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Algebraic Surfaces of General Type with Small c_1^2. III

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Introduction

In this Part III, we shall study minimal algebraic surfaces defined over $\mathbb{C}$ with $p_g = 4$, $q=0$, and $c_1^2=6$. Our purpose is to determine the structures and deformations of these surfaces. We shall use the terminology and notation of our preceding papers [10, 11, I and II], which are respectively referred to as [Q], Part I, and Part II.

Minimal surfaces with $p_g=4$, $q=0$, and $c_1^2=6$ are classified into five types: I to V. These are further divided into eleven types: Ia, Ib, II, IIIa, IIIb, IVa−1, IVa−2, IVb−2, V−1, and V−2. A surface of type Ia is birationally equivalent to a sextic surface in $\mathbb{P}^3$ which has a double curve along a plane cubic curve, while a surface of type Ib is birationally equivalent to a double covering of a cubic surface in $\mathbb{P}^3$. A surface of type II is birationally equivalent to a triple covering of a quadratic cone in $\mathbb{P}^3$. A surface of type IIIa or IIIb has a pencil of curves of genus 3 with a base point, and a surface of type IV or V has a pencil of curves of genus 2. We have the following hierarchy among them:

$$\begin{tikzcd}
 & & & & \mathrm{IIIa} \arrow[dl] \arrow[d] & \\
 & \mathrm{IVa}-1 \arrow[dl] \arrow[d] & \mathrm{Ia} \arrow[dl] \arrow[dd] & \mathrm{V}-1 \arrow[d] & \mathrm{IIIb} \arrow[dl] & \mathrm{II} \\
\mathrm{IVa}-2 \arrow[d] & \mathrm{IVb}-1 & & \mathrm{V}-2 & & \\
\mathrm{IVb}-2 & & \mathrm{Ib} & & &
\end{tikzcd}$$

Here $A \to B$ means that B is a specialization of A, i.e. there exists a 1-parameter family with a special fibre being of type B and a generic fibre being of type A. Several points are left unsettled. In particular, we can not prove nor disprove the following specializations: $\mathrm{II} \to \mathrm{IIIb}$ or V, $\mathrm{Ia} \to \mathrm{V}$, and $\mathrm{IV} \to \mathrm{V}-2$[1]. The number of moduli for the generic surfaces are 38 for Ia, II, IVa−1, and 39

[1] This is not the case as we shall prove in Part IV, §6

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for IIIa. These surfaces except for those of type IV and V were already known to Enriques (see [6], pp. 271–273).

Here we mention two points about deformations. First, the generic surfaces S of type II have non-reduced Kuranishi spaces of deformations. This phenomenon is connected with the existence of a non-singular rational curve G with $G^2 = -2$ on such S, which is stable under small deformations. In other words, for any sufficiently small deformation S_t of S, the canonical bundle of S_t is not ample. Secondly, the surfaces of type Ia and those of type IVa-1 both specialize to surfaces of type IVb-1. This fact as well as its proof is quite similar to that in the case of numerical quintic surfaces [Q].

In addition to the three papers cited above, we shall often use the result of our paper on pencils of curves of genus 2 [12], which will be referred to as [P]. Other surfaces with $c_1^2 = 2p_g - 2$ will be studied in a subsequent paper.

Among others we shall use the following notation. For any non-negative integer d, Σ_d denotes the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(d))$ over $\mathbb{P}^1$, and we call it the Hirzebruch surface of degree d. We let Γ denote a fibre of Σ_d and Δ_0 a section with $\Delta_0^2 = -d$. The latter will be called the 0-section of Σ_d. For any sheaf $\mathscr{F}$ on a compact complex manifold X, we set $h^i(\mathscr{F}) = \dim H^i(X, \mathscr{F})$, and $\chi(\mathscr{F}) = \sum(-1)^i h^i(\mathscr{F})$. If $\mathscr{F} = \mathcal{O}(D)$ with a divisor or a line bundle D, we set $h^i(D) = h^i(\mathcal{O}(D))$ and $\chi(D) = \chi(\mathcal{O}(D))$. Finally Θ_X denotes the sheaf of germs of holomorphic vector fields on X.

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Contents

§1. Canonical Maps of Algebraic Surfaces

Let S be a minimal algebraic surface of general type defined over $\mathbb{C}$ and let K be the canonical bundle of S. As usual, we denote by p_g the geometric genus of S. Then the complete linear system $|K|$ defines a rational map $\Phi_K: S \to \mathbb{P}^n$ with $n = p_g - 1$. We shall say that $|K|$ is composite with a linear pencil if the image of Φ_K is a rational curve.

Theorem 1.1. *Let c_1 be the first Chern class of S. If $c_1^2 \leqq 3p_g - 4$ with $p_g \geqq 3$, then $|K|$ is not composite with a linear pencil.*

Proof. Suppose that $|K|$ is composite with a linear pencil $|D|$. Then we have $|K| = |nD| + F$, where F is the fixed part of $|K|$. Since S is of general type, we have $KF \geqq 0$. Hence

$$nKD \leqq K^2 < 3n.$$

It follows that $KD=2$, because $KD=1$ would imply $K^2=1$ and $p_g \leqq 2$ (see [16] or [3]). Therefore, we have the equality

$$2 = nD^2 + DF.$$

Since D^2 is even and since $n \geqq 2$, we conclude $D^2=0$ and $DF=2$. Hence $|D|$ defines a holomorphic map $g: S \to \mathbb{P}^1$ whose general fibre is of genus 2.

We now recall the result of [P]. By the construction there, we obtain a "canonical" rational map f of degree 2 of S onto some Hirzebruch surface Σ_d. We let B denote the branch locus of f. Its singularities are classified in [P], Lemma 6 into six types: (0), (I_k), (II_k), (III_k), (IV_k), and (V). Let $\nu(T)$ denote the number of singular fibres of type (T). Then we have, by [P], Theorem 3,

$$\begin{aligned} &c_1^2 - (2p_a - 4) \\ &= \sum_k (2k-1)\{\nu(\mathrm{I}_k) + \nu(\mathrm{III}_k)\} + \sum_k 2k\{\nu(\mathrm{II}_k) + \nu(\mathrm{IV}_k)\} + \nu(\mathrm{V}), \end{aligned} \tag{1.1}$$

where $p_a = p_g - q$ is the arithmetic genus of S.

On the other hand, assume that B is linearly equivalent to $6\Delta_0 + 2(m+d+2)\Gamma$. Then we have

$$\begin{aligned} p_g &= \dim H^0(\Sigma_d, \mathcal{O}(\Delta_0 + (m - \deg \mathfrak{c})\Gamma)), \\ q &= \dim H^1(\Sigma_d, \mathcal{O}(\Delta_0 + (m - \deg \mathfrak{c})\Gamma)), \end{aligned}$$

where $\mathfrak{c}$ is a divisor on $\mathbb{P}^1$ determined in terms of singular fibres (see [P], p. 87). Since $|K|$ is composite with a pencil, we have the equalities $m - \deg \mathfrak{c} = p_g - 1$, and $d = p_g + q$.

Since B has no multiple component, we have

$$-d \leqq \Delta_0 B = 2\deg \mathfrak{c} - 2d + 2 - 2q.$$

From Theorem 3 in [P], we obtain

$$2\deg \mathfrak{c} - (c_1^2 - 2p_a + 4) = \sum_k (\nu(\mathrm{I}_k) + \nu(\mathrm{III}_k)) + \nu(\mathrm{V}).$$

We call the right hand side r. Then B contains r fibres $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$. So let us write $B = B_0 + \Gamma_1 + \cdots + \Gamma_r$. Then we have

$$\Delta_0 B_0 = c_1^2 - (4p_g - 6) - 2q,$$

and, by assumption, $\Delta_0 B_0$ is negative. Therefore B_0 contains Δ_0 as a component. If we set $C=B_0-\Delta_0$, then we have

$$\Delta_0 C=c_1^2-3p_g+6-q. \tag{1.2}$$

Next we prove that $\nu(\mathrm{III}_k)=\nu(\mathrm{IV}_k)=0$. In fact, suppose that Γ is a fibre of type (III_k) or (IV_k). Then the curve B_0 intersects Γ at a unique point s, and B_0 has infinitely near triple points at s. Since Δ_0 is a component of B_0, s is on Δ_0. Let $\pi: W_1 \to \Sigma_d$ be the quadratic transformation with center s, $E=\pi^{-1}(s)$, and let $\tilde{\Gamma}$ be the proper transform of Γ. Then, by definition, the proper transform $\tilde{B}_0$ of B_0 has a triple point at $E\cap\tilde{\Gamma}$. On the other hand, $\tilde{B}_0$ contains the proper transform of Δ_0. These facts imply that $\tilde{B}_0 E \geqq 4$, which contadicts that s is a triple point.

Suppose that Γ is a singular fibre of type (I_k). Then, by definition, $B_0=C+\Delta_0$ has at least $(2k-1)$-fold triple point at $s=\Gamma\cap\Delta_0$. This implies that C has a contact of order $\geqq 4k-2$ with Δ_0 at s, i.e. $(\Delta_0 C)_s \geqq 4k-2$. In a similar way, if Γ is of type (III_k), we have $(\Delta_0 C)_s \geqq 4k$, and, if Γ is of type (V), we have $(\Delta_0 C)_s \geqq 3$. From these facts we obtain

$$\Delta_0 C \geqq \sum_k (4k-2)\,\nu(\mathrm{I}_k)+\sum_k 4k\,\nu(\mathrm{II}_k)+3\nu(\mathrm{V}).$$

Combining with (1.1) and (1.2), we obtain $c_1^2 \leqq p_g-2-5q$. This contradicts the inequality $c_1^2 \geqq 2p_g-4$ (see [Q], Lemma 2). Q.E.D.

Theorem 1.2. *Assume that* $p_g \geqq 5$ *and* $c_1^2 \leqq 4p_g-7$. *Then* $|K|$ *is not composite with a linear pencil.*

Proof. We suppose that $|K|$ is composite with a linear pencil, and write $|K| = |nD|+F$ as before. Then we have

$$nKD \leqq K^2 \leqq 4n-3.$$

This implies $KD=2$ or 3. But, if $KD=3$, we have $3=nD^2+DF$, and D^2 is odd. It follows that $n \leqq 3$, which contradicts the assumption $p_g \geqq 5$. Thus we have $KD=2$ and we can repeat the above proof. Q.E.D.

Theorem 1.3. *Assume that* $c_1^2=2p_g-2$ *or* $2p_g-1$ *and* $p_g \geqq 3$. *Then* $|K|$ *is not composite with a pencil.*

Proof. If $|K|$ is composite with a pencil $\{D\}$, then K is algebraically equivalent to $\mu D+F$ for some integer $\mu \geqq n$, where F denotes the fixed part of $|K|$. From the inequalities $\mu KD \leqq K^2 \leqq 2n+1$, we infer that $\mu=n$. This implies that $\{D\}$ is a linear pencil. But this is impossible by Theorem 1.1. Q.E.D.

§2. Holomorphic Maps of Degree 2

The purpose of this section is to supplement the result of [Q], §2. Let $f: S \to W$ be a surjective holomorphic map of degree 2 of compact complex surfaces S and

W, both being non-singular. We assume that no exceptional curve (of the first kind) on S is mapped into a point by f. We define the ramification divisor R on S, and the branch locus B on W as in [Q], §2. The following lemma proves that the assumptions in [Q], Lemma 4 are always satisfied.

Lemma 2.1. *There exists a line bundle F on W which satisfies the following conditions:*

(i) $[B]=2F$;

(ii) *S is the minimal resolution of the singularities of the double covering S' of W in F with branch locus B* (*see* [P], §2);

(iii) *there exists an effective divisor Z* (*possibly* 0) *in $|f^*F-R|$ such that $f^*B-2R=2Z$.*

Proof. Let $S\to S'\to W$ be the Stein decomposition of f, and let $\pi: S\to S'$, $f': S'\to W$ be the induced maps. Then S' is normal and $\pi_*\mathcal{O}_S=\mathcal{O}_{S'}$. Hence $f_*\mathcal{O}_S=f'_*\mathcal{O}_{S'}$ is locally free of rank 2, and we have an exact sequence

$$0\to\mathcal{O}_W\to f_*\mathcal{O}_S\to\mathcal{O}(-F)\to 0$$

for some line bundle F on W (see [P], Lemma 4). This F satisfies the condition (i) and S' is nothing but the double covering of W in F with branch locus B.

Let w_i be a fibre coordinate on F over an open set $U_i\subset W$, and let $b_i=0$ be an equation of B on U_i. Then S' is defined by $w_i^2=b_i$ over U_i (see [Q], p. 48). Moreover, the equations $w_i=0$ define a Cartier divisor on S', and it induces one on S. We call this R'. Then we have $R'\in|f^*F|$ and $2R'=f^*B$. If we set $Z=R'-R$, then $2Z$ is equal to f^*B-2R, and is effective by [Q], Lemma 3. Therefore Z itself is effective. The second assertion is proved in [Q], Lemma 4. Q.E.D.

§3. Surfaces of Type I

From now on, S denotes a minimal algebraic surface with $p_g=4$, $q=0$, and $c_1^2=6$. Let $\pi: \tilde{S}\to S$ be a composition of quadratic transformations such that the variable part $|L|$ of $|\pi^*K|$ has no base point. We assume that π is the shortest one among such compositions. We let F denote the fixed part of $|\pi^*K|$, and we write the canonical bundle $\tilde{K}$ of $\tilde{S}$ in the form $\pi^*K+[E]$ with an effective divisor E.

Lemma 3.1. *We have $L^2=6$ or 4.*

Proof. By Theorem 1.3, $|L|$ is not composite with a pencil. Hence, by [Q], Lemma 2, we have $4\leqq L^2\leqq 6$. It remains to exclude the possibility $L^2=5$.

Suppose $L^2=5$. Then $|L|$ defines a birational map of S onto a quintic surface in $\mathbb{P}^3$. Let C be a general member of $|L|$, which we assume to be non-singular. Then, by the adjunction formula, C is of genus $6+L(E+F)/2$. On the other hand, the image of C is a plane quintic curve, which is at most of genus 6. Therefore we have $LF=0$. But this implies $F^2=1$, which contradicts Hodge's index theorem. Q.E.D.

In the rest of this section, we assume $L^2=6$, i.e., by [Q], Lemma 2, the canonical system $|K|$ has no base point. The following theorem is obvious.

Theorem 3.1. *Suppose that the canonical system $|K|$ has no base point. Such surfaces are classified into the following three types:*

(Ia) *the canonical map Φ_K induces a birational map of S onto a sextic surface in $\mathbb{P}^3$;*

(Ib) *Φ_K induces a map of degree 2 onto a cubic surface in $\mathbb{P}^3$;*

(II) *Φ_K induces a map of degree 3 onto a quadric in $\mathbb{P}^3$.*

We shall study surfaces of type I in this section, and those of type II in the next section.

Theorem 3.2. *Let S be a surface of type* Ia, *and let $S'=\Phi_K(S)$ be its canonical image. Then S' is defined in $\mathbb{P}^3$ by the equation of the form*

$$g^2+Agh+Bh^2=0,$$

where g, h, A, and B are homogeneous forms of degree 3, 1, 2, *and* 4, *respectively. Moreover, S is the minimal resolution of the singularities of S'.*

Proof. Let $\{\varphi_0,\varphi_1,\varphi_2,\varphi_3\}$ be a basis of $H^0(S,\mathcal{O}(K))$. Then the products $\varphi_i\varphi_j$ generate a 10-dimensional subspace of $H^0(S,\mathcal{O}(2K))$. Since we have $h^0(2K)=11$ (see [16, 3]), there exists $\psi\in H^0(S,\mathcal{O}(2K))$ which is independent from the $\varphi_i\varphi_j$.

The products $\varphi_i\varphi_j\varphi_k$ and $\varphi_i\psi$ determine 24 elements of $H^0(S,\mathcal{O}(3K))$. Because of the equality $h^0(3K)=23$, there exists a relation of the form

$$h\psi=g, \tag{3.1}$$

where h and g are linear and cubic forms in $(\varphi_0,\varphi_1,\varphi_2,\varphi_3)$, respectively. We may assume $h=\varphi_0$. Then, by the choice of ψ, g is not divisible by φ_0.

Suppose that we have another relation $h'\psi=g'$ of the same form. Then we have $hg'=h'g$. Since g is not divisible by φ_0, we conclude that $h'=\alpha h$ and $g'=g/\alpha$ for some constant $\alpha\neq 0$. This implies that the collection

$$\varphi_i\varphi_j\varphi_k \quad (0\leqq i\leqq j\leqq k\leqq 3), \quad \varphi_i\psi \quad (1\leqq i\leqq 3)$$

forms a basis of $H^0(S,\mathcal{O}(3K))$.

Next we prove that the following 41 elements

$$\begin{aligned} &\varphi_i\varphi_j\varphi_k\varphi_l && (0\leqq i\leqq j\leqq k\leqq l\leqq 3),\\ &\varphi_i\varphi_j\psi && (1\leqq i\leqq j\leqq 3)\end{aligned} \tag{3.2}$$

are linearly independent in $H^0(S,\mathcal{O}(4K))$. Suppose we have a relation $p\psi=q$, where p and q are respectively of degree 2 and 4 in the φ_i's. Then we have $hq=pg$. This is a relation of degree 5 in the φ_i's. Hence p is divisible by φ_0. This proves that there is no relation among the elements in (3.2).

