hypotheses of Proposition 3.3 yields a facet that can be lifted to (6).

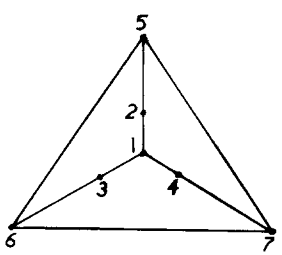


Fig. 3.

We do not know whether all facets of $\mathcal{B}_G$ can be obtained by using Propositions 3.3 and 3.4 in conjunction with Theorem 3.1. If such were the case, however, and it could be shown that every set $S$ that is not a clique, but is facet producing, contains an odd hole or odd anti-hole, this would prove the strong perfect graph conjecture which states that $G$ is perfect if and only if $G$ contains no odd holes and no odd anti-holes.

The following corollary shows that facets obtained from odd holes and cliques using Theorem 3.1 remove all *original* fractional vertices from $\mathcal{L}_G$. Although this result is interesting, it may be of little consequence, since the introduction of constraints of the form (3) generally will produce new fractional vertices as in Example 3.8.

**Corollary 3.10.** *Let $x$ be a fractional extreme point of $\mathcal{L}_G$. Then $x$ violates a facet of $\mathcal{B}_G$ of the form* (3) *due to an odd hole or a clique in $G$.*[6]

**Proof.** Let $\hat{x}$ be any elementary component of $x$. Then there exists $\hat{S} \subseteq V$ such that $\hat{x} = x^{\hat{S}}$ as defined in (1); furthermore, by Proposition 2.3, $\hat{S} \supseteq S$, an odd cycle in $G$. Without loss of generality, assume $S$ contains no chord. Hence $S$ is a 3-clique or an odd hole. If $S$ is an odd hole, then $\Sigma_{v_j \in S}\, \hat{x}_j = \frac{1}{2}|S| > \frac{1}{2}(|S| - 1)$, so $x$ violates any facet of $\mathcal{B}_G$ due to $S$ (see Corollary 3.6). If $S$ is a 3-clique, then $S$ is contained in a maximal clique $\overline{S}$ and $\Sigma_{v_j \in \overline{S}}\, \hat{x}_j \geq \Sigma_{v_j \in S}\, \hat{x}_j = \frac{3}{2}$ which shows (Corollary 3.5) that $x$ violates the facet of $\mathcal{B}_G$ due to $\overline{S}$.

## 4. Independence systems

Let $\mathcal{I}$ be a family of subsets of the set $I = \{1, \ldots, n\}$ with the property that $I_1 \subset I_2 \in \mathcal{I}$ implies $I_1 \in \mathcal{I}$. The pair $S = (I, \mathcal{I})$ is called an *independence system* and the individual members of $\mathcal{I}$ are referred to as *independent sets.* Sets $J \subseteq I$ for which $J \notin \mathcal{I}$ are said to be *dependent* and minimal such sets are called *circuits.* For a graph $G = (V, E)$ the family $\mathcal{P}_G$ of all v-packings in $G$ naturally induces the independence system $(V, \mathcal{P}_G)$ whose family of circuits is given by $E$.

Define an independence system $(I, \mathcal{I})$ to be *graphical* if there exists a graph $G = (V, E)$ such that $(I, \mathcal{I}) = (V, \mathcal{P}_G)$. Not all independence systems are graphical since

[6] In [13], it is shown that these facets exclude those extreme points of $\mathcal{L}_G$ that are adjacent to integer extreme points.

**Theorem 4.1.** *Let $(I, \mathcal{I})$ be an independence system. Then $(I, \mathcal{I})$ is graphical if and only if*

(i) $\{i\} \in \mathcal{I}$, *for all* $i \in I$,

(ii) $I_0 \setminus \{i\} \in \mathcal{I}$, *for all* $i \in I_0 \Rightarrow I_0 \in \mathcal{I}$, *for all* $I_0 \subseteq I$, $|I_0| > 2$.

