EMBEDDING THEOREMS FOR WEIGHTED CLASSES OF HARMONIC AND ANALYTIC FUNCTIONS

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We establish conditions on the measure μ, under which for any analytic (harmonic) function in some domain Ω we have the inequality

$$\left(\int_{\Omega}|u|^q d\mu\right)^{1/q} \leqslant c\left(\int_{\Omega}|u|^p \rho\, d\lambda\right)^{1/p}, \tag{1}$$

where $\rho > 0$ is a continuous function and λ is the Lebesgue measure in Ω. In addition, one finds out under what restrictions on μ is the embedding (1) compact. For many concrete Ω and ρ one finds necessary and sufficient conditions.

Introduction

In many problems of analysis the question arises of the comparative strength of the norms on a set of analytic functions. Thus, e.g., Carleson in his well-known paper on the corona problem [1] makes use of the following statement (see also [2, 3]).

THEOREM A. Let D be the open unit circle, let H^p be the Hardy class (see [4]) of functions which are regular in D, let μ be a positive measure in the closed circle $\bar{D}$, let λ_1 be the Lebesgue measure on the circumference ∂D, let $0 < p \leqslant q < \infty$, $d_z = 1 - |z|$; we set for $z \in D$

$$B(z) = \{\zeta \in \bar{D} : |e^{i\arg z} - \zeta| < d_z\}.$$

In order that for all functions $u \in H^p$ we should have the inequality

$$\left(\int_{\bar{D}}|u|^q d\mu\right)^{1/q} \leqslant C\left(\int_{\partial D}|u(e^{i\theta})|^p d\lambda_1\right)^{1/p} \tag{0.1}$$

it is necessary and sufficient that

$$\mu(B(z))^{1/q} \leqslant c\lambda_1(B(z) \cap \partial D)^{1/q}. \tag{0.2}$$

Thus, condition (0.2) is a necessary and sufficient condition for the boundedness of the embedding of H^p in $L_q(\mu)$.

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In this paper we shall determine the conditions on the measure μ in order that for any analytic (harmonic) function in some domain Ω we should have an inequality of the type:

$$\left(\int_\Omega |u|^q d\mu\right)^{1/q} \leq C\left(\int_\Omega |u|^p \rho\, d\lambda\right)^{1/p}, \tag{0.3}$$

where ρ is a positive and continuous weight function, λ is the Lebesgue measure in Ω and C is a constant independent of u. In addition, we establish under what conditions on μ will the set of all analytic (harmonic) functions for which the integral in the right-hand side of the inequality (0.3) is equal to unity be a compact set in the metric of $L_q(\mu)$. The latter means that the embedding (0.3) is compact. We formulate a typical result (maintaining the notations of Theorem A).

<u>THEOREM B.</u> Let μ be a positive measure in D, let λ_2 be the planar Lebesgue measure in D, let $0<p\leq q<\infty$, $\alpha>0$,

$$\rho(z)=\exp(-d_z^{-2\alpha}), D(z)=\{\zeta : |z-\zeta|<d_z^{1+\alpha}\}.$$

The inequality

$$\left(\int_D |u\rho|^q d\mu\right)^{1/q} \leq C\left(\int_D |u\rho|^p d\lambda_2\right)^{1/p} \tag{0.4}$$

holds for any function u which is regular in D if and only if

$$\mu(D(z))^{1/q} \leq C\lambda_2(D(z))^{1/p}. \tag{0.5}$$

In order that embedding (0.4) be compact, it is necessary and sufficient that for $|z|\to 1$ we have

$$\mu(D(z))^{1/q} = o(\lambda_2(D(z))^{1/q}). \tag{0.6}$$

If in Theorem A we extend the measure λ_1 into D by zero, then condition (0.2) can be written in the form $\mu(B(z))^{1/q} \leq c\lambda_1(B(z))^{1/p}$, which coincides formally with (0.5). Both in Theorem A and Theorem B one indicates a class of sets $B(z)$ [$D(z)$, respectively] on which the measure μ is "subordinate" to the measure λ. However the sets $B(z)$ border on the boundary ∂D, while the circles $D(z)$ are included compactly in D. The fact is that the functions from H^p have boundary values on ∂D, but this cannot be said about the functions summable over an area. We also note that the larger is the singularity of the functions at the boundary $D(z)$, the smaller is the radius d_z^{1+2} of the circle ∂D.

Our method of proof of the inequalities of type (0.3) is based on the mean value theorem; it can be applied in those cases when the functions u can increase sufficiently fast when one approaches the boundary of the domain Ω. In this case it is necessary that the function of the radii [in Theorem B this is $t(z)=d_z^{1+\alpha}$], which is determined by the weight

ρ, should satisfy some additional condition.

The first section of the paper is devoted to the proof of inequality (0.3) for (logarithmically) subharmonic functions. Here we consider both cases: $p \leqslant q$ and $q < p$. It should be mentioned that for $q < p$ the conditions on the measure differ qualitatively from conditions (0.2) and (0.5). In the second section we consider the classes of analytic and harmonic functions: one obtains conditions for the compactness of embedding (0.3). In the first two sections one considers only sufficient conditions on the measure μ, for which inequality (0.3) holds. The necessity of these conditions for specific examples is established in the last section.