Comparing the dimensions, we see that (3.2) forms a basis. Therefore, we can write ψ^2 in the form

$$\psi^2=-A\psi-B, \tag{3.3}$$

where A and B are homogeneous forms in the φ_i's of degree 2 and 4, respectively. Combining with (3.1), we obtain

$$g^2 + Agh + Bh^2 = 0.$$

This is a non-trivial relation of degree 6 among the φ_i's. Hence S' is defined by this equation.

Next we consider the line bundle $\varpi: V \to \mathbb{P}^3$ of degree 2, and let θ be a fibre coordinate on V. By this, we mean that θ is a section of $\varpi^* \mathcal{O}(2)$ which defines the 0-section of ϖ. Then, setting $\theta = \psi$, we obtain a well-defined holomorphic map $\Psi: S \to V$ which satisfies $\varpi \circ \Psi = \Phi_K$. Since (3.2) is a basis of $H^0(S, \mathcal{O}(4K))$, the image $\Psi(S)$ has only rational double points (see [3], Main Theorem). From (3.1) and (3.3), it follows that $\Psi(S)$ is defined by

$$\begin{aligned} &\theta^2 + A\theta + B = 0, \\ &\qquad h\theta - g = 0. \end{aligned} \tag{3.4}$$

Since the projection $\Psi(S) \to S'$ is finite, $\Psi(S)$ is the normalization of S'. Hence S is the minimal resolution of the singularities of S'. Q.E.D.

Conversely, if S'' is a subvariety of V of the form (3.4), and if S'' has only rational double points, then its minimal resolution of singularities is a minimal surface with $p_g = 4$, $q = 0$, and $c_1^2 = 6$ which is of type Ia. These surfaces were already known to M. Noether [21] in the generic case. However, the double curve $g = h = 0$ may be singular.

Corollary. *The image of the bicanonical map Φ_{2K} is normal and has at most rational double points.*

Next we shall study the structure of surfaces of type Ib.

Theorem 3.3. *Let S be a surface of type* Ib. *Then*

(i) *the canonical map is of degree 2 onto a cubic surface $g = 0$ in $\mathbb{P}^3$,*

(ii) *the bicanonical map Φ_{2K} is birational and its image is defined by*

$$\theta^2 + A\theta + B = 0, \qquad g = 0 \tag{3.5}$$

in the line bundle $\varpi: V \to \mathbb{P}^3$ of degree 2, where θ denotes a fibre coordinate on V, and A and B are forms of degree 2 and 4 on $\mathbb{P}^3$, respectively.

(i) is obvious by definition. To prove (ii), we first prove the following

Lemma 3.2. (i) *The canonical image $W = \Phi_K(S)$ has only isolated singularities.*

(ii) *A general member C of $|K|$ is a non-hyperelliptic curve of genus 7.*

(iii) *The bicanonical map Φ_{2K} is birational.*

Proof. (i) Suppose that W has a singular locus of dimension 1. Let D be a hyperplane section of W and let C be its inverse image on S. We assume that C is irreducible and non-singular. Let $\tilde{D} \to D$ be the normalization of D. Then the induced map $C \to D$ factors through $\tilde{D}$. If $K_{|C}$ denotes the restriction of K to C, then we have $\dim H^0(C, \mathcal{O}(K_{|C})) > 3$, because D is a rational curve of degree 3.

On the other hand, from the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(K) \to \mathcal{O}(K_{|C}) \to 0,$$

and $H^1(S, \mathcal{O})=0$, it follows that $\dim H^0(C, \mathcal{O}(K_{|C}))=3$. This is a contradiction.

(ii) Let D be a generic hyperplane section of W, and let C be as above. Then the induced map $C \to D$ is 2-sheeted and the composition $C \to D \to \mathbb{P}^2$ is defined by the linear system $|K_{|C}|$. Since $2K_{|C}$ is the canonical bundle of C, the canonical image of C dominates D. Since D is an elliptic curve, this implies that C is not hyperelliptic.

(iii) We consider the exact sequence

$$0 \to \mathcal{O}(K) \to \mathcal{O}(2K) \to \mathcal{O}(2K_{|C}) \to 0.$$

From the assumption $q=0$, it follows that the restriction map

$$H^0(S, \mathcal{O}(2K)) \to H^0(C, \mathcal{O}(2K_{|C}))$$

is surjective. Combining with (ii), we conclude that Φ_{2K} induces a birational map of C onto its image. Hence Φ_{2K} induces a birational map of S onto its image. Q.E.D.

Proof of Theorem 3.3. We take a basis $\{\varphi_0, \varphi_1, \varphi_2, \varphi_3\}$ of $H^0(S, \mathcal{O}(K))$ and a section $\psi \in H^0(S, \mathcal{O}(2K))$ in the same way as in Theorem 3.2. We claim that the products

$$\varphi_i \varphi_j \varphi_k \quad (0 \leqq i \leqq j \leqq k \leqq 3), \qquad \varphi_i \psi \quad (0 \leqq i \leqq 3)$$

generate $H^0(S, \mathcal{O}(3K))$. In fact, the products $\varphi_i \varphi_j \varphi_k$ generate a 19-dimensional subspace. Hence, if our claim is false, there is a non-trivial relation $p\psi = q$, where p and q are linear and cubic forms in φ_i's. But this contradicts that Φ_{2K} is birational.

In the same manner, we can prove that

$$\varphi_i \varphi_j \varphi_k \varphi_l \quad (0 \leqq i \leqq j \leqq k \leqq l \leqq 3), \qquad \varphi_i \varphi_j \psi \quad (0 \leqq i \leqq j \leqq 3) \tag{3.6}$$

generate $H^0(S, \mathcal{O}(4K))$. It follows that there exist homogeneous polynomials A and B in the φ_i's of degree 2 and 4, respectively, such that

$$\psi^2 + A\psi + B = 0.$$

Moreover, ψ defines a holomorphic map $\psi: S \to V$ whose image is contained in (3.5). Since the collection (3.6) generates $H^0(S, \mathcal{O}(4K))$, the image $\Psi(S)$ is birationally equivalent to S (see [3], Main Theorem). Hence the projection $\Psi(S) \to W$ is two-sheeted and $\Psi(S)$ is defined by (3.5).

Conversely, if (3.5) defines a normal surface with at most rational double points, then its minimal resolution is a minimal surface with $p_g=4$, $q=0$, and $c_1^2=6$ which is of type Ib. By Bertini's theorem, these surfaces actually exist.

§4. Surfaces of Type II

Let S be a surface of type II. That is, $\Phi_K: S \to \mathbb{P}^3$ induces a holomorphic map of degree 3 onto a quadric W.

Lemma 4.1. *W is singular.*

Proof. Suppose that W is non-singular. Then W is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Hence the canonical bundle is of the form $[C+D]$ with $C^2 = D^2 = 0$, $CD = 3$. This contradicts that $KC + C^2$ is even. Q.E.D.

Let Σ_2 be the Hirzebruch surface of degree 2. Then Σ_2 is the minimal resolution of the singularity of W. We let Γ and Δ_0 denote a fibre and the 0-section of Σ_2, respectively.

Theorem 4.1. *Let $\varpi: V \to \Sigma_2$ be the line bundle associated with $2\Delta_0 + 3\Gamma$, and let w be a fibre coordinate on V. Then S is birationally equivalent to a surface S' in V defined by the equation*

$$w^3 + \alpha\zeta w^2 + \beta\zeta w + \gamma\zeta^2 = 0, \tag{4.1}$$

where

$$\zeta \in H^0(\Sigma_2, \mathcal{O}([\Delta_0])), \qquad \alpha \in H^0(\Sigma_2, \mathcal{O}([\Delta_0 + 3\Gamma])),$$
$$\beta \in H^0(\Sigma_2, \mathcal{O}([3\Delta_0 + 6\Gamma])), \qquad \gamma \in H^0(\Sigma_2, \mathcal{O}([4\Delta_0 + 9\Gamma])).$$

More precisely, let $\tilde{S}$ be the minimal resolution of S'. Then $\tilde{S}$ contains an exceptional curve E over Δ_0 and S is obtained from $\tilde{S}$ by contracting E to a point.

Proof. Since $\Phi_K(S)$ is a quadratic cone, we can find a pencil $|D|$ and an effective divisor G such that $|K| = |2D + G|$ and $KG = 0$ (see [Q], p. 46). From $K^2 = 6$, it follows that $KD = 3$, which in turn implies $D^2 = DG = 1$ and $G^2 = -2$. In particular, the pencil $|D|$ has a unique base point b. We let $\pi: \tilde{S} \to S$ be the quadratic transformation with center b, and set $E = \pi^{-1}(b)$. The variable part $|\tilde{D}|$ of $|\pi^* D|$ defines a holomorphic map $\tilde{S} \to \mathbb{P}^1$. Since $|K|$ has no base point, there exists a section $\eta \in H^0(\tilde{S}, \mathcal{O}(\pi^* K))$ which does not vanish on $\pi^* G \cup E$. Furthermore, we take a non-zero section ω of $[\pi^* G + 2E]$ over $\tilde{S}$, which is unique up to a constant. Then the pair (ω, η) defines a holomorphic map $h: \tilde{S} \to \Sigma_2$ such that $h^* \Delta_0 = \pi^* G + 2E$. We note that the canonical bundle $\tilde{K}$ of $\tilde{S}$ is given by $h^*[\Delta_0 + 2\Gamma] + [E]$.

Next we shall lift the map h to $\tilde{h}: \tilde{S} \to V$. For this purpose, we consider the line bundle

$$h^*[2\Delta_0 + 3\Gamma] = \tilde{K} + [\tilde{D} + \pi^* G + E]. \tag{4.2}$$

By the Riemann-Roch theorem and the vanishing of $H^1(\tilde{S}, \mathcal{O}(-\tilde{D}))$ (see [3], Theorem A), we have

$$\dim H^0(\tilde{S}, \mathcal{O}(\tilde{K} + \tilde{D})) = \tfrac{1}{2}\tilde{D}(\tilde{K} + \tilde{D}) + 5 = 7.$$

If ξ is a non-zero section of $[\pi^* G+E]$, then, for any $\varphi \in H^0(\tilde{S}, \mathcal{O}(\tilde{K}+[\tilde{D}]))$, the product $\varphi\xi$ determines a section of $h^*[2\Delta_0+3\Gamma]$ over $\tilde{S}$. Since we have $\dim H^0(\Sigma_2, \mathcal{O}(2\Delta_0+3\Gamma))=6$, there exists φ such that $\varphi\xi$ is not induced by a section of $[2\Delta_0+3\Gamma]$ over Σ_2. Setting $w=\varphi\xi$, we obtain a holomorphic map $\tilde{h}: \tilde{S} \to V$.

Lemma 4.2. *The holomorphic map $\tilde{h}$ induces a birational map of $\tilde{S}$ onto its image.*

Proof. Let S' be the image of $\tilde{h}$. Since h is of degree 3, if S' is not birationally equivalent to $\tilde{S}$, then the projection $\varpi: S' \to \Sigma_2$ is birational. Hence we have an isomorphism

$$H^0(\Sigma_2, \mathcal{O}([2\Delta_0+3\Gamma])) \xrightarrow{\sim} H^0(S', \mathcal{O}(\varpi^*[2\Delta_0+3\Gamma])).$$

But this contradicts the choice of φ. Q.E.D.

Lemma 4.3. *The image of $\tilde{h}$ is defined by the equation of the form* (4.1).

Proof. We consider the subspace of $H^0(\tilde{S}, \mathcal{O}(h^*[6\Delta_0+9\Gamma]))$ which consists of those sections vanishing on $2\pi^*G+3E$. It contains

$$\begin{array}{ll} \varphi^3\xi^3 & \\ \varphi^2\xi^2\alpha\zeta & \text{with } \alpha \in H^0(\Sigma_2, \mathcal{O}([\Delta_0+3\Gamma])), \\ \varphi\xi\beta\zeta & \text{with } \beta \in H^0(\Sigma_2, \mathcal{O}([3\Delta_0+6\Gamma])), \\ \gamma\zeta^2 & \text{with } \gamma \in H^0(\Sigma_2, \mathcal{O}([4\Delta_0+9\Gamma])). \end{array} \tag{4.3}$$

When α, β, and γ move in bases of the corresponding spaces, the above list represents 53 sections. On the other hand, we have

$$[h^*(6\Delta_0+9\Gamma)-(2\pi^*G+3E)]=\tilde{K}+h^*[3\Delta_0+7\Gamma].$$

Hence, by the Riemann-Roch theorem and the Kodaira vanishing theorem (see [16, 3]), we have

$$h^0(h^*(6\Delta_0+9\Gamma)-(2\pi^*G+3E))=52.$$

Therefore we can find a non-trivial relation of the form

$$\delta(\varphi\xi)^3+\alpha\zeta(\varphi\xi)^2+\beta\zeta(\varphi\xi)+\gamma\zeta^2=0,$$

where $\delta \in \mathbb{C}$ and α, β, γ, ζ are as above. From Lemma 4.2, we infer that $\delta \neq 0$ and that this is the only relation among the sections in (4.3). Q.E.D.

Lemma 4.4. *Let Λ be the curve defined by $w=\zeta=0$. Then $S'=\tilde{h}(\tilde{S})$ has a double curve along Λ, but other singular points of S' are rational double points.*

Proof of Lemma 4.4 consists of several lemmas. Let $\mu: V^* \to V$ be the monoidal transformation with center Λ and let S^* be the proper transform of S' by μ.

Lemma 4.5. *The holomorphic map $\tilde{h}: \tilde{S} \to S'$ factors through S^*.*

Proof. Let $\mathscr{I}$ be the ideal sheaf of Λ on S' and let $\mathscr{J}$ be the ideal of $\mathcal{O}_{\tilde{S}}$ generated by $\tilde{h}^* \mathscr{I}$. We shall show that $\mathscr{J}$ is invertible. In fact, we have

$$\tilde{h}^* w = \varphi \xi, \quad \tilde{h}^* \zeta = e \xi,$$

where e is a non-zero section in $H^0(\tilde{S}, \mathcal{O}([E]))$. It suffices to prove that φ and e have no common zero. To see this, we first note that, since $(\tilde{K}+\tilde{D})E=0$, φ is constant on E. Secondly, we have the exact sequence

$$0 \to \mathcal{O}(\tilde{K}) \to \mathcal{O}(\tilde{K}+[\tilde{D}]) \to \mathcal{O}(K_{\tilde{D}}) \to 0,$$

where $K_{\tilde{D}}$ denotes the canonical bundle of $\tilde{D}$, and the restriction map

$$H^0(\tilde{S}, \mathcal{O}(\tilde{K}+[\tilde{D}])) \to H^0(\tilde{D}, \mathcal{O}(K_{\tilde{D}}))$$

is surjective. Since $|K_{\tilde{D}}|$ has no base point, it follows that $|\tilde{K}+\tilde{D}|$ has no base point on $\tilde{D}$.

Thirdly, by virtue of the choice of φ, we can find a basis $\{\varphi, \psi_1, \ldots, \psi_6\}$ of $H^0(\tilde{S}, \mathcal{O}(\tilde{K}+[\tilde{D}]))$ such that the $\psi_i \xi$ are induced by the elements of $H^0(\Sigma_2, \mathcal{O}([2\Delta_0+3\Gamma]))$. In view of the fact that Δ_0 is a fixed component of $|2\Delta_0+3\Gamma|$ and the equality $h^*\Delta_0 = \pi^* G + 2E$, the $\psi_i \xi$ vanish twice along E. Hence the ψ_i vanish along E. Therefore, if φ is 0 on E, the linear system $|\tilde{K}+\tilde{D}|$ has base points along E. This contradicts what we have proved above. Q.E.D.

Lemma 4.6. *The singular locus of S' consists of the double curve Λ and a finite number of points.*

Proof. Let Δ be a general member of $|\Delta_0+2\Gamma|$ on Σ_2 and let C be its inverse image on $\tilde{S}$. We may assume that C is irreducible and non-singular. Since C is a member of $|\pi^* K|$, it is of genus 7. Let C' be the inverse image of Δ on S'. Then we have the sequence of maps $C \to C' \to \Delta$. We consider the restriction $V_{|\Delta} \to \Delta$ of V to Δ, which is a line bundle of degree 3. From the form of (4.1), we readily infer that C' is of arithmetic genus 7. Hence C' is non-singular. This implies that there is no multiple curve other than Λ. Q.E.D.