**Proof.** If $(I, \mathcal{I})$ is graphical, then $\mathcal{P}_G$ obviously satisfies the conditions given. On the other hand, suppose $\mathcal{I}$ satisfies (i) and (ii) and consider $G = (V, E)$, where $V = I$ and

$$E = \{(i, j) \colon i \in I,\ j \in I,\ i \neq j \text{ and } \{i, j\} \notin \mathcal{I}\}\ .$$

Clearly $J \in \mathcal{I}$, $|J| \geq 2$ implies $\{i, j\} \in \mathcal{I}$ for each 2-subset $\{i, j\} \subseteq J$, so that $J \in \mathcal{P}_G$. In addition, if $|J| = 1$, then $J \in \mathcal{P}_G$ since $V = I$. Hence $\mathcal{I} \subseteq \mathcal{P}_G$ and we need only show that $\mathcal{P}_G \subseteq \mathcal{I}$. Choose $P \in \mathcal{P}_G$. If $|P| \leq 2$, then $P \in \mathcal{I}$ by (i) and the construction of $E$. If $|P| \geq 3$, then every 2-subset of $P$ is in $\mathcal{I}$, since $P \supset \{i, j\} \notin \mathcal{I}$ implies $(i, j) \in E$ which contradicts $P \in \mathcal{P}_G$. Inductively applying condition (ii), we obtain first that every 3-subset of $P$ is in $\mathcal{I}$, then that every 4-subset of $P$ is in $\mathcal{I}, \ldots,$ and finally, that $P \in \mathcal{I}$.

Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be the circuits of an independence system $S = (I, \mathcal{I})$. The problem of determining a maximum weighted member of $\mathcal{I}$ is a proper generalization of vertex packing and can be formulated as the linear program

$$\max cx, \quad x \in \mathcal{B}_S\ ,$$

$$\mathcal{B}_S = \text{convex hull}\{x \in \mathbf{R}^n \colon Ax \leq b,\ x \text{ binary}\}$$

in which $n = |I|$, $m = |\mathcal{C}|$, $c$ is an arbitrary $n$-vector, $A$ is the incidence matrix of the members of $\mathcal{C}$ with $I$, and $b = (b_1, \ldots, b_m)$, where $b_i = |C_i| - 1$, $i = 1, \ldots, m$.

Theorem 3.1 and Propositions 3.3 and 3.4 are special cases of more general results about facets for $\mathcal{B}_S$ which we now present. This generality appears only in the statement of these results, not in the techniques required for their proof. Consequently, the process of defining facets was given in the more familiar context of Theorem 3.1; similar proofs are omitted here.

Suppose $I' \subseteq I$ and let $\mathcal{I}' = \{J \in \mathcal{I} : J \subseteq I'\}$. Then we say that the independence system $S' = (I', \mathcal{I}')$ is a subsystem of $S = (I, \mathcal{I})$ generated by $I'$. A generalization of Theorem 3.1 to independence systems is given by

**Theorem 4.2.** *Suppose* $S' = (I', \mathcal{I}') \subseteq S = (I, \mathcal{I})$ *and*

$$\sum_{j \in I'} \alpha_j x_j \leq \alpha_0 \tag{7}$$

*is a facet of* $\mathcal{B}_{S'}$. *Then there exist* $\alpha_j$ *for* $j \in N(I')$, *where*

$$N(I') = \{k \in I \setminus I' \colon \{k\} \cup J \notin \mathcal{I} \text{ for some } J \in \mathcal{I}'\},$$

*so that*

$$\sum_{j \in I'} \alpha_j x_j + \sum_{j \in N(I')} \alpha_j x_j \leq \alpha_0$$

*is a facet of* $\mathcal{B}_S$.

Thus for the independence system $S = (I, \mathcal{I})$, some of the facets of $\mathcal{B}_S$ may be lifted from facets due to subsystems of $S$. As is the case with the system $(V, \mathcal{P}_G)$, certain of these classes of facets can be characterized. Analogous to Proposition 3.3, we have

**Proposition 4.3.**[7] *Suppose* $S = (I, \mathcal{I})$, $\alpha_0$ *is the cardinality of a maximum independent set in* $S$ *and* $S$ *contains* $n$ *maximum independent sets* $I_1, \ldots, I_n$ *with corresponding affinely independent vectors* $x^1, \ldots, x^n$. *Then* (7) *is a facet of* $\mathcal{B}_S$ *with* $\alpha_j = 1$ *for all* $j \in I' = I$.