Lemma 4.7. *The proper transform S^* by μ has only isolated singularities.*

Proof. In view of Lemma 4.6, it suffices to consider $\mu^{-1}(\Lambda)$. Let U be a coordinate neighborhood on Σ_2 which meets Δ_0, and let (z, ζ) be a system of coordinates on U such that Δ_0 is defined by $\zeta=0$. For a while, we let w denote a fibre coordinate on $\tilde{U} = \varpi^{-1}(U)$. Then $S' \cap \tilde{U}$ is defined by the equation of the form (4.1), where α, β, γ are regarded as functions of (z, ζ). By replacing w by $w + \alpha\zeta/3$, we may assume $\alpha=0$. The inverse image $\mu^{-1}(\tilde{U})$ is covered by two open sets $V_i, i=1,2$, and the V_i are covered by systems of coordinates (z_i, u_i, v_i), which satisfy

$$z = z_1 = z_2, \quad \zeta = u_1 v_1 = u_2, \quad w = u_1 = u_2 v_2.$$

Suppose first that at least one of β and γ does not vanish identically on Δ_0. Then the proper transform S^* of S' is defined by

$$\begin{aligned} &u_1 + \beta v_1 + \gamma v_1^2 = 0 \quad \text{on } V_1, \\ &u_2 v_2^3 + \beta v_2 + \gamma = 0 \quad \text{on } V_2. \end{aligned} \tag{4.4}$$

It is clear that $S^* \cap V_1$ is non-singular. On the other hand, the intersection $\mu^{-1}(\Lambda) \cap S^* \cap V_2$ is defined by $u_2 = \beta v_2 + \gamma = 0$, and is non-singular. Hence $S^* \cap V_2$ is non-singular in a neighborhood of $\mu^{-1}(\Lambda) \cap S^* \cap V_2$.

Next suppose that β and γ both vanish on Δ_0. Then Λ is a triple curve of S'. We shall prove that this is impossible. For this purpose, we take a non-singular rational curve $\bar{C} \in |\Delta_0 + 3\Gamma|$ on Σ_2, and let C be its preimage on $\tilde{S}$. We may assume C is non-singular. Then, being a member of $|\pi^* K + \tilde{D}|$, C is of genus 12. Let C' be the image of C on S'. Then C' is defined by the Equation (4.1) in the restriction of V over $\bar{C}$. This restriction is a line bundle of degree 5 over $\bar{C}$. From these facts, it follows that C' is of arithmetic genus 13. This implies that C' has only one double point. Hence Λ is not a triple curve. Q.E.D.

Lemma 4.8. *S^* has only rational double points as singularities.*

Proof. In this proof we let V denote the $\mathbb{P}^1$-bundle associated with $[2\Delta_0 + 3\Gamma]$. Since the normal sheaf $\mathcal{N} = \mathcal{N}_{\Lambda/V}$ of Λ in V is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$, the exceptional divisor $\mathcal{E} = \mu^{-1}(\Lambda)$ is isomorphic to Σ_1. Furthermore, the normal bundle of $\mathcal{E}$ in V is given by $[-b-2f]$, where b and f denote the 0-section and a fibre of Σ_1, respectively. This latter fact can be seen, for example, as follows. We note that $\bigwedge^2 \mathcal{N} = \mathcal{O}(-3)$. Hence the canonical bundle K_V induces a line bundle of degree 1 on Λ. The canonical bundle of $\mathcal{E}$ is induced by $\mu^* K_V + [2\mathcal{E}]$, and it coincides with $[-2b-3f]$. Since $\mathcal{E}$ has no torsion line bundle, we conclude $[\mathcal{E}]_{|\mathcal{E}} = [-b-2f]$.

From (4.4), we infer that $[S^*]_{|\mathcal{E}} = [2b+f]$. Hence, combined with the above observation, it follows that $[\mu^* S']$ induces $[-3f]$ on $\mathcal{E}$. Hence we obtain the exact sequence

$$0 \to \mathcal{O}(-\mu^* S' + (k-1)\mathcal{E})) \to \mathcal{O}(-\mu^* S' + k\mathcal{E}) \to \mathcal{O}_{\mathcal{E}}(-kb-(2k-3)f) \to 0 \qquad (4.5)$$

for any k. First, by a standard calculation, we prove $\chi(\mathcal{O}_V(-S')) = -3$. Then using (4.5), we obtain $\chi(\mathcal{O}(-\mu^* S' + 2\mathcal{E})) = -4$. Finally we obtain $\chi(\mathcal{O}_{S^*}) = 5$.

Thus we have proved $\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{S^*})$. Since S^* is normal by Lemma 4.7, it follows that S^* has only rational singularities. Since S^* is embedded in a non-singular threefold, it has only rational double points (see [1], Corollary 6). This completes the proof of Lemma 4.8, and is the end of the proof of Lemma 4.4.

As we have seen in the proof of Lemma 4.7, S^* is non-singular in a neighborhood of $\mu^{-1}(\Lambda) \cap S^*$. Now we shall prove that S^* contains an exceptional curve in $\mu^{-1}(\Lambda) \cap S^*$. We divide the case according to whether β vanishes on Δ_0 or not.

If β does not vanish on Δ_0, $\mu^{-1}(\Lambda) \cap S^*$ consists of the two disjoint curves E and $\tilde{G}$, where E is defined by $v_1 = 0$ and $\tilde{G}$ is defined by $\beta + \gamma v_1 = 0$ on V_1 and $\beta v_2 + \gamma = 0$ on V_2, in the previous notation. We claim $E^2 = -1$. In fact, let U' be another coordinate neighborhood on Σ_2 and let ζ', w', v_1' etc. be the corresponding coordinates as above. Then we have

$$\zeta/\zeta' = (w/w')(v_1/v_1').$$

We identify E with Λ by the map μ, and note that ζ/ζ' and w/w' are respectively the transition functions of $\varpi^*[\Delta_0]$ and the line bundle associated with the 0-section of V. On Λ, the former is of degree -2, and the latter -1. This implies that $E^2=-1$. Moreover, from the equality $\zeta=-v_1^2(\beta+\gamma v_1)$, it follows that the pull-back of Δ_0 on S^* is $2E+\tilde{G}$. Hence we have $\tilde{G}^2=-2$.

Next we suppose that β vanishes identically on Δ_0. Then $\mathscr{E}_{|S^*}$ is of the form $2E+G'$, where E is defined by $v_1=0$ and G' by $\gamma=0$. In the same way as above, we can prove $E^2=-1$. On the other hand, since γ has a simple zero on Δ_0, G' is a fibre of the projection $\mu^{-1}(\Lambda)\to\Lambda$. Hence we have $EG'=1$. Furthermore, the inverse image of Δ_0 is $3E+G'$. This proves that $G'^2=-3$. We set $\tilde{G}=G'+E$ in accordance with the preseding case.

In the first case, S^* is the normalization of S', while, in the second case, S^* is obtained by blowing up a point on the normalization of S'. In either case $\tilde{S}$ is the minimal resolution of the singularities of S'.

Conversely, we start with S' defined by (4.1) and assume it has only rational double points except for the double curve Λ. Let μ: $S^*\to S'$ be the monodial transformation with center Λ and let $\tilde{S}$ be the minimal resolution of S^*. Then $\tilde{S}$ contains an exceptional curve E over Δ_0. We let π: $\tilde{S}\to S$ be the contraction of E. We now prove S is a minimal surface with $p_g=4$, $q=0$, and $c_1^2=6$. First, by a standard calculation using (4.5), we obtain $p_g=4$ and $q=0$. Secondly, the canonical bundle of $\tilde{S}$ is induced by $h^*(2\Delta_0+2\Gamma)-\mu^{-1}(\Lambda)$, where h: $\tilde{S}\to\Sigma_2$ denotes the natural map. As we have seen above, $h^*\Delta_0=2E+\tilde{G}$ and $\mu^{-1}(\Lambda)=E+\tilde{G}$. Hence, $\pi^*K=h^*[\Delta_0+2\Gamma]$. This implies $K^2=6$, and that $|K|$ has no base point. In particular, S is minimal. This completes the proof of Theorem 4.1.

It remains to prove the existence of the Equations (4.1) which define S' with the properties required above. For this purpose, we regard (4.1) as a linear system of divisors on V. It can be easily checked that it has no base point outside of Λ. Furhtermore we have already shown that (4.4) is non-singular provided one of β and γ does not vanish identically on Δ_0. Therefore, for a generic choice of (4.1), S^* is non-singular. This proves the existence of surfaces of type II.

§5. Surfaces of Type III

In this and the next sections we shall study the case in which $|K|$ has base points. We let π: $\tilde{S}\to S$, L, E, and F be the same as in §3.

Lemma 5.1. *Assume that $|K|$ has base points. Then we have $L^2=4$, $LF=2$, $F^2=-2$, and $LE=0$ or 2.*

Proof. By Lemma 3.1, we have $L^2=4$ and $(L+\pi^*K)F=K^2-L^2=2$. The second equality implies that $LF=1$ or 2. On the other hand, by the Riemann-Roch theorem and the Kodaira vanishing theorem, we have

$$h^0(\tilde{K}+L)=L^2+\tfrac{1}{2}L(E+F)+5, \tag{5.1}$$

which is at most equal to $h^0(2\tilde{K})=11$. We also note that $F-E$ is effective, and hence $LF\geqq LE$. This is obvious if π is a single quadratic transformation at a base point of $|K|$. The general case immediately follows from this special case.

If $LF=1$, we have $LE=1$, and hence E is irreducible (see [Q], p. 45). From the equalities $LE=LF=-EF=1$, we infer that $F=E+F'$ with $LF'=EF'=0$ (see [Q], p. 46). From $F^2=K^2-L^2-2LF=0$, it follows that $F'^2=1$, which contradicts Hodge's index theorem. Therefore, we have $LF=2$ and $F^2=-2$. Since $L(E+F)$ is even by (5.1), we have $LE=0$ or 2. Q.E.D.

We shall call S of *type* III if $|K|$ has base points and if S does not admit a pencil $|D|$ whose general member is a non-singular curve of genus 2. We refer to such a pencil as a pencil of curves of genus 2. Surfaces with such pencil will be studied in the next section. Furthermore, we call S of *type* IIIa or IIIb according to whether $LE=2$ or 0.

Theorem 5.1. *Let S be a surface of type* IIIa. *Then $|L|$ defines a holomorphic map of degree 2 onto a non-singular quadric W in $\mathbb{P}^3$. If we identify W with Σ_0, the branch locus B is in $|8\Delta_0+8\Gamma|$, and, in the generic case, it consists of a fibre Γ, a section Δ_0, and a curve B_0 which has a quadruple point at $x=\Gamma\cap\Delta_0$ and two triple points, one each on Γ and Δ_0.*

We first prove the following lemma.

Lemma 5.2. *The equality $LE=2$ implies that $F=E$ and $E^2=-2$.*

Proof. By (5.1) we have $h^0(\tilde{K}+L)=h^0(2K)$. Since $F-E$ is effective, we have

$$|2\pi^* K|=|\tilde{K}+L|+(F-E).$$

Since $|2K|$ has no base point (see [3], Theorem 2), we obtain $F=E$. Q.E.D.

Corollary. *π is the composition of two quadratic transformations.*

Let $f\colon \tilde{S}\to\mathbb{P}^3$ be the holomorphic map defined by $|L|$. We first suppose that W is singular and derive a contradiction. In the same manner as in [Q], p. 46, we can find a pencil $|D|$ and an effective divisor G such that $2D+G\in|L|$ and $LG=0$. From $L^2=4$, we get

$$2=LD=2D^2+DG. \tag{5.2}$$

Hence $\tilde{K}D$ and D^2 are even integers. By (5.2), we obtain $D^2=0$. $DG=2$, and $G^2=-4$. Using the inequality

$$6=(\pi^* K)L\geqq 2(\pi^* K)D=4+2DE,$$

we obtain $DE=0$ or 1. But $DE=0$ would imply that $|D|$ is a pencil of genus 2. Hence we have $DE=1$, and hence $GE=0$. Let E_0 be the unique component of E such that $DE_0=1$, and set $E_1=E-E_0$. Then, by the above Corollary, $E_1\neq 0$, and

$$2\geqq LE_0=2+GE_0, \quad LE_1=GE_1.$$

Suppose first that $GE_0=0$. Then, combining with $GE=0$, we get $GE_1=LE_1=0$. This implies that π is the blowing up of two infinitely near points, for otherwise, we would have $LE_1>0$. But, then E is of the form $2E_0'+E_1'$, and we necessarily have $E_0=E_0'$. This contradicts the equality $DE=1$.

Next suppose $GE_0<0$. Then G contains E_0. Hence, from $LG=0$, we obtain $LE_0=0$. This again implies that π is the blowing up of two infinitely near points. This time we have $E_0^2=-2$ and $E_1=2E_1'$ with $E_0E_1'=-E_1'^2=1$. We also have

$$DE_1'=0, \quad LE_1'=1. \tag{5.3}$$

By the same method as in [Q], p. 46, f induces a holomorphic map $h: \tilde{S}\to\Sigma_2$ of degree 2 such that $h^*\Delta_0=G$, and the ramification divisor R of h is linearly equivalent to $6D+3G+2E$. Hence $DR=8$, $GR=0$. It follows that the branch locus B is linearly equivalent to $8\Delta_0+16\Gamma$. We consider an involution ϕ of $\tilde{S}$ which commutes with h. Such ϕ exists because $\tilde{S}$ is the minimal resolution of the double covering of Σ_2 with branch locus B (see Lemma 2.1 and [Q], Lemma 4). Since ϕ induces an involution on the minimal model S, and since it fixes the base points of $|K|$, it follows that $\phi(E_0)=E_0$ and $\phi(E_1')=E_1'$. Moreover, the equalities $DE_0=LE_1'=1$ imply that h is generically one-to-one on E_0 and E_1'. Therefore $h(E_0)$ and $h(E_1')$ are contained in the branch locus B. By construction, E_0 is mapped onto Δ_0, and, by (5.3), E_1' is mapped onto a fibre Γ_0. Since B is in $|8\Delta_0+16\Gamma|$, the divisor $B-\Delta_0-\Gamma_0$ intersects Δ_0 transversally at one point. This implies that B has no infinitely near triple points in a neighborhood of Δ_0. Hence the minimal resolution coincides with the canonical resolution in a neighborhood of the preimage of Δ_0 (see [Q], §2, Lemma 5). But, by the canonical resolution, we have to blow up the intersection of Δ_0 and Γ_0, and it turns out that E_0 and E_1' cannot intersect. This contradicts our previous observation.

Thus we have proved that $f(S)$ is a non-singular quadric W. We identify W with Σ_0, and use the same symbol f to denote the induced map $\tilde{S}\to\Sigma_0$. From the equality $L=f^*[\Delta_0+\Gamma]$, we obtain

$$\tilde{K}f^*\Delta_0+\tilde{K}f^*\Gamma=8.$$

Since S has no pencil of genus 2, it follows that $\tilde{K}f^*\Delta_0=\tilde{K}f^*\Gamma=4$, and hence $Ef^*\Delta_0=Ef^*\Gamma=1$. We take an irreducible component E_0 of E such that $E_0f^*\Gamma=1$ and set $E_1=E-E_0$. Note that E_0 is a simple component of E. Hence, by Corollary to Lemma 5.2, E_1 is either an exceptional curve or is double in case the two base points are infinitely near. Anyway, E_1 is not mapped into a point by f. Therefore we have $LE_1>0$. Since $E_1f^*\Gamma=0$, it follows that $E_1f^*\Delta_0=1$ and $E_0f^*\Delta_0=0$. Eventually, E_1 is reduced and irreducible.

By a straightforward calculation we see that the ramification divisor of f is linearly equivalent to $3L+2E$ and the branch locus B is linearly equivalent to $8\Delta_0+8\Gamma$. By Lemma 2.1, there exists a divisor $Z\in|f^*(\Delta_0+\Gamma)-2E|$. This implies the existence of

$$Z_0\in|f^*\Delta_0-2E_0|, \quad\text{and}\quad Z_1\in|f^*\Gamma-2E_1|.$$

Therefore B contains $f(E_0)$ and $f(E_1)$ as components.

Let S' be the double covering of Σ_0 with branch locus B, and let S^* be the canonical resolution of S'. Then S^* has the following numerical characters:

$$p_a(S^*)=9-\tfrac{1}{2}\sum[m_i/2]([m_i/2]-1),$$
$$c_1^2(S^*)=16-2\sum([m_i/2]-1)^2,$$

where the m_i denote the multiplicities appearing in the process of the canonical resolution, and the brackets denote the integral part (see [Q], §2). We recall $p_a(S^*)=4$ and $c_1^2(S^*)\leqq c_1^2(\tilde{S})=4$. Therefore, we have $[m_i/2]=3$ for one i and $[m_i/2]=2$ for two i's. Hence, we have $c_1^2(S^*)=c_1^2(\tilde{S})$, and $\tilde{S}$ is the canonical resolution of the singularities of S'.

Let $\tilde{x}$ be one of the intersection $E_0\cap Z_0$ and set $x=f(\tilde{x})$. Let q: $W_1\to\Sigma_0$ be the quadratic transformation with center x. Then, by the above observation, f can be lifted to f_1: $\tilde{S}\to W_1$. Let $E_x=q^{-1}(x)$ and let R_1 be the ramification divisor of f_1. Then $(f^*E_x)R_1$ is 6 or 4 according to whether x is on $f(E_1)$ or not. Applying a similar consideration to E_1, we see that B has a sextuple point at $f(E_0)\cap f(E_1)$. Now we take x to be $f(E_0)\cap f(E_1)$. Then the branch locus B_1 of f_1 has two quadruple points, one each on the proper transforms of $f(E_0)$ and $f(E_1)$.

Conversely, we take a section Δ_0 and a fibre Γ on Σ_0, and we let x be the intersection $\Delta_0\cap\Gamma$. Let q: $W_1\to\Sigma_0$ be the blowing up of x, and let q: $\tilde{W}\to W_1$ be the blowing up of two points y and z, which are on the proper transforms of Δ_0 and Γ, respectively. Here y and z may be over x. We consider the composition $\mu=q_1\circ q$, and a divisor

$$\tilde{B}_0\in|\mu^*(7\Delta_0+7\Gamma)-4E_x-3E_y-3E_z|,$$

where $E_x=q_1^*(q^{-1}(x))$, $E_y=q_1^{-1}(y)$, and $E_z=q_1^{-1}(z)$. We assume that $\tilde{B}_0$ has no multiple component nor infinitely near triple points, and does not contain the proper transforms $\tilde{\Delta}_0$ and $\tilde{\Gamma}$ of Δ_0 and Γ. Let $\tilde{S}$ be the double covering of $\tilde{W}$ with branch locus $\tilde{B}_0+\tilde{\Delta}_0+\tilde{\Gamma}$. Then $\tilde{S}$ contains two exceptional curves E_0 and E_1 over $\tilde{\Delta}_0$ and $\tilde{\Gamma}$. Contracting E_0 and E_1, we obtain a minimal surface S with $p_g=4$, $q=0$, and $c_1^2=6$.

We prove that S is of type IIIa. For this purpose, we first note that the canonical bundle $\tilde{K}$ of $\tilde{S}$ is defined by $f^*(\Delta_0+\Gamma)+2E_0+2E_1$ (see [Q], §2). Hence the canonical system $|K|$ of S has two base points. Next we shall prove that there exists no pencil $|D|$ of genus 2 on S. In fact, if such $|D|$ exists, it has no base point (see [P], Theorem 5). Hence we have

$$2=(\pi^*D)\tilde{K}=(\pi^*D)(f^*\Delta_0)+(\pi^*D)(f^*\Gamma),$$

where π denotes the contraction map $\tilde{S}\to S$. If $(\pi^*D)(f^*\Gamma)=1$, then D is mapped birationally onto $\mathbb{P}^1$, which is impossible. Hence, one of the above summands vanishes. Suppose, for instance, $(\pi^*D)(f^*\Gamma)=0$. Then, since $(\pi^*D)^2=0$, π^*D is numerically equivalent to a rational multiple of $f^*\Gamma$ (see, e.g. [23], p. 92). But this is impossible. This completes the proof of Theorem 5.1.

Remarks 1. The existence of $\tilde{B}_0$ can be checked by a method similar to [Q], pp. 52–53. Here we note that y and z may be both infinitely near to x, in which case B_0 has a quintuple point at x.

2. We can contract on W_1 the proper transforms of Δ_0 and Γ. This transforms W_1 into $\mathbb{P}^2$, and B into a curve of degree 10 which has two 2-fold triple points (cf. Enriques [6], p. 272). We shall use this fact in §11.

Theorem 5.2. *Let S be a surface of type* IIIb. *Then S is birationally equivalent to a double covering of Σ_2 whose branch locus B consists of the 0-section Δ_0 and $B_0 \in |7\Delta_0 + 14\Gamma|$ which has a quadruple point at $x \in \Gamma$ and a 2-fold triple point at $y \in \Gamma$ on a fibre Γ, with x and y being possibly infinitely near.*

Proof. Using the previous notation, we have $E=0$. Let $f: S \to \mathbb{P}^3$ be the holomorphic map defined by the variable part $|L|$ of $|K|$. We first prove that the image W of f is singular. If not, then $W = \Sigma_0$, and $L = f^*[\Delta_0 + \Gamma]$. From $KL = 6$, we obtain $K(f^*\Delta_0 + f^*\Gamma) = 6$. Since $Kf^*\Delta_0$ and $Kf^*\Gamma$ are both even, one of them must be equal to 2. This contradicts that S has no pencil of genus 2.

Thus W is a quadratic cone, and hence $|L|$ is of the form $|2D+G|$ with a pencil $|D|$ and $LG=0$ (see [Q], p. 46). It follows that

$$2D^2 + DG = DL = 2, \qquad 2DF + GF = 2, \tag{5.4}$$

where F denotes the fixed part of $|K|$. Hence we have $D^2 = 0$ or 1. But, if $D^2 = 0$, then KD is even, and hence so is DF. Since $GF = KF$ is non-negative, from the second formula in (5.4), we obtain $DF = 0$. This contradicts that S has no pencil of genus 2. Thus we have proved $D^2 = 1$, $DG = 0$, and $G^2 = 0$. By Hodge's index theorem, we obtain $G = 0$, and hence $DF = 1$. In particular, $|D|$ has a base point b.

Let $\pi: \tilde{S} \to S$ be the quadratic transformation with center b, and set $E = \pi^{-1}(b)$, $\tilde{D} = \pi^* D - E$. Then we obtain a holomorphic map $f: \tilde{S} \to \Sigma_2$ of degree 2 such that $f^*\Delta_0 = 2E$. The ramification divisor R and the branch locus B are linearly equivalent to $\pi^*(3L+F)+E$ and $8\Delta_0 + 14\Gamma$, respectively. In particular, B is a disjoint sum $\Delta_0 + B_0$ with $B_0 \in |7\Delta_0 + 14\Gamma|$. By Lemma 2.1, there exists an effective divisor $Z \in |\tilde{D} + E - \pi^* F|$. Since $f(Z)$ is a finite set, Z does not contain E. On the other hand, E is a fixed component of $|\tilde{D} + E|$. Hence $\pi^* F$ is of the form $E + F'$ with $F' \geqq 0$.

Let S^* be the canonical resolution of the double covering S' of Σ_2 with branch locus B. Then S^* has the following numerical characters:

$$p_a = 6 - \tfrac{1}{2} \sum [m_i/2]([m_i/2] - 1),$$
$$c_1^2 = 8 - 2\sum([m_i/2] - 1)^2$$

(see [Q], §2). It follows that there are exactly two i's with $[m_i/2] = 2$. Hence S^* is a quadratic transform of $\tilde{S}$. Moreover, since Z is linearly equivalent to $\tilde{D} - F'$, the essential singularities of B are on a single fibre Γ (cf. [Q], Lemma 5).

We have the following four possibilities:

1) B has two quadruple points.

2) B has a quadruple point and a 2-fold triple point.

3) B has a quadruple point x which, after a quadratic transformation at x, gives a 2-fold triple point.

4) B has a 4-fold triple point.

Among these, only 2) and 3) are possible. In the case 1), S^* would contain no exceptional curve which is mapped to a point on S'. In the case 4), S^* would contain two of such exceptional curves. Neither of them is the case.

First we consider the case 2). Let x and y be two points on a fibre Γ and not on Δ_0, and let $q\colon W_1 \to \Sigma_2$ be the blowing up of x and y. We set $E_x = q^{-1}(x)$, $E_y = q^{-1}(y)$. Then, take a point y' on E_y which is not on the proper transform of Γ, and let $q_1\colon \tilde{W} \to W_1$ be the blowing up of y'. We set $\mu = q_1 \circ q$, $E'_y = q_1^{-1}(y')$, and $\tilde{E}_y = q_1^* E_y - E'_y$, and for simplicity we write E_x instead of $q_1^* E_x$. We consider divisors $\tilde{B}_0$ on $\tilde{W}$ linearly equivalent to

$$\mu^*(7\Delta_0 + 14\Gamma) - 4E_x - 3\tilde{E}_y - 6E'_y.$$

Let $\Delta_x \in |\Delta_0 + 2\Gamma|$ be an irreducible curve through x, and let $\Delta_y \in |\Delta_0 + 2\Gamma|$ be through y at the direction of y'. We let $\tilde{\Delta}_x$ and $\tilde{\Delta}_y$ be their proper transforms by μ. Finally we denote by $\tilde{\Gamma}$ the proper transform of Γ by μ. Then $|\tilde{B}_0|$ contains

$$4\tilde{\Delta}_x + 3\tilde{\Delta}_y, \quad 6\tilde{\Gamma} + 2E_x + 3\tilde{E}_y + 7\mu^*\Delta_0 + |8\mu^*\Gamma|, \quad \text{and}$$
$$4\tilde{\Gamma} + 3\tilde{\Delta}_y + 4\tilde{E}_y + 4E'_y + |\mu^*(4\Delta_0 + 4\Gamma)|.$$

It follows that $|\tilde{B}_0|$ has no base point on $\tilde{W}$. Using the Riemann-Roch theorem, we obtain $\dim|\tilde{B}_0| = 41$.

Remark 3. The above argument can be applied with a slight modification to the case where x and y are not on a single fibre, and $\dim|\tilde{B}_0|$ remains constant. But in this case, we obtain a surface of type IIIa instead of type IIIb. This fact will be used in constructing a family of deformations of a surface of type IIIb (see §9).

Next we shall prove that the singularities of 3) actually occur. We distinguish the case according to whether the 2-fold triple point is on the proper transform of the fibre Γ or not. Let z and η be inhomogeneous coordinates on the base curve and on Γ, respectively. We may assume that x is at $z = \eta = 0$. Then any section of $[7\Delta_0 + 14\Gamma]$ over Σ_2 is represented by a linear combination of

$$\eta^i z^j \quad \text{with} \quad j \leqq 14 - 2i. \tag{5.5}$$

Let Λ be the subsystem spanned by $\eta^i z^j$ with $3i + j \geqq 10$. Then a generic member B of Λ has the singularity of type 3) at x. In fact, after a quadratic transformation at x, the proper transform of B is defined by a linear combination of $(\eta/z)^i z^{i+j-4}$ with $2i + (i+j-4) \geqq 6$. This has a 2-fold triple point at $z = \eta/z = 0$ (which is not on the proper transform of Γ). We can also check that Λ has no base point other than x. Hence B is non-singular at other points.

Next we consider the linear subsystem Λ' of (5.5) spanned by

$$\eta^i z^j \quad \text{with} \quad i + 2j \geqq 10$$
$$\eta(\eta^2 + z)^3, \quad z^2(\eta^2 + z)^2, \quad \eta^3 z^2(\eta^2 + z).$$

Then a member of Λ' has, after a quadratic transformation at $z = \eta = 0$, a 2-fold triple point at $\eta = z/\eta = 0$, with the tangent being $\eta + (z/\eta) = 0$. Moreover, Λ' has no base point other than x. Hence we can also find a seeked branch locus in this case.

Conversely, if we start with a branch locus B of the above form, then we obtain a minimal surface with $p_g = 4$, $q = 0$, and $c_1^2 = 6$. We can easily prove that

S has no pencil of genus 2 (cf. the final part of the proof of Theorem 5.1). Hence S is of type III. In the case 2), S is of type IIIb (but, see Remark 3). However, in the case 3), S is of type IIIb or IIIa according to whether the 2-fold triple point is on the proper transform of Γ or not.

Remark 4. A surface of type IIIb is birationally equivalent to a double covering of $\mathbb{P}^2$ with branch locus of degree 10 which has a 4-fold triple point. To obtain this, we apply the elementary transformation to Σ_2 at the quadruple point x. Then we obtain Σ_1 and this can be transformed into $\mathbb{P}^2$ by contracting an exceptional curve.

§6. Surfaces of Types IV and V

In this section we shall study the case in which S has a pencil $|D|$ of curves of genus 2. We apply the result of [P] to such surfaces. By [P], Theorem 5, $|D|$ defines a surjective holomorphic map $g: S \to \mathbb{P}^1$. For each integer m, we set

$$b(m) = h^0(K + mD) - h^0(K + (m-1)D).$$

Then we have $b(m) \leqq b(m+1) \leqq 2$. Since $h^1(-D)$ vanishes (see [3], Theorem A), we have $h^0(K+D) = p_g + 2$. Hence it follows that $b(1) = 2$. We set

$$m_j = \operatorname{Min}\{m \mid b(m) \geqq j\}, \quad j = 1, 2,$$

and $d = m_2 - m_1$. We consider the rational map Φ defined by $|K + mD|$ for sufficiently large m. By [P], Theorems 1 and 2, the image of Φ is isomorphic to Σ_d with $d \equiv p_g \bmod 2$. Since $p_g = 4$, and since $|K|$ is not composite with a pencil (Theorem 1.3), we have $-2 \leqq m_1 \leqq m_2 \leqq 1$. This implies $d = 0$ or 2. Therefore, Φ induces a rational map $f: S \to \Sigma_d$ of degree 2, and $d = 0$ or 2. If $d = 0$, S is called of *type* IVa. If $d = 2$, S is called of *type* V or of *type* IVb according to whether the 0-section Δ_0 is contained in the branch locus or not. The singularities of the branch locus B are classified in [P], Lemma 6. In the present case, by [P], Theorem 3, one of the following holds:

(i) S has two singular fibres each of which is of type (I_1), (III_1) or (V).
(ii) S has one singular fibre of type (II_1) or (IV_1).

We shall add -1 or -2, e.g. IVa-1 or IVa-2, to indicate that S has singular fibers of the above type (i) or (ii).

Let us first suppose that S has singular fibres of (i). Then, employing the notation of [P], Lemmas 9 and 10, we have $\deg \mathfrak{c} = 2$, and, by [P], Theorem 2, $\deg \mathfrak{f} = 3 + d/2$. Hence B is linearly equivalent to $6\Delta_0 + (10 + 3d)\Gamma$.

Theorem 6.1. *Let S be a surface of type* IVa-1, IVb-1, *or* V-1. *Set $d=0$ in the first case, and $d=2$ in the second and third cases. Then S is birationally equivalent to a double covering of Σ_d whose branch locus B is linearly equivalent to $6\Delta_0 + (10 + 3d)\Gamma$ and is of the following form: B contains two fibers Γ_1 and Γ_2, and $B_0 = B - \Gamma_1 - \Gamma_2$ has two triple points on each of Γ_1 and Γ_2, which may be infinitely near. In this case $|K|$ has two simple base points.*

Next suppose that B has a singular fibre of type (ii). In this case, we have $\deg \mathfrak{c}=1$ and $\deg \mathfrak{f}=2+d/2$. Hence B is linearly equivalent to $6\Delta_0+(8+3d)\Gamma$.

Theorem 6.2. *Let S be a surface of type* IVa-2, IVb-2, *or* V-2. *Set $d=0$ in the first case, and $d=2$ in the second and third cases. Then S is birationally equivalent to a double covering of Σ_d whose branch locus B is linearly equivalent to $6\Delta_0+(8+3d)\Gamma$ and has two 2-fold triple points on a fibre Γ, which may be infinitely near. In this case, $|K|$ has a unique fixed component F which is a non-singular rational curve with $F^2=-2$.*

These are all in [P], except that we regarded a singular fibre of type (V) as a degenerate case of a singular fibre of type (III_1) to simplify the statement. Especially, the last statement about the fixed component follows from Lemma 10 in [P].

Remark. The pencil of genus 2 is unique on each surface of type IV or V, because it is induced from the canonical map in a natural way.

§7. Deformations of Surfaces of Type I

Theorem 7.1. *Let S be a surface of type* I a, *and assume that K is ample. Then the number of moduli $m(S)$ is defined and is equal to* $\dim H^1(S, \Theta_S)=38$, *where Θ_S denotes the sheaf of germs of holomorphic vector fields on S.*

Proof. See [9].

Theorem 7.2. *The surfaces of type* I *have one and the same deformation type.*

Proof. In view of the simultaneous resolution of the rational double points [4], it follows from the constructions in §3 that each type I a or I b has one and the same deformation type. Therefore, it remains to show that a surface of type I b is a deformation of a surface of type I a.

We let t be a parameter which moves in a neighborhood of the origin in $\mathbb{C}$. Using the notation in §3, we define S_t in V by the equations

$$\begin{aligned} &\theta^2+A\theta+B=0,\\ &\quad th\theta-g=0. \end{aligned} \tag{7.1}$$

Then S_0 is of type I b. If $t\neq 0$, (7.1) reduces to $g^2+tAgh+t^2Bh^2=0$. Hence the $S_t, t\neq 0$, are of type I a (cf. [6], p. 280). Q.E.D.

§8. Deformations of Surfaces of Type II

The main result of this section is the following.

Theorem 8.1. (i) *Let S be a surface of type* II. *Then any sufficiently small deformation of S is of type* II.

(ii) *If S is of type* II *and generic, then the number of moduli $m(S)$, in the sense of Kodaira-Spencer* [17], §11, *is defined and is equal to* 38.

(iii) *We have* $\dim H^1(S, \Theta_S)=39$ *and the Kuranishi family of deformations of* S *is parametrized by a non-reduced space* M *for which* M_{red} *is non-singular of dimension* 38.

Remark. The assertion (i), combined with the result of §3, implies that S cannot be directly deformed to a surface with ample canonical bundle. This seems to be a new example to the author. We shall find a similar example in [14], with $p_g=6$, $q=0$, and $c_1^2=11$.

Proof of (i). We let $\mathbf{D}$ denote a small disc at the origin of $\mathbb{C}$ with coordinate t, and let $p: \mathscr{S}\to\mathbf{D}$ be a family of surfaces such that $S=p^{-1}(0)$ is of type II. We take a generic member C of $|K|$ on S. Then, since $H^1(S, \mathcal{O}(C))$ vanishes, C can be extended to a family $q: \mathscr{C}\to\mathbf{D}$ of curves of genus 7. This family carries a line bundle $\mathscr{L}$ such that $2\mathscr{L}$ is the canonical bundle $\mathscr{K}$ of $\mathscr{C}$. The following lemma corresponds to our classification of surfaces with $p_g=4$, $q=0$, and $c_1^2=6$.