If $S = (I, \mathcal{I})$ is a matroid (i.e., in each subsystem all maximal independent sets are of the same cardinality), the work of Edmonds [7] shows that Proposition 4.3, when applied to all subsystems of $S$, characterizes all the non-trivial facets of $\mathcal{B}_S$.

An independence system will be called *k-regular* if each of its circuits is of size $k$ ($k \geq 2$). (Note that $(V, \mathcal{P}_G)$ is 2-regular.) Define $I' \subseteq I$ to be a *clique* if $|I'| \geq k$ and all $\binom{|I'|}{k}$ $k$-subsets of $I'$ are circuits of $S = (I, \mathcal{I})$. A generalization of Corollary 3.5 is

**Corollary 4.4.** *Suppose* $I' \subseteq I$ *is a maximal clique in the k-regular independence system* $S = (I, \mathcal{I})$. *Then*

$$\sum_{j \in I'} x_j \leq k - 1 \tag{8}$$

*is a facet of* $\mathcal{B}_S$.

[7] There is a similar generalization of Proposition 3.4.

**Proof.** Since any selection of more than $k - 1$ distinct elements of $I'$ contains a circuit, it is clear that (8) is a support of $\mathcal{B}_S$. Hence we need only display $n$ independent sets whose corresponding incidence vectors satisfy (8) at equality and are affinely independent. Without loss of generality assume that $I' = \{1, \ldots, t\}$ and consider the incidence matrix of independent sets with elements of $I'$, where $u$ is an identity matrix.

$$M = \begin{array}{c}
\begin{array}{cc} \overbrace{\qquad\qquad\qquad\qquad}^{k} & \overbrace{\qquad}^{t-k} \end{array} \\
\left[\begin{array}{ccccc|c}
0 & 1 & \cdots & \cdots & 1 & \\
1 & 0 & 1 & \cdots & 1 & \\
\vdots & & \ddots & \ddots & \vdots & 0 \\
1 & \cdots & 1 & 0 & 1 & \\
1 & \cdots & \cdots & 1 & 0 & \\
\hline
1 & \cdots & 1 & 0 & 0 & \\
\vdots & & \vdots & \vdots & \vdots & u \\
1 & \cdots & 1 & 0 & 0 &
\end{array}\right]
\end{array}
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} k \\
\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} t-k
\end{array}$$

Each row of $M$ represents a $(k - 1)$-subset of $I'$, which is independent because $S$ is $k$-regular. Furthermore, $M$ is easily shown to be nonsingular, so that $M$ represents $|I'| = t$ of the $n$ independent sets required. The remaining $n - t$ independent sets are chosen by combining an element of $I \setminus I'$ with a $(k - 1)$-subset of $I'$. Since $I'$ is a maximal clique, such a $(k - 1)$-subset of $I'$ exists for each element of $I \setminus I'$.

**Example 4.5.** Given the graph $G = (V, E)$, define the independence system $S = (I, \mathcal{I})$, where $I = V$ and $J \in \mathcal{I}$ if and only if $J$ generates a bipartite subgraph of $G$. From Theorem 2.4 we note that the non-integer extreme points of $\mathcal{L}_G$ correspond to dependent subsets of $S$ and the circuits of $S$ are the chordless odd cycles of $G$. If $G$ is perfect, then $G$ contains no odd holes (chordless odd cycles of length $\geq 5$). Thus, $G$ perfect implies that $S$ is 3-regular. Furthermore, the maximal cliques of $G$ of size greater than 2 correspond to the maximal cliques of $S$ and so, by Corollary 4.4, $\sum_{j \in J} x_j \leq 2$ is a facet of $\mathcal{B}_S$ when $|J| \geq 3$ and $J$ is a maximal clique in $G$.

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