Lemma 8.1. *Let* C *be a non-singular curve of genus* 7, *and suppose that there exists a line bundle* L *on* C *such that* $2L$ *is the canonical bundle* K_C *and such that* $\dim|L|=2$. *Then one of the following is true.*

(Ia) *C is birationally equivalent to a plane sextic curve with three double points on a line.*

(Ib) *C is a double covering of an elliptic curve.*

(II) *C is a triple covering of $\mathbb{P}^1$ and L is induced by $\mathcal{O}(2)$.*

(III) *C is a hyperelliptic curve and L is induced by $5Q_1+Q_2$ with two Weierstrass points Q_1 and Q_2.*

Proof is parallel to §§3–5, and is much simpler.

If C is a canonical curve on a surface of type II, then C is of type II in Lemma 8.1. We need some more precise informations. We first prove a general lemma.

Lemma 8.2. *Let C be a non-singular curve which is a triple covering of another non-singular curve D. Then C can be embedded in a $\mathbb{P}^1$-bundle over D.*

Proof. Obviously, we can find a possibly singular model C' of C in the product $D\times\mathbb{P}^1$, which intersects a general fibre Γ at three points. If C' has a singular point s, we apply the elementary transformation at s. Since $C'\Gamma=3$, and since s is a singular point, it does not produce any new singularity, and it has the same effect on C' as the blowing up of s. Hence, after a finite number of such elementary transformations, C' is transformed into a non-singular curve in a $\mathbb{P}^1$-bundle over D. Q.E.D.

Lemma 8.3. *Let C be a non-singular curve of genus 7 which is a triple covering of $\mathbb{P}^1$. Then C can be embedded either in Σ_1 or in Σ_3. Moreover, C carries a line bundle L as in Lemma* 8.1, *if and only if C can be embedded in Σ_3.*

Proof. By Lemma 8.2, C can be embedded in Σ_d for some d. If C is linearly equivalent to $3\Delta_0+m\Gamma$, then K_C is induced by $\Delta_0+(m-d-2)\Gamma$. Hence, we have the equality $m=(3d+9)/2$. Since C is irreducible, we have $m\geqq 3d$. Hence d is 1 or 3. Conversely, such C certainly exists for $d=1$ and 3.

If $d=3$, then C is disjoint from Δ_0. Therefore K_C is induced by 4Γ. Hence $L=[2\Gamma]_{|C}$ satisfies the condition of Lemma 8.1. Conversely, if C is of type II in

Lemma 8.1, then L can be written as $2L_0$, and, by the Riemann-Roch theorem, we have $h^0(3L_0)=5$. If $\{\varphi_0, \varphi_1\}$ is a basis of $H^0(\mathcal{O}(L_0))$, then $H^0(\mathcal{O}(3L_0))$ is generated by the products $\varphi_0^i \varphi_1^{3-i}$, $0 \leqq i \leqq 3$, together with an element ψ. Hence we can define a holomorphic map Ψ: $C \to \Sigma_3$ such that ψ corresponds to the ∞-section of Σ_3. To prove that Ψ is an embedding, it is enough to note that

$$\varphi_0^i \varphi_1^{4-i} \quad (0 \leqq i \leqq 4), \qquad \varphi_0 \psi, \quad \varphi_1 \psi$$

form a basis of $H^0(\mathcal{O}(4L_0)) = H^0(\mathcal{O}(K_C))$, and that C is not hyperelliptic. Q.E.D.

Let $\mathcal{M}_0 \to M_0$ be the family of the triple coverings of $\mathbb{P}^1$ in Σ_3 which are close to C, and let $T_0(M_0)$ be the tangent space of M_0 at the point 0 corresponding to C (see [15]). This determines an infinitesimal deformation map

$$\rho_0: T_0(M_0) \to H^1(C, \Theta_C).$$

In a similar way, let $\mathcal{M}_1 \to M_1$ be the family of all triple coverings of $\mathbb{P}^1$ (not necessarily in Σ_3) which are close to g: $C \to \mathbb{P}^1$ (see [8], I). We denote its infinitesimal deformation map by

$$\rho_1: T_0(M_1) \to H^1(C, \Theta_C).$$

Lemma 8.4. *We have* $\dim \operatorname{Im} \rho_0 = 13$ *and* $\dim \operatorname{Im} \rho_1 = 15$.

Proof. The tangent space $T_0(M_0)$ can be identified with $H^0(C, \mathcal{N})$, where $\mathcal{N}$ is the normal sheaf of C in $W = \Sigma_3$. Since C is linearly equivalent to $3\Delta_0 + 9\Gamma$, we have $\dim H^0(\mathcal{N}) = 21$. In view of the exact sequence

$$0 \to H^0(\Theta_{W|C}) \to H^0(\mathcal{N}) \to H^1(\Theta_C),$$

we need to calculate $\dim H^0(\Theta_{W|C})$. For this purpose, we consider the exact sequence

$$0 \to \Theta_{W/\mathbb{P}^1} \to \Theta_W \to p^* \Theta_{\mathbb{P}^1} \to 0, \tag{8.1}$$

where p: $W \to \mathbb{P}^1$ is the projection and $\Theta_{W/\mathbb{P}^1}$ is the sheaf of vector fields along the fibres. As is well known, we have the isomorphisms $\Theta_{W/\mathbb{P}^1} \cong \mathcal{O}(2\Delta_0 + 3\Gamma)$ and $p^* \Theta_{\mathbb{P}^1} \cong \mathcal{O}(2\Gamma)$. The sequence (8.1) induces the exact sequence

$$0 \to \mathcal{O}(3L_0) \to \Theta_{W|C} \to \mathcal{O}(2L_0) \to 0$$

on C, where L_0 denotes the restriction of $[\Gamma]$ to C. We note that the two maps

$$H^0(W, \Theta_W) \to H^0(W, \mathcal{O}(2\Gamma)), \qquad H^0(W, \mathcal{O}(2\Gamma)) \to H^0(C, \mathcal{O}(2L_0))$$

are surjective. Hence, it follows that

$$\dim H^0(C, \Theta_{W|C}) = h^0(3L_0) + h^0(2L_0) = 8.$$

This proves the first equality $\dim \operatorname{Im} \rho_0 = 13$.

To prove the second equality we use the theory of deformations of holomorphic maps [8], and consider the exact sequence

$$0 \to \Theta_C \to g^* \Theta_{\mathbb{P}^1} \to \mathcal{T}_{C/\mathbb{P}^1} \to 0. \tag{8.2}$$

By [8], I, Theorem 3.1, we have

$$\dim \operatorname{Im} \rho_1 = \dim H^0(\mathcal{T}_{C/\mathbb{P}^1}) - \dim H^0(g^* \Theta_{\mathbb{P}^1}).$$

Using (8.2) and the fact that Supp $\mathcal{T}_{C/\mathbb{P}^1}$ is discrete, we obtain $h^0(\mathcal{T}_{C/\mathbb{P}^1}) = 18$. Then, it immediately follows $\dim \operatorname{Im} \rho_1 = 15$. Q.E.D.

The embedding of $\mathbb{P}^1$ into $\mathbb{P}^2$ as a conic depends on 5 parameters. Adding these parameters to M_1, we obtain a family $\mathcal{M}_2 \to M_2$ and a holomorphic map Ψ: $\mathcal{M}_2 \to \mathbb{P}^2 \times M_2$. Let f: $C \to \mathbb{P}^2$ be the holomorphic map defined by $|2L_0|$. Then $(\mathcal{M}_2, \Psi)$ contains those small deformations of f which factor through conics. We now change the meaning of M_0 and let it denote the set of those points of M_2 which correspond to the triple coverings of $\mathbb{P}^1$ contained in Σ_3.

Let q: $\mathscr{C} \to \mathbf{D}$ be the family of curves of genus 7 which was obtained form the family of surfaces at the beginning of the proof. Then f: $C \to \mathbb{P}^2$ extends to a holomorphic map Φ: $\mathscr{C} \to \mathbb{P}^2 \times \mathbf{D}$ over $\mathbf{D}$ provided $\mathbf{D}$ is sufficiently small.

Lemma 8.5. *There exists a holomorphic map s: $\mathbf{D} \to M_0$ with $s(0) = 0$ such that $\mathscr{C}$ and Φ are respectively induced from $\mathcal{M}_2$ and Ψ by s.*

Proof. We set $\mathbf{D}_\mu = \operatorname{Spec}(\mathbb{C}[t]/t^{\mu+1})$ and prove the existence of holomorphic maps s_μ: $\mathbf{D}_\mu \to M_0$, $\mu = 1, 2, \ldots$ such that $(\mathscr{C}, \Phi)$ modulo $t^{\mu+1}$ is induced from $(\mathcal{M}_2, \Psi)$ by s_μ.

For $\mu = 1$, $\mathscr{C} \times \mathbf{D}_1$ (the product being taken over $\mathbf{D}$) corresponds to an element ρ of $H^1(C, \Theta_C)$. The existence of Φ implies that ρ is in the image of the coboundary map

$$\delta\colon H^0(C, \mathcal{T}_{C/\mathbb{P}^2}) \to H^1(C, \Theta_C)$$

of the exact sequence

$$0 \to \Theta_C \to f^* \Theta_{\mathbb{P}^2} \to \mathcal{T}_{C/\mathbb{P}^2} \to 0. \tag{8.3}$$

In order to calculate $h^i(f^* \Theta_{\mathbb{P}^2})$, we use the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}(2L_0)^3 \to f^* \Theta_{\mathbb{P}^2} \to 0. \tag{8.4}$$

The second arrow induces the map $H^1(\mathcal{O}_C) \to H^1(\mathcal{O}(2L_0))^3$ which is dual to the map $H^0(\mathcal{O}(2L_0))^3 \to H^0(\mathcal{O}(4L_0))$ defined by

$$(a, b, c) \to a\,\varphi_0^2 + b\,\varphi_0\,\varphi_1 + c\,\varphi_1^2,$$

where $\{\varphi_0, \varphi_1\}$ is a basis of $H^0(\mathcal{O}(L_0))$. Since the image of f is a conic, this map has a 5-dimensional image. Therefore, we have

$$\dim \operatorname{Ker}(H^1(\mathcal{O}_C) \to H^1(\mathcal{O}(2L_0))^3) = 2.$$

Hence, from (8.4), we obtain $h^0(f^* \Theta_{\mathbb{P}^2}) = 10$, $h^1(f^* \Theta_{\mathbb{P}^2}) = 4$.

On the other hand, we have the exact sequence

$$0 \to \mathcal{T}_{C/\mathbb{P}^1} \to \mathcal{T}_{C/\mathbb{P}^2} \to g^* \mathcal{N}_{\mathbb{P}^1} \to 0,$$

where $\mathcal{N}_{\mathbb{P}^1}$ denotes the normal sheaf of $\mathbb{P}^1$ in $\mathbb{P}^2$ as a conic. Using $h^0(\mathcal{T}_{C/\mathbb{P}^1})=18$ and $h^0(g^*\mathcal{N}_{\mathbb{P}^1})=h^0(4L_0)=7$, we obtain $h^0(\mathcal{T}_{C/\mathbb{P}^2})=25$ and $h^1(\mathcal{T}_{C/\mathbb{P}^2})=1$. Then, using (8.3), we conclude $\dim \operatorname{Im}\delta=15$. Since $\operatorname{Im}\delta$ contains $\operatorname{Im}\rho_1$, these two spaces coincide (Lemma 8.4). This implies that there exists a holomorphic map $s_1: \mathbf{D}_1 \to M_2$ such that $\mathscr{C}\times \mathbf{D}_1$ is isomorphic to $s_1^*\mathscr{M}_2$.

Here we insert the following lemma.

Lemma 8.6. *Let $s: N \to M_2$ be a holomorphic map with N being possibly non-reduced and $s(0)=0$, and let $\mathscr{C}_1 \to N$ be the family of deformations of C induced by s from $\mathscr{M}_2$. Suppose that $\mathscr{C}$ carries a line bundle $\mathscr{L}_1$ inducing L on C such that $2\mathscr{L}_1$ is the relative canonical bundle of $\mathscr{C}_1 \to N$ and such that any section ψ of L over C extends to a section of $\mathscr{L}_1$ over $\mathscr{C}_1$. Then s factors through M_0.*

Proof. We define a line bundle $\mathscr{L}$ on $\mathscr{M}_2$ by the conditions $\mathscr{L}_{|C}=L$ and that $2\mathscr{L}$ is the canonical bundle of $\mathscr{M}_2$. We note that such $\mathscr{L}$ is uniquely determined. Recall that the obstruction for extending $\psi\in H^0(C,\mathcal{O}(L))$ to a section of $\mathscr{L}$ lies in $H^1(C,\mathcal{O}(L))$. We define M_0^* to be the maximal subspace of M_2 over which any ψ extends (cf. [20, 20a]). Then, by assumption, s factors through M_0^*. We remark that M_0^* is possibly non-reduced.

Sublemma 1. $(M_0^*)_{\mathrm{red}}=M_0$.

Proof. Let C_t denote the curve over $t\in M_0^*$, and let L_t be the restriction to C_t of the line bundle $\mathscr{L}$. Then, for any $t\in M_0^*$, we have $\dim H^0(C_t,\mathcal{O}(L_t))=3$, and $2L_t$ is the canonical bundle of C_t. Hence, by Lemma 8.3, t belongs to M_0. Q.E.D.

Sublemma 2. *M_0^* is reduced.*

Proof. By Lemma 8.4, M_0 is a non-singular submanifold of M_2 of codimension 2. Hence, it suffices to find two linearly independent tangent vectors on M_2 at 0 for which some $\psi\in H^0(C,\mathcal{O}(L))$ does not extend.

Recall that C is a triple covering $g: C\to\mathbb{P}^1$ defined by the linear system $|L_0|$ such that $2L_0=L$. We take a point P_1 at which g ramifies. We assume that g has ramification index 2 at P_1. The case where the index equals 3 will be discussed later. We further take two points P_2 and P_3 with $g(P_1)=g(P_2)\neq g(P_3)$ such that g is unramified at P_2 and P_3. We set $P=P_1+P_2+P_3$, and consider the elements in the kernel of the natural map

$$i_P: H^1(C,\Theta)\to H^1(C,\Theta(P)), \tag{8.5}$$

where $\Theta=\Theta_C$ denotes the tangent sheaf to C. To investigate the kernel, we consider the exact commutative diagram

$$\begin{array}{ccccccc}
0\to\Theta & \longrightarrow & \mathcal{O}(2L_0) & \longrightarrow & \mathcal{T}_{C/\mathbb{P}^1} & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0\to\Theta(P) & \longrightarrow & \mathcal{O}(2L_0+P) & \longrightarrow & \mathcal{T}_{C/\mathbb{P}^1}(P) & \to 0.
\end{array}$$

It induces the exact commutative diagram

$$\begin{array}{ccccc} 0\to H^0(\mathcal{O}(2L_0)) & \longrightarrow & H^0(\mathcal{T}_{C/\mathbb{P}^1}) & \longrightarrow & H^1(\Theta) \\ \downarrow & & \downarrow & & \downarrow \\ 0\to H^0(\mathcal{O}(2L_0+P)) & \longrightarrow & H^0(\mathcal{T}_{C/\mathbb{P}^1}(P)) & \longrightarrow & H^1(\Theta(P)). \end{array} \tag{8.6}$$

First, we note that $h^0(\mathcal{T}_{C/\mathbb{P}^1})=h^0(\mathcal{T}_{C/\mathbb{P}^1}(P))=18$, and, in fact, this is true for any divisor P. Secondly, by the choice of P, we have $h^1(2L_0+P)=1$, and hence, by the Riemann-Roch theorem, $h^0(2L_0+P)=4$. Since we have $h^0(2L_0)=3$, this implies that there exists a non-zero element $\rho\in H^1(\Theta)$ which is the image of an element $\tau\in H^0(\mathcal{T}_{C/\mathbb{P}^1})$ and which is annihilated by i_P. Such ρ is unique up to a constant factor. By specifying $\tau\in H^0(\mathcal{T}_{C/\mathbb{P}^1})$ over ρ, we may consider that ρ corresponds to a tangent vector on M_2.

Let $S_\rho\to \mathbf{D}_1$ be the infinitesimal deformation corresponding to ρ. We shall prove that some $\psi\in H^0(C,\mathcal{O}(L))$ does not extend over S_ρ.

Let $\{U_i\}$ be an open covering of C by sufficiently small discs U_i. We assume that each P_i belongs only to U_i, $1\leqq i\leqq 3$, and that U_j, $j\geqq 4$ intersects at most one of the U_i, $1\leqq i\leqq 3$. We let z_i be a coordinate on U_i, and assume that it is centered at P_i for $1\leqq i\leqq 3$. Then, the above cohomology class ρ is represented by the 1-cocycle $\{\rho_{ij}\}$ on the nerve of the covering $\{U_i\}$ with

$$\rho_{ij}=\frac{\alpha_j}{z_j}\frac{\partial}{\partial z_j}-\frac{\alpha_i}{z_i}\frac{\partial}{\partial z_i},$$

where $\alpha_j=0$ for $j\geqq 4$. Let $\{l_{ij}\}$ *and* $\{\kappa_{ij}\}=\{\partial z_j/\partial z_i\}$ be systems of transition functions for L and K_C, respectively. Then, we have a 0-cochain $\{u_i\}$ of non-vanishing holomorphic functions such that

$$l_{ij}^2=u_j\kappa_{ij}u_i^{-1}\quad \text{on}\quad U_i\cap U_j.$$

Let K_ρ be the canonical bundle of S_ρ. Then, it is defined by the transition functions

$$\kappa_{ij}+\kappa_{ij}(-\operatorname{div}\rho_{ij})\,t,$$

where $\operatorname{div}\rho_{ij}=\partial\beta_{ij}/\partial z_i$ if $\rho_{ij}=\beta_{ij}(\partial/\partial z_i)$ (see [Q], p. 68). Let L_ρ be the extension of L over S_ρ such that $2L_\rho=K_\rho$. Then L_ρ is defined by the transition functions

$$v_{ij}=l_{ij}+\tfrac{1}{2}l_{ij}(-\operatorname{div}\rho_{ij}-\rho_{ij}\cdot\log u_i)\,t.$$

In fact, it can be easily verified that

$$u_j\kappa_{ij}(1-\operatorname{div}\rho_{ij}\,t)=v_{ij}^2u_i(z_j+\beta_{ij}\,t)$$

over $\mathbf{D}_1$, where the last factor means the u_i evaluated at $z_i=z_j+\beta_{ij}t$.

Let ψ be a section of L which vanishes at the P_i's. We represent ψ by a 0-cochain $\{\psi_i\}$ with $\psi_i=l_{ij}\psi_j$. If ψ extends to a section of L_ρ, then we can find a 0-cochain $\{h_i\}$ such that

$$\rho_{ij}\cdot\psi_i+\tfrac{1}{2}\psi_i(-\operatorname{div}\rho_{ij}-\rho_{ij}\cdot\log u_i)=l_{ij}h_j-h_i \tag{8.7}$$

(see [Q], §5, or [13]). We take $i \leqq 3$ and $j \geqq 4$, with $U_i \cap U_j \neq \emptyset$. Then, we have

$$\rho_{ij} = -\frac{\alpha_i}{z_i} \frac{\partial}{\partial z_i}, \qquad \operatorname{div} \rho_{ij} = \frac{\alpha_i}{z_i^2}.$$

Hence, (8.7) reduces to

$$-\frac{\alpha_i}{z_i} \frac{\partial \psi_i}{\partial z_i} - \frac{1}{2} \psi_i \left(\frac{\alpha_i}{z_i^2} - \frac{\alpha_i}{z_i} \frac{\partial}{\partial z_i} \log u_i \right) = l_{ij} h_j - h_i. \tag{8.8}$$

The expression on the left side does not depend on j. So we call it γ_i, and set $\gamma_j = 0$ for $j \geqq 4$. Then (8.8) implies

$$h_i + \gamma_i = l_{ij}(h_j + \gamma_j) \qquad \text{on } U_i \cap U_j.$$

This obviously holds for any pair (i, j). Since ψ vanishes to second order at P_1, it follows that γ_1 is holomorphic. On the other hand, γ_2 and γ_3 may have poles of order 1 at P_2 and P_3, respectively. Hence, the collection $\{h_i + \gamma_i\}$ defines an element of $H^0(C, \mathcal{O}(L + P_2 + P_3))$. But, by the Riemann-Roch theorem, we have $h^0(L) = h^0(L + P_2 + P_3) = 3$. Therefore, γ_2 and γ_3 actually have no poles. This implies $\alpha_2 = \alpha_3 = 0$. Recalling the choice of ρ, we conclude that ρ is annihilated by (8.5) with P being replaced by P_1. But, in the diagram (8.6) with $P = P_1$, we have $h^0(2L_0) = h^0(2L_0 + P_1) = 3$. Therefore, there is no such ρ except 0. This proves that ψ does not extend over S_ρ.

Next suppose that g has ramification index 3 at P_1. In this case, we consider i_P with $P = 2P_1$ and $P = 3P_1$. From the equalities $h^0(2L_0 + 2P_1) = 4$, $h^0(2L_0 + 3P_1) = 5$, it follows that there exists an element $\rho \in H^1(\Theta)$ which is the image of an element $\tau \in H^0(\mathcal{T}_{C/\mathbb{P}^1})$ such that $i_{2P_1}(\rho) \neq 0$ and $i_{3P_1}(\rho) = 0$. We take as ψ a section of L which vanishes exactly to order 3 at P_1. We repeat the above argument. In this case, we have

$$\rho_{1j} = -\left(\frac{\alpha_1}{z_1^3} + \frac{\alpha_2}{z_1^2} + \frac{\alpha_3}{z_1} \right) \frac{\partial}{\partial z_1}.$$

Hence, (8.8) becomes

$$\begin{aligned} &-\left(\frac{\alpha_1}{z_1^3} + \frac{\alpha_2}{z_1^2} + \frac{\alpha_3}{z_1} \right) \frac{\partial \psi_1}{\partial z_1} - \frac{1}{2} \psi_1 \left[\frac{3\alpha_1}{z_1^4} + \frac{2\alpha_2}{z_1^3} + \frac{\alpha_3}{z_1^2} - \left(\frac{\alpha_1}{z_1^3} + \frac{\alpha_2}{z_1^2} + \frac{\alpha_3}{z_1} \right) \frac{\partial}{\partial z_1} \log u_1 \right] \\ &= l_{1j} h_j - h_1. \end{aligned}$$

We call the left side γ_1, and set $\gamma_j = 0$ for $j \neq 1$. Then we obtain an element $\{h_i + \gamma_i\}$ of $H^0(C, \mathcal{O}(L + P_1))$. Since it actually has no pole at P_1, it follows that α_1 vanishes. This contradicts $i_{2P_1}(\rho) \neq 0$.

In any case, we can find a divisor P of degree 3 and $\rho \in H^1(\Theta)$ with $i_P(\rho) = 0$ such that some $\psi \in H^0(C, \mathcal{O}(L))$ does not extend over S_ρ. We choose a similar divisor Q of degree 3 which is disjoint from P, and let ρ' be the corresponding element of $H^1(\Theta)$.

Sublemma 3. *The two elements ρ and ρ' are linearly independent.*

Proof. Using the natural injections $\mathcal{O}\to\mathcal{O}(P)$ and $\mathcal{O}\to\mathcal{O}(Q)$ we obtain the exact sequence

$$0\to\mathcal{O}\to\mathcal{O}(P)\oplus\mathcal{O}(Q)\to\mathcal{O}(P+Q)\to 0.$$

This induces

$$0\to\Theta\to\Theta(P)\oplus\Theta(Q)\to\Theta(P+Q)\to 0.$$

If ρ and ρ' are linearly dependent, we have $i_P(\rho)=i_Q(\rho)=0$. But, since $H^0(\Theta(P+Q))$ vanishes, this is impossible.

This completes the proof of Lemma 8.6.

We continue the proof of Lemma 8.5. By Lemma 8.6, the map $s_1:\mathbf{D}_1\to M_2$ factors through M_0. Then, the two holomorphic maps induced by Φ and $s_1^*\,\Psi$ are both defined by the same line bundle. Hence they differ by an automorphism of $\mathbb{P}^2\times\mathbf{D}_1$. Therefore, by modifying s_1, we may assume these two maps coincide.

Now suppose that $s_\mu:\mathbf{D}_\mu\to M_0$ is constructed in such a manner $(\mathscr{C},\Phi)$ over $\mathbf{D}_\mu$ is induced by s_μ from $(\mathscr{M}_2,\Psi)$. We take an arbitrary extension $s_{\mu+1}:\mathbf{D}_{\mu+1}\to M_2$. Then the two families $\mathscr{C}\times\mathbf{D}_{\mu+1}$ and $s_{\mu+1}^*\,\mathscr{M}_2$ differ by an element of $\delta H^0(C,\mathscr{T}_{C/\mathbb{P}^2})$. Since $\operatorname{Im}\rho_1$ coincides with $\operatorname{Im}\delta$, we can modify $s_{\mu+1}$ so that these two families should be isomorphic over $\mathbf{D}_{\mu+1}$ (cf. proof of the theorem of completeness [18]). By Lemma 8.6, $s_{\mu+1}$ factors through M_0. Then, we can modify $s_{\mu+1}$ so that Φ and $s_{\mu+1}^*\,\Psi$ should coincide over $\mathbf{D}_{\mu+1}$. (This part may be replaced by a general theory [24].)

The existence of a formal solution implies that of an analytic solution (see [2], or proof of Theorem 2.1 in [8], I). Thus, we obtain a holomorphic map $s:\mathbf{D}\to M_0$ such that $q:\mathscr{C}\to\mathbf{D}$ is induced by s from $\mathscr{M}_2$. It follows that any member of $\mathscr{C}$ is of type II. This completes the proof of the assertion (i).

Proof of (ii) *and* (iii). We assume that S is generic, and consider the family $\tilde{\mathscr{S}}$ of deformations of $\tilde{S}$ defined by (4.1), in which we regard α, β, and γ as parameters. We further assume that β does not vanish on Δ_0. Using the notation of §4, we have $\dim H^0(\tilde{S},\mathscr{N}_{\tilde{S}/V^*})=52$, where $\mathscr{N}_{\tilde{S}/V^*}$ denotes the normal sheaf of $\tilde{S}$ in V^*.

Next we use the exact sequences

$$0\to\mathscr{N}_{\tilde{S}/V^*}\to\mathscr{T}_{\tilde{S}/V}\to\mathscr{T}_{V^*/V|\tilde{S}}\to 0,$$
$$0\to\mathscr{N}_{\mathscr{E}/V^*}\to\mu^*\,\mathscr{N}_{A/V}\to\mathscr{T}_{V^*/V}\to 0.$$

The second one results from the exact commutative diagram

$$\begin{array}{ccccccc}
0\to & \Theta_{\mathscr{E}} & \longrightarrow & \Theta_{V^*|\mathscr{E}} & \longrightarrow & \mathscr{N}_{\mathscr{E}/V^*} & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & \mu^*\,\Theta_A & \longrightarrow & \mu^*\,\Theta_{V|\mathscr{E}} & \longrightarrow & \mu^*\,\mathscr{N}_{A/V} & \to 0\\
 & & & \downarrow & & \downarrow & \\
 & & & \mathscr{T}_{V^*/V}\otimes\mathcal{O}_{\mathscr{E}} & = & \mathscr{T}_{V^*/V}\otimes\mathcal{O}_{\mathscr{E}} & \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

Here, using the standard coordinates on $\mathscr{E}$, we can easily verify $\mathscr{T}_{V^*/V}\otimes\mathcal{O}_{\mathscr{E}}=\mathscr{T}_{V^*/V}$ and that $\mathscr{N}_{\mathscr{E}/V^*}\to\mu^*\mathscr{N}_{A/V}$ is injective.

From these exact sequences we infer that $h^0(\mathscr{T}_{V^*/V|S})=0$ and $h^0(\mathscr{T}_{\tilde{S}/V})=52$. We let $\tilde{h}:\tilde{S}\to V$ be the natural map and calculate $h^0(\tilde{h}^*\Theta_V)$. This sheaf fits in the exact sequence

$$0\to\mathcal{O}(h^*(2\Delta_0+3\Gamma))\to\tilde{h}^*\Theta_V\to h^*\Theta_{\Sigma_2}\to 0,$$

where h is the projection $\tilde{S}\to\Sigma_2$ (cf. (8.1)). We also use the exact sequence

$$0\to\mathcal{O}(h^*(2\Delta_0+2\Gamma))\to h^*\Theta_{\Sigma_2}\to\mathcal{O}(h^*(2\Gamma))\to 0.$$

From the second one, we get $h^0(h^*\Theta_{\Sigma_2})=7$. To calculate $h^i(h^*(2\Delta_0+3\Gamma))$, we use the equality (4.2). Since $\tilde{D}+\pi^*G+E$ is a reduced connected curve, we have $h^1(h^*(2\Delta_0+3\Gamma))=0$ (see [3], Theorem A). Hence, by the Riemann-Roch theorem, we get $h^0(h^*(2\Delta_0+3\Gamma))=7$. Using the first exact sequence, we obtain $h^0(\tilde{h}^*\Theta_V)=14$. Therefore, from the exact sequence

$$0\to\Theta_{\tilde{S}}\to\tilde{h}^*\Theta_V\to\mathscr{T}_{\tilde{S}/V}\to 0,$$

it follows that our family $\tilde{\mathscr{S}}$ contains 38-dimensional infinitesimal deformations. Let $\mathscr{S}$ be the family of the minimal models of the members of $\tilde{\mathscr{S}}$ (see [8], III). Then $\mathscr{S}$ contains the same number of infinitesimal deformations (cf. [Q], Lemma 26)[2].

We now want to prove $h^1(\Theta_S)=39$. First recall that S contains a rational curve G with $G^2=-2$ and that the natural map $\zeta_*:H^1(S,\Theta_S)\to H^1(G,\mathscr{N}_{G/S})\cong\mathbb{C}$ is surjective (see [5]). Since G extends to a family of curves on $\mathscr{S}$, the infinitesimal deformations in $\mathscr{S}$ are annihilated by ζ_*. Conversely, suppose that $\rho\in H^1(S,\Theta_S)$ is annihilated by ζ_*. Then the divisor G extends over the corresponding deformation $S_\rho\to\mathbf{D}_1$. Since the canonical bundle extends automatically, it follows that the line bundle $[2D]$, and hence $[D]$ extend over S_ρ. Moreover, we can easily prove $h^1(D)=0$, and $h^1(\tilde{K}+\tilde{D}+\pi^*G+E)=0$ as remarked above. Therefore, the construction in Theorem 4.1 can be carried out over S_ρ. This implies that ρ is contained among the infinitesimal deformations in $\mathscr{S}$. Thus we conclude $h^1(\Theta_S)=39$ and $h^2(\Theta_S)=1$.

To complete the proof of (ii) and (iii), we take the Kuranishi family $\mathscr{S}_0\to M_0$ of deformations of S. Then M_0 is defined at the origin of $\mathbb{C}^{39}$ by a single equation (see [19]). It contains a non-singular component M_1 of dimension 38 which corresponds to our family $\mathscr{S}$. We shall prove $(M_0)_{\text{red}}=M_1$. In fact, let $\mathscr{S}'\to M'$ be any family of deformations of S with reduced base space M'. Then, by virtue of (i), any member of $\mathscr{S}'$ is of type II.

Here we use the following lemma.

Lemma 8.7. *Suppose that the canonical image of each member of $\mathscr{S}'\to M'$ is a quadric. Then the images form a flat family.*

Proof. We take a basis $\{\varphi_0,\varphi_1,\varphi_2,\varphi_3\}$ of $H^0(S,\mathcal{O}(K))$ and assume that $\varphi_0^2=Q(\varphi_1,\varphi_2,\varphi_3)$, where Q is a quadratic form. We extend each φ_i to a section $\tilde{\varphi}_i$ of the canonical bundle of $\mathscr{S}'$. Let $\{\psi_0,\psi_1,\ldots,\psi_{10}\}$ be a basis of $H^0(S,\mathcal{O}(2K))$ and suppose that these sections are already extended to those over $\mathscr{S}'$. We may further

[2] See Part IV, §5 for a detailed proof

assume that the nine of them, for instance, $\psi_2, \psi_3, \ldots, \psi_{10}$ are quadratic forms in $\tilde{\varphi}_0, \ldots, \tilde{\varphi}_3$. Then we can write

$$\tilde{\varphi}_0^2 = \sum_{i=0}^{10} \alpha_i(t)\, \psi_i,$$

where the $\alpha_i(t)$ are holomorphic functions on M'. By assumption, the rank of the space generated by the products $\tilde{\varphi}_i\, \tilde{\varphi}_j$ is 9 at each point t of M'. Therefore, we have $\alpha_0(t) = \alpha_1(t) = 0$. Hence $\tilde{\varphi}_0, \ldots, \tilde{\varphi}_3$ satisfy a quadratic relation which depends holomorphically on t. Q.E.D.

Changing the basis of $H^0(S, \mathcal{O}(K))$, if necessary, we may assume $\tilde{\varphi}_0^2 = \tilde{\varphi}_1\, \tilde{\varphi}_2$ on $\mathcal{S}'$. Then, as is proved in §4, the locus $\tilde{\varphi}_0 = \tilde{\varphi}_1 = \tilde{\varphi}_2 = 0$ has two connected components, because we assumed that β does not vanish on Δ_0. One of them induces G on S, and the other is the locus of the base point of $|D|$. This implies that G extends to a family of divisors on $\mathcal{S}'$. Therefore, by the same argument as before, we can prove that $\mathcal{S}'$ is induced from $\mathcal{S}$. This completes the proof of the equality $(M_0)_{\mathrm{red}} = M_1$. Since the equation of M_0 is at least of order 2, it follows that M_0 is non-reduced (see [19]).

Remark. Enriques' proof of (i) is as follows (see [6], p. 273). Let $\{C_t\}$ be a 1-parameter family of plane sextic curves with three double points on a line for $t \neq 0$, and suppose that C_t tends to a triple conic $3Q$ as $t \to 0$. Then the line passing through the three double points tends to a tangent line to Q. Hence these three points come together to a point P. If t is sufficiently close to 0, C_t is close to a union of three conics in a neighborhood of P. If two of them intersect like as in Figure 1, then the third component must be reducible, which is a contradiction.

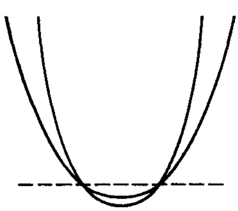


Fig. 1

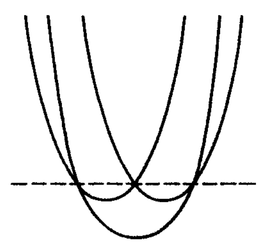


Fig. 2

Enriques did not mention the possibility of Figure 2. Some examples indicate that, in this case, the limit of C_t is a curve of arithmetic genus 7 with one double point. But the author could not produce a rigorous proof of (i) on this line.

§9. Deformations of Surfaces of Type III

We first assume that S is of type IIIa. Let $\pi: \tilde{S} \to S$ be the blowing up of the two base points of $|K|$. Then $|K|$ defines a holomorphic map $f: \tilde{S} \to \Sigma_0$ of degree 2. It follows that $|K|$ is composed of two pencils $|C|$ and $|D|$ of genus 3 with $C^2 = D^2 = 1$.

Lemma 9.1. *Let $g: S \to \mathbb{P}^1$ be the rational map associated with $|C|$. Then, for any family $p: \mathcal{S} \to M$ of deformations of S, g extends to a family $\Psi: \mathcal{S} \to \mathbb{P}^1 \times M$ of deformations of g, provided M is sufficiently small. More precisely, the line bundle $[C]$ and a basis $\{g_0, g_1\}$ of $H^0(S, \mathcal{O}([C]))$ extend to a line bundle on $\mathcal{S}$ and to a pair $\{\Psi_0, \Psi_1\}$ of its sections*

Proof. We apply [13], Theorem 5. Recall that the sheaf $\mathcal{K}_g$ is defined by the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}([C])^2 \to \mathcal{K}_g \to 0,$$

where the first map is defined by $\alpha \to (\alpha g_0, \alpha g_1)$. By the theorem cited above, it suffices to prove $H^1(S, \mathcal{K}_g) = 0$, or equivalently, $H^1(\tilde{S}, \pi^* \mathcal{K}_g) = 0$. For this purpose, we write $\pi^* C = \tilde{C} + E_1$ on $\tilde{S}$, where $\tilde{C}$ is the proper transform of C and E_1 is an exceptional curve. From the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathcal{O} & \longrightarrow & \mathcal{O}(\tilde{C}+E_1)^2 & \longrightarrow & \pi^* \mathcal{K}_g & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{O}(E_1) & \longrightarrow & \mathcal{O}(\tilde{C}+E_1)^2 & \longrightarrow & \mathcal{O}(2\tilde{C}+E_1) & \to 0, \end{array}$$

we obtain the exact sequence

$$0 \to \mathcal{O}_{E_1}(-1) \to \pi^* \mathcal{K}_g \to \mathcal{O}(2\tilde{C}+E_1) \to 0.$$

Hence, it suffices to prove $h^1(2\tilde{C}+E_1) = 0$. To prove this, we use the exact sequence

$$0 \to \mathcal{O}(\tilde{C}+E_1) \to \mathcal{O}(2\tilde{C}+E_1) \to \mathcal{O}_{\tilde{C}}(E_1) \to 0.$$

From $\tilde{C}E_1 = 1$, it follows that $\dim H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(E_1)) = 1$. On the other hand, we have $h^0(\tilde{C}+E_1) = 2$. Therefore, $h^0(2\tilde{C}+E_1) \leqq 3$. Since the converse inequality is obvious, we have $h^0(2\tilde{C}+E_1) = 3$. Then by using the Riemann-Roch theorem, we obtain $h^1(2\tilde{C}+E_1) = 0$. Q.E.D.

Theorem 9.1. *Let S be a surface of type* III a. *Then any sufficiently small deformation of S is of type* III a.

Proof. Let $p: \mathcal{S} \to M$ be a family of deformations of S. Then, by Lemma 9.1, the rational map $g: S \to \mathbb{P}^1$ extends to a rational map $\Psi: \mathcal{S} \to \mathbb{P}^1 \times M$. Similarly, the rational map $h: S \to \mathbb{P}^1$ defined by $|D|$ extends. Therefore, we have two pencils $|C_t|$ and $|D_t|$ on each $S_t = p^{-1}(t)$. If we take the bases $\{g_{1t}, g_{2t}\}$ of $H^0(S_t, \mathcal{O}([C_t]))$ and $\{h_{1t}, h_{2t}\}$ of $H^0(S_t, \mathcal{O}([D_t]))$, then the products $g_{it} h_{jt}, i, j = 1, 2$, form a basis of $H^0(S_t, \mathcal{O}(K_t))$, where K_t denotes the canonical bundle of S_t. Hence, $|K_t|$ is composed of two pencils of genus 3. This proves that S_t is of type III a. Q.E.D.

Theorem 9.2. *Let S be a generic surface of type* III a. *Then the number of moduli $m(S)$ of S is defined and is equal to* $\dim H^1(S, \Theta_S) = 39$.

Proof. Let $\tilde{W} \to W = \Sigma_0$ be the blowing up of the three points as in §5. Then we can prove

$$h^0(\mathcal{T}_{\tilde{S}/\tilde{W}}) = 41, \quad h^1(\mathcal{T}_{\tilde{S}/\tilde{W}}) = 2, \quad \text{and} \quad h^0(h^* \mathcal{T}_{\tilde{W}/W}) = 6,$$

where $h: \tilde{S} \to \tilde{W}$ is the induced map (cf. [Q], §4). Next we consider the exact sequence

$$0 \to \mathcal{T}_{\tilde{S}/\tilde{W}} \to \mathcal{T}_{\tilde{S}/W} \to h^* \mathcal{T}_{\tilde{W}/W} \to 0.$$

In the same way as in [Q], Lemma 15, we can prove that the coboundary map

$$H^0(\tilde{S}, h^* \mathcal{T}_{\tilde{W}/W}) \to H^1(\tilde{S}, \mathcal{T}_{\tilde{S}/\tilde{W}})$$

is surjective. Therefore, we have $h^0(\mathcal{T}_{\tilde{S}/W}) = 45$ and $h^1(\mathcal{T}_{\tilde{S}/W}) = 0$.
Finally, we use the exact sequence

$$0 \to \Theta_{\tilde{S}} \to f^* \Theta_W \to \mathcal{T}_{\tilde{S}/W} \to 0.$$

We note that $h^0(f^* \Theta_W) = 6$, $h^1(f^* \Theta_W) = 4$, and $h^2(f^* \Theta_W) = 0$. Hence we obtain $h^1(\Theta_{\tilde{S}}) = 43$ and $h^1(\Theta_S) = 39$.

We take a complete family $\tilde{\mathcal{S}} \to \Sigma_0 \times M$ of deformations of $\tilde{S} \to \Sigma_0$ such that the characteristic map $\tau: T_0(M) \to H^0(\tilde{S}, \mathcal{T}_{\tilde{S}/W})$ is surjective (see [8], I, Theorem 3.1). Then $\tilde{\mathcal{S}}$ contains 39-dimensional infinitesimal deformations of $\tilde{S}$. We let $\mathcal{S}$ be the family obtained from $\tilde{\mathcal{S}}$ by contracting exceptional curves (see [8], III). Then, using [Q], Lemma 26, it is not difficult to prove that $\mathcal{S}$ contains 39-dimensional infinitesimal deformations[3]. Hence the map $\rho: T_0(M) \to H^1(S, \Theta_S)$ is surjective. Since $H^0(S, \Theta_S)$ vanishes, there exists a 39-dimensional family of deformations of S, which is complete and effectively parametrized. Q.E.D.

As to the deformations of surfaces of type IIIb, we only prove the following

Theorem 9.3. *A generic surface of type* IIIb *is a specialization of surfaces of type* IIIa.

Proof. Let S be a generic surface of type IIIb. Then, it is birationally equivalent to a double covering of Σ_2 whose branch locus B has a quadruple point x and a 2-fold triple point y on a fibre Γ (see Theorem 5.2). Let x_t be a point depending holomorphically on a parameter t with $x_0 = x$, and consider the curves in $|8\Delta_0 + 14\Gamma|$ on Σ_2 having the above singularities at x_t and y. Then, by Remark 3 in §5, and by Grauert's theorem (see [7], Satz 5), we can extend B to a family $\{B_t\}$ of such curves. Let $\{\tilde{S}_t\}$ be the family of the canonical resolutions of the double coverings of Σ_2 with branch loci B_t. Contracting the exceptional curves on $\tilde{S}_t$, we obtain a family $\{S_t\}$ of deformations of S (see [8], III). Suppose that the x_t, $t \neq 0$, are not on the same fibre as y. Then, by Remark 3, in §5, the S_t, $t \neq 0$, are of type IIIa. Q.E.D.

With some more efforts, we can prove that any surface of type IIIb is not a specialization of surfaces of type I, IV, nor V. But we do not go into details. On the other hand, we do not know whether a surface of type IIIb is a specialization of surfaces of type II. This is not the case for the generic surfaces of type IIIb, as follows from the fact that the generic surfaces of type IIIb depend on 38 parameters which is equal to the number of moduli of the generic surfaces of type II.

[3] See Part IV, §5 for a detailed proof

§10. Deformations of Surfaces of Type IV

We begin with the following two theorems.

Theorem 10.1. *Let S be a surface of type* IVa. *Then any sufficiently small deformation of S is of type* IVa.

Proof. Let $|D|$ be a pencil of curves of genus 2 on S. Then we have $h^1(2D)=0$. This implies that the pencil $|D|$ is stable under small deformations (cf. [Q], proof of Proposition 1). Hence, if $\{S_t\}$ is a family of deformations of $S=S_0$, then each S_t is of type IV or V. By Lemma 8.7, the canonical images form a flat family. Hence they are non-singular quadrics. This proves that S_t is of type IVa. Q.E.D.

Theorem 10.2. *A generic surface of type* IVa $-$ 2 *is a specialization of surfaces of type* IVa $-$ 1.

Proof. Let S be a generic surface of type IVa $-$ 2. For simplicity, we assume that S has one singular fibre of type (III_1). Then the corresponding branch locus B has two 2-fold triple points x, y on a fibre Γ. As in the proof of Theorem 9.3, we consider x_t with $x_0=x$. By the same argument as there, B extends to a family $\{B_t\}$ of curves having 2-fold triple points at x_t and y. This gives rise to a family of deformations of S. If x_t and y are not on a single fibre, then the corresponding member S_t is of type IVa $-$ 1. This can be seen by applying the elementary transformations at x_t and y. The proof in the case where S has a singular fibre of type (IV_1) is similar. Q.E.D.

Next we study deformations of surfaces of type IVb.

Theorem 10.3. *Let S be a generic surface of type* IVb $-$ 1, *and let $p:\mathscr{S}\to M$ be the Kuranishi family of deformations of S* (see [19]). *Then M has two 38-dimensional non-singular components M_0 and M_1, which intersect transversally along a 37-dimensional manifold N. The points of M_0-N and M_1-N respectively correspond to surfaces of type* Ia *and* IVa $-$ 1, *and the points of N to surfaces of type* IVb $-$ 1.

Proof is similar to [Q], §§4–6. So we shall give only an outline. Let $g: S\to \mathbb{P}^1$ be the unique pencil of genus 2 on S. The following two propositions can be proved in the same way as in [Q], §4.

Proposition 10.1. *Let $p:\mathscr{S}\to M$ be a family of deformations of S which is of type* IVb $-$ 1. *Suppose that each fibre $S_t=p^{-1}(t)$ is of type* III, IV, *or* V. *Then g extends to a holomorphic map $\Psi:\mathscr{S}\to\mathbb{P}^1\times M$ over M (shrinking M if necessary).*

Proposition 10.2. *Let S be a generic surface of type* IVb $-$ 1. *Then, there exists a family $p_1:\mathscr{S}_1\to M_1$ of deformations of $S=p^{-1}(0)$ with $0\in M_1$ such that*

i) *M_1 is a 38-dimensional manifold,*

ii) *the infinitesimal deformation map $\rho: T_0(M_1)\to H^1(S,\Theta_S)$ induces an isomorphism of $T_0(M_1)$ onto the kernel of the canonical homomorphism*

$G: H^1(S,\Theta_S)\to H^1(S, g^*\Theta_{\mathbb{P}^1})$,

iii) *each fibre S_t is of type* IVa $-$ 1 *or* IVb $-$ 1, *and the generic members are of type* IVa $-$ 1.

The calculation in [Q], §5 can be carried out with the following modifications. First, $|K|$ is of the form $|2D+\bar{G}|$ with $\bar{G}^2=-2$ and $K\bar{G}=2$. Hence $\bar{G}$ is an elliptic curve. With minor changes in Lemmas 21 and 22, we obtain $h^1(\Theta_S)=39$ and $h^2(\Theta_S)=1$. The analogue of Lemma 24 can be proved in a similar way. That is, the canonical homomorphism

$$\zeta_*: H^1(S,\Theta_S)\to H^1(\bar{G},\mathcal{N}_{\bar{G}})$$

is surjective, where $\mathcal{N}_{\bar{G}}$ denotes the normal sheaf of $\bar{G}$ in S. By the same way as in [Q], pp. 71–72, we define a linear map

$$\gamma: H^1(S,\Theta_S)\to H^0(\bar{G},\mathcal{O}(K_{|\bar{G}})),$$

which has the following property.

Lemma 10.1. *Let b_1, $b_2\in\bar{G}$ be the base points of $|K|$. Then*
1) *$G\rho=0$ if and only if $\gamma(\rho)$ vanishes at b_1 and b_2,*
2) *$\zeta_*\rho=0$ if and only if $\gamma(\rho)$ vanishes identically on $\bar{G}$.*

We now take a basis $\{\rho_1,\rho_2,\dots,\rho_{39}\}$ of $H^1(S,\Theta_S)$ such that

$$G\rho_1\neq 0,\quad G\rho_2=0,\quad \zeta_*\rho_2\neq 0,\quad \zeta_*\rho_\lambda=0 \qquad (\lambda\geqq 3),$$

and let $(t_1,t_2,\dots,t_{39})$ be the corresponding coordinates on $\mathbb{C}^{39}$.

Lemma 10.2. *We fix a hermitian metric on S and let $\varphi(t)$ be a $(0,1)$-form with coefficients in Θ_S which defines the Kuranishi family. Then we have*

$$\mathbf{H}[\varphi(t),\varphi(t)]=\sum_{\lambda=1}^{39} a_\lambda t_1 t_\lambda+(\text{higher terms}),$$

with $a_2\neq 0$, where $\mathbf{H}$ denotes the projection to the harmonic part with respect to the given hermitian metric on S.

Proof. Lemmas 30 and 31 can be applied without change. Hence, it suffices to prove that the product $\gamma(\rho_1)\gamma(\rho_2)\in H^0(\bar{G},\mathcal{O}(2K_{|\bar{G}}))$ is not a restriction of an element of $H^0(S,\mathcal{O}(2K))$. For this purpose, let $f:\tilde{S}\to\Sigma_2$ be the map of degree 2 defined in §6. Then the branch locus B has four quadruple points x_i, $1\leqq i\leqq 4$. From the construction in §6, we infer that $|2\pi^*K|$ contains the linear system

$$\left|2\Delta_0+6\Gamma-\sum_{i=1}^{4}x_i\right|. \tag{10.1}$$

Comparing the dimension of these two linear systems, we see that they coincide. Let Δ_1 and $\Delta_2\in|\Delta_0+2\Gamma|$ be irreducible curves through x_1,x_2, and x_3,x_4, respectively. Then, $|2\Gamma|+\Delta_1+\Delta_2$ is a subsystem of (10.1), and induces a 2-dimensional linear system on $\bar{G}$. On the other hand, since $h^0(K+2D)=8$, the image of the restriction map

$$\text{res}: H^0(S,\mathcal{O}(2K))\to H^0(\bar{G},\mathcal{O}(2K_{|\bar{G}}))$$

is 3-dimensional. Hence it coincides with the image of the pull-back

$$H^0(\mathbb{P}^1, \mathcal{O}(2)) \to H^0(\bar{G}, \mathcal{O}([2b_1+2b_2]))$$

by the restriction $\bar{f}: \bar{G} \to \mathbb{P}^1$ of f. We note that $\bar{f}$ is a double covering which is ramified at 4 points including b_1 and b_2. Therefore, if $\gamma(\rho_1)\,\gamma(\rho_2)$ is in the image of res, its divisor is $2b_1+2b_2$, for, it vanishes at b_1+b_2. But this is impossible by Lemma 10.1.

We shall prove that S is not a specialization of surfaces of type II in Theorem 10.4 below, and the remaining assertions of Theorem 10.3 can be proved in the same way as in [Q], Theorem 3. Q.E.D.

In order to complete the proof of the hierarchy in Introduction, we need to prove the following theorems.

Theorem 10.4. *Let S be a surface of type* IV. *Then, any sufficiently small deformation of S is either of type* I *or* IV.

Theorem 10.5. *Generic surfaces of type* IVb -1 *and* IVb -2 *are respectively the specializations of surfaces of type* IVa -1 *and* IVa -2.

We first prepare with the following lemma.

Lemma 10.3. *Let S be a surface of type* IVb, *and let $|D|$ be a pencil of genus* 2 *on S. Then there exists a unique divisor $G \in |K-2D|$, and G is one of the following.*

If S is of type IVb -1, *then*

(i) *G is a reduced irreducible curve, which is an elliptic curve or a rational curve with one node, and $G^2=-2$, or*

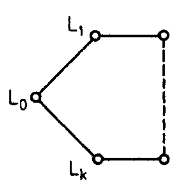
(ii) *G is a sum $\sum_{i=0}^{k} L_k$ of non-singular rational curves L_i, with $k \geqq 1$, $L_0^2=-4$, $DL_0=2$, and $L_i^2=-2$ for $1 \leqq i \leqq k$.*

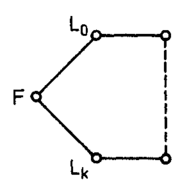
If S is of type IVb -2, *then $|K|$ has a unique fixed component F which is a non-singular rational curve with $F^2=-2$. In this case,*

(iii) *$G=F+L_0$ where L_0 is a non-singular rational curve with $L_0^2=-4$, and $FL_0=DL_0=2$, or*

(iv) *$G=F+\sum_{i=0}^{k} L_i$ with $k \geqq 1$, where the L_i are non-singular rational curves with $L_0^2=L_k^2=-3$, $DL_0=DL_k=1$, and $L_i^2=-2$, $DL_i=0$ for $1 \leqq i \leqq k-1$.*

In the cases (ii) *and* (iv), *the curves form the configurations whose dual graphs are as follows:*

(ii) 

(iv) 

These can be proved by examining the possible singularities of the branch locus given in §6.

Proof of Theorem 10.4. Let $p: \mathcal{S} \to \mathbf{D}$ be a 1-parameter family of deformations of $S = p^{-1}(0)$, where $\mathbf{D}$ is a small disc centered at $0 \in \mathbb{C}$. In view of Theorem 10.1, we may assume that S is of type IVb. We suppose that the general fibres $S_t = p^{-1}(t)$ are not of type I. Then, the canonical images of S_t form a flat family of quadrics in $\mathbb{P}^3$ (see Lemma 8.7). Therefore, we can find a basis $\{\varphi_0(t), \ldots, \varphi_3(t)\}$ of $H^0(S_t, \mathcal{O}(K_t))$[4] such that

$$\varphi_2(t)^2 = \varphi_0(t)\,\varphi_1(t) + \alpha(t)\,\varphi_3(t)^2,$$

where $\alpha(t)$ is a holomorphic function in t with $\alpha(0)=0$.

First suppose that $\alpha(t)$ is identically 0. In this case, $g = \varphi_0/\varphi_2 = \varphi_2/\varphi_1$ defines a meromorphic function on $\mathcal{S}$. We write the divisor (g) in the form $\mathcal{D} - \mathcal{D}_1$, where $\mathcal{D}$ and $\mathcal{D}_1$ are effective divisors without common component on $\mathcal{S}$. Furthermore, there exists an effective divisor $\mathcal{G}$ on $\mathcal{S}$ such that

$$(\varphi_0) = 2\mathcal{D} + \mathcal{G}, \quad (\varphi_1) = 2\mathcal{D}_1 + \mathcal{G}, \quad (\varphi_2) = \mathcal{D} + \mathcal{D}_1 + \mathcal{G} \tag{10.2}$$

(cf. [Q], p. 46). The restrictions $\mathcal{D}_{|S}$ and $\mathcal{D}_{1|S}$ may have some common components. So we express them as

$$\mathcal{D}_{|S} = D + C, \quad \mathcal{D}_{1|S} = D_1 + C,$$

where D and D_1 have no common component. By construction, $|D|$ is the unique pencil of genus 2 on S. Moreover, it follows that $2C + \mathcal{G}_{|S}$ coincides with the unique divisor G in $|K - 2D|$. But, by Lemma 10.3, G has no multiple component. Hence $|D|$ extends to pencils on S_t. Since we have assumed $\alpha(t) \equiv 0$, this implies that each S_t is of type IVb or V. To exclude the second possibility, we consider the family $\{G_t\} = \{\mathcal{G}_{|S_t}\}$ of divisors on S_t. As remarked above, G_0 has no multiple component. Hence neither G_t has one, provided $|t|$ is sufficiently small. But, if S_t is of type V, then Δ_0 is in the branch locus. Hence, G_t contains a double component. This proves that S_t is not of type V, and settles the case when $\alpha(t) \equiv 0$.

Next suppose that $\alpha(t)$ is not identically 0. In this case, taking a square root of t, we may assume that $\alpha(t)$ is of the form $\beta(t)^2$. Then we have two meromorphic functions

$$g = \frac{\varphi_0}{\varphi_2 + \beta\varphi_3} = \frac{\varphi_2 - \beta\varphi_3}{\varphi_1}, \qquad h = \frac{\varphi_0}{\varphi_2 - \beta\varphi_3} = \frac{\varphi_2 + \beta\varphi_3}{\varphi_1} \tag{10.3}$$

on $\mathcal{S}$. We write $(g) = \mathcal{D} - \mathcal{D}_1$ and $(h) = \mathcal{D}' - \mathcal{D}_1'$ as above. Then we can write

$$\begin{aligned} &(\varphi_0) = \mathcal{D} + \mathcal{G}_1, && (\varphi_2 + \beta\varphi_3) = \mathcal{D}_1 + \mathcal{G}_1, \\ &(\varphi_2 - \beta\varphi_3) = \mathcal{D} + \mathcal{G}_2, && (\varphi_1) = \mathcal{D}_1 + \mathcal{G}_2. \end{aligned}$$

By (10.3), we have

$$(g) + (h) = (\varphi_0) - (\varphi_1) = (\mathcal{D} - \mathcal{D}_1) + (\mathcal{G}_1 - \mathcal{G}_2).$$

Hence, we have $\mathcal{G}_1 - \mathcal{G}_2 = \mathcal{D}' - \mathcal{D}_1'$, and we can write

$$\mathcal{G}_1 = \mathcal{D}' + \mathcal{G}_0, \quad \mathcal{G}_2 = \mathcal{D}_1' + \mathcal{G}_0$$

[4] K_t denotes the canonical bundle of S_t

with an effective divisor $\mathscr{G}_0$. Finally, we get

$$\begin{aligned} &(\varphi_0)=\mathscr{D}+\mathscr{D}'+\mathscr{G}_0, && (\varphi_2+\beta\varphi_3)=\mathscr{D}_1+\mathscr{D}'+\mathscr{G}_0, \\ &(\varphi_2-\beta\varphi_3)=\mathscr{D}+\mathscr{D}_1'+\mathscr{G}_0, && (\varphi_1)=\mathscr{D}_1+\mathscr{D}_1'+\mathscr{G}_0. \end{aligned} \tag{10.4}$$

We write as before

$$\begin{aligned} &\mathscr{D}_{|S}=D+C, && \mathscr{D}_{1|S}=D_1+C, \\ &\mathscr{D}'_{|S}=D'+C', && \mathscr{D}'_{1|S}=D'_1+C'. \end{aligned}$$

Then $|D|$ and $|D'|$ both coincide with the unique pencil of genus 2 on S. Hence, it follows that $C+C'+\mathscr{G}_{0|S}$ coincides with $G\in|K-2D|$.

We now suppose that a general member S_t is not of type I nor IV. Then, since we have assumed $\alpha(t)\not\equiv 0$, S_t is necessarily of type IIIa. This implies that $|D|+C$ and $|D|+C'$ both extend to pencils of genus 3 with one base point on S_t. Therefore, we have

$$\begin{aligned} &K(D+C)=K(D+C')=3, \\ &(D+C)^2=(D+C')^2=1. \end{aligned}$$

The first line implies $KC=KC'=1$. By Lemma 10.3, this is possible only in the case (iv). But then we have $DC=1$ and $C^2\leqq -3$, which contradicts the above second line. Q.E.D.

Proof of Theorem 10.5. The assertions can be proved by the same method as in the proof of [Q], Proposition 2.

Remark. If we apply the construction of Theorem 10.2 to a surface of type IVb-2, we obtain a family of surfaces of type IVa-1 which specialize to a surface of type IVb-2.

§11. Deformations of Surfaces of Type V

In this section we shall prove the following three theorems concerning the deformations of surfaces of type V.

Theorem 11.1. *Surfaces of type* V-1 *and* V-2 *are specializations of surfaces of type* IIIa *and* IIIb, *respectively.*

Theorem 11.2. *A surface of type* V-1 *is not a specialization of surfaces of type* IV.

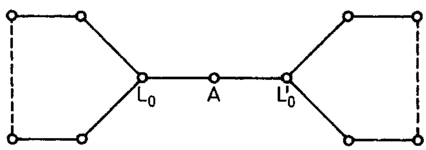
Theorem 11.3. *A surface of type* V-2 *is a specialization of surfaces of type* V-1.

Proof of Theorem 11.1. We recall that a surface of type III is birationally equivalent to a double covering of $\mathbb{P}^2$ with branch locus of degree 10 which has two 2-fold triple points or a 4-fold triple point (see §5, Remarks 2 and 4). On the other hand, a surface of type V is birationally equivalent to a double covering of Σ_4 with branch locus B in $|6\Delta_0+20\Gamma|$ which has the same singularities as above. This can be seen as follows. First, it is birationally equivalent to a double covering of Σ_2 as in §6. Then, we apply the elementary transformations at the two singular points on Δ_0 of the branch locus.

Using these facts, we can construct, by the same method as in Lemma 4.4 in Part I, a family in which surfaces of type IIIa (or IIIb) specialize to a surface of type V−1 (or V−2). Q.E.D.

Before starting the proof of Theorem 11.2, we prepare with the following lemma.

Lemma 11.1. *Let S be a surface of type* V−1, $|D|$ *the unique pencil of genus* 2 *on S, and let G be the unique divisor in* $|K-2D|$. *Then, G contains a unique double component A, which is a non-singular rational curve with* $A^2=-2$, *and two components L_0 and L'_0 which satisfy* $KL_0=KL'_0=1$. *Other components, if any, are non-singular rational curves with self-intersection number* −2. *They form the configuration whose dual graph is as follows.*



Moreover, A is the unique component of G which intersects D.

This lemma, like Lemma 10.3, can be proved by examining the singularities of the branch locus. Here A is the unique component over Δ_0, and, in the generic case, L_0 and L'_0 are elliptic curves with $L_0^2=L_0'^2=-1$, and there are no other components. The above notation for the curves on S will be used in the following.

Proof of Theorem 11.2. Let $p\colon \mathscr{S}\to\mathbf{D}$ be a 1-parameter family of deformations of $S=p^{-1}(0)$ of type V−1, and suppose that a general member $S_t=p^{-1}(t)$ is of type IV. Since $|K|$ has no fixed component, S_t is of type IVa−1 or IVb−1. Hence, the canonical system $|K_t|$ of S_t has two base points. Therefore, we can consider the family $\tilde{\mathscr{S}}$ obtained from $\mathscr{S}$ by blowing up the locus of base points, and we have a holomorphic map $\Phi\colon \tilde{\mathscr{S}}\to\mathbb{P}^3\times\mathbf{D}$, which induces the canonical map on each fibre. By Lemma 8.7, we can find a basis $\{\varphi_0(t), \ldots, \varphi_3(t)\}$ of $H^0(S_t, \mathcal{O}(K_t))$ such that

$$\varphi_2(t)^2=\varphi_0(t)\,\varphi_1(t)+\alpha(t)\,\varphi_3(t)^2, \qquad \alpha(0)=0.$$

First suppose that $\alpha(t)\equiv 0$. Then, a pencil of the form $|D|+C$ extends to a pencil $|D_t|$ on S_t (cf. proof of Theorem 10.4). If $C\neq 0$, C is the unique double component A of G in Lemma 11.1. Since S_t is of type IVb−1, $|D_t|$ is a pencil of genus 2. Hence, in (10.2), the divisor $\mathscr{G}_{|S_t}$ is connected for $t\neq 0$, and is disconnected for $t=0$. This contradiction proves that $C=0$, i.e. the pencil $|D|$ extends to $|D_t|$.

From this fact, we readily infer that Φ can be lifted to $\tilde{\Phi}\colon \tilde{\mathscr{S}}\to\Sigma_2\times\mathbf{D}$. Now let $\tilde{W}\to\Sigma_2$ be the blowing up of four quadruple points of the branch locus of $\tilde{S}\to\Sigma_2$ (Theorem 6.1). Then, we can prove that $\tilde{\Phi}$ can be lifted to $\tilde{\mathscr{S}}\to\tilde{\mathscr{W}}$ extending the natural map $h\colon \tilde{S}\to\tilde{W}$, where $q\colon \tilde{\mathscr{W}}\to\mathbf{D}$ is a family of deformations of $\tilde{W}$. In fact, by [8], II, Proposition 7.1, it suffices to prove that the natural map

$$H^0(\tilde{W}, \mathscr{T}_{\tilde{W}/\Sigma_2})\to H^0(\tilde{S}, h^*\mathscr{T}_{\tilde{W}/\Sigma_2})$$

is surjective. This can be proved to be bijective by a method similar to [Q], Lemma 13.

In this way, we get a family of maps $h_t\colon \tilde{S}_t \to \tilde{W}_t = q^{-1}(t)$ of degree 2. We note that the branch locus of h has no infinitely near triple points. Hence, it follows that the branch loci B_t of the h_t form a flat family (see Part I, p. 369). But, while B_0 has 4 connected components, B_t, $t \neq 0$ has three, which is a contradiction.

Next suppose that $\alpha(t)$ is not identically 0. Then the general fibres are of type IVa -1. By the second half of the proof of Theorem 10.4, we may assume that two pencils $|D|+C$ and $|D|+C'$ extend to those on S_t. Since $|K_t|$ has no fixed component, we have $C+C'=G$ (cf. (10.4)). Moreover, since a generic member S_t is of type IVa, we may assume that $|D|+C$ extends to a pencil $|D_t|$ of genus 2. This implies that $KC=0$ and $KC'=2$. Suppose $C \neq 0$. Then, C consists of non-singular rational curves with self-intersection number -2. If $DC=0$, then from $(D+C)^2=0$, we have $C^2=0$, and hence $C=0$, by Hodge's index theorem. So we have $DC \geqq 1$. Similarly, $DC' \geqq 1$. Combined with $DG=2$, these imply $DC=DC'=1$. Then from $(D+C)^2=0$, we get $C^2=-2$. By Lemma 11.1,

$$C = A + (\text{rational curves}),$$

and the equality $KC=0$ implies that this is a disjoint sum. Hence, using $C^2=A^2=-2$, we conclude $C=A$.

We now want to apply an elementary operation to A (see Part II, Appendix B). For this purpose, we need to show that A does not extend to a family $\{A_t\}$ of divisors on S_t. Suppose A extends. Then, A_t is the rational curve on S_t with $K_t A_t = 0$, and is away from the base points of $|K_t|$. Hence A_t is mapped into a point by the canonical map Φ_{K_t} on S_t. This contradicts the equality $D_t A_t = 1$.

By the elementary operation, we may assume $|D|$ itself extends to $|D_t|$. Then, we have

$$(\varphi_0) = \mathscr{D} + \mathscr{D}', \qquad (\varphi_2 + \beta\varphi_3) = \mathscr{D}_1 + \mathscr{D}', \tag{11.1}$$

where $\mathscr{D}$ and $\mathscr{D}_1$ are disjoint (cf. (10.4)). Recall that the image of $\Phi\colon \tilde{\mathscr{S}} \to \mathbb{P}^3$ is defined by the equation

$$(Z_2 + \beta Z_3)(Z_2 - \beta Z_3) = Z_0 Z_1,$$

for a system of homogeneous coordinates (Z_0, Z_1, Z_2, Z_3) on $\mathbb{P}^3$. By blowing up the curve $Z_0 = Z_2 + \beta Z_3 = 0$, we obtain a family $\mathscr{W}$ of deformations of Σ_2. From (11.1), it follows that the ideal sheaf generated by φ_0 and $\varphi_2 + \beta\varphi_3$ on $\tilde{\mathscr{S}}$ is invertible. Therefore, the map Φ factors through $\mathscr{W}$. This leads us to a contradiction as before. Q.E.D.

Proof of Theorem 11.3. A surface of type V -2 is birationally equivalent to a double covering of Σ_2 such that its branch locus has two 2-fold triple points on a fibre, one of them being on Δ_0. By displacing the other triple point out of the fibre, we obtain a family of double coverings of Σ_2 (cf. Theorem 10.2). Applying elementary transformations, we see that the general members are of type V -1. Q.E.D.

§ 12. Simply-Connectedness

Theorem 12.1. *Any minimal algebraic surface S with $p_g=4$, $q=0$, and $c_1^2=6$ is simply connected.*

Proof. It suffices to prove the theorem, for instance, for surfaces of type Ib, II, and IIIa. First, it can be easily proved that S has no finite non-trivial unramified covering (see [3], p. 212). Hence, it is enough to show that the fundamental group $\pi_1(S)$ is abelian. For surfaces of type Ib and IIIa, this last fact follows from [22], Proposition 3 (cf. the proof of Theorem 3.4 in Part I). The proof for surfaces of type II is as follows. Since they have one and the same deformation type, we may assume, in (4.1), that $\alpha=\beta=0$ and that the divisor $\gamma=0$ is non-singular. Then the branch locus B of $\tilde{S}\to\Sigma_2$ is set-theoretically defined by $\gamma\zeta=0$. By the result cited above, the fundamental group of Σ_2-B is abelian. This implies that $\pi_1(S)$ is abelian. Q.E.D.

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