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1. Inequalities for (Logarithmic) Subharmonic Functions

Let Ω be an open subset of the m-dimensional Euclidean space R^m, let $\partial\Omega$ be the boundary of Ω and for $x\in\Omega$ we set $d_x = \mathrm{dist}(x,\partial\Omega)$.* We introduce the notations: $D(x,\tau)$ is the open sphere of radius $\tau>0$ with center at the point x; $t(x)$ is a function continuous in Ω such that $0<t(x)\leqslant d_x$; $D(t(x)) = D(x,t(x))$; for $G\subset\Omega$, $G^t = \bigcup_{x\in G} D(t(x))$; $\rho(x)$ is a positive and continuous function in Ω; μ is a nonnegative Borel measure, finite on any compactum situated in Ω; for $p>0$ we set

$$\|u\|_{p,\mu} = \left(\int |u|^p\, d\mu\right)^{1/p}.$$

If $d\mu = \rho(x)\,dx$, then $\|u\|_{p,\mu} = \|u\|_{p,\rho}$. Here and in the sequel, integration without the indication of the limits denotes integration over Ω.

We denote the mean value of the function ρ in the sphere $D = D(x,\tau)\subset\Omega$ by $\mathcal{M}(\rho, D)$, i.e.,

$$\mathcal{M}(\rho,D) = \frac{1}{\tau^m\omega_m}\int_D \rho\, dy,\quad \omega_m = \pi^{m/2}/\Gamma(m/2+1);$$

for $D = D(t(x))$, $\mathcal{M}(\rho, D(t(x))) = \mathcal{M}(\rho, t(x))$.

Assume that for the functions ρ and t there exists a number $M\geqslant 1$ such that for all $x\in\Omega$ one of the following conditions holds:

1) for all $y\in D(t(x))$, $\rho(y)\leqslant M\rho(x)$;

2) $\mathcal{M}(\rho, t(x)) \leqslant M\rho(x)$;

3) $\mathcal{M}(\log\rho, t(x)) \leqslant \log[M\rho(x)]$.

Then we shall say that the pair (ρ,t) belongs to the class $\Sigma_\infty = \Sigma_\infty(\Omega)$, $\Sigma = \Sigma(\Omega)$ or $\Sigma_{\log} = \Sigma_{\log}(\Omega)$, respectively. We note that $\Sigma_\infty \subset \Sigma \subset \Sigma_{\log}$. The first inclusion is obvious and the second one follows from the inequality between the geometric and arithmetic means (see [5]). In addition, if† $\rho_1 \asymp \rho_2$, then (ρ_1,t) and (ρ_2,t) belong simultaneously to the class Σ_∞, Σ or $\Sigma_{\log}$.

*In the case $\Omega = R^m$ we set $d_x = |x|$.

†As usual, $\rho_1 \asymp \rho_2$ mean that $0 < c \leqslant \rho_1(x)\rho_2(x)^{-1} \leqslant C < \infty$ for all $x\in\Omega$.

Finally, we consider the sequence of open subsets $\Omega_i \subset \Omega$, $i=1,2,\dots$, such that $\bigcup_i \Omega_i = \Omega$ and such that each point of the domain Ω belongs to at most $N (N<\infty)$ sets Ω_i. We shall say that such a sequence $\{\Omega_i\}$ forms a covering of the domain Ω of finite multiplicity. For a covering $\{\Omega_i\}$, finite positive numbers p and q, a measure , and functions and t we set $\varkappa^{-1} = q^{-1} - p^{-1}$; $d_i^q = \int_{\Omega_i} (\rho t^m)^{-q/p} d\mu$; $d = \sup_i d_i$ for $\varkappa \leqslant 0$ and $d = \left(\sum d_i^{\varkappa}\right)^{1/\varkappa}$ for $\varkappa > 0$.

THEOREM 1.1. Let $0<q<\infty$, $1 \leqslant p < \infty$; $(\rho^{1/1-p}, t) \in \Sigma$ for $p \neq 1$ and $(\rho^{-1}, t) \in \Sigma_\infty$ for $p=1$; the sets Ω_i and Ω_i^t, $i=1,2,\dots$, form a covering of Ω of finite multiplicity. If $d<\infty$, then for any nonnegative subharmonic function u in Ω we have the inequality

$$\| u \|_{q,\mu} \leqslant C d \| u \|_{p,\rho}, \tag{1.1}$$

where the constant C does not depend on u and on μ.

Let us formulate the corresponding result for the logarithmically subharmonic functions, i.e., such $u \geqslant 0$ for which $\log u$ is a function subharmonic in Ω.

THEOREM 1.2. Let $0<p, q<\infty$; $(\rho^{-1}, t) \in \Sigma_{\log}$ and assume that the sets Ω_i^t, $i=1,2,\dots$, form a covering of Ω of finite multiplicity. If $d<\infty$, then for any logarithmically subharmonic function u in Ω inequality (1.1) holds.

Proof of Theorem 1.1. Let u be a nonnegative subharmonic function in Ω. 1) We consider the case $p>1$. Since $t(x) \leqslant d_x$ and $(\rho^{1/1-p}, t) \in \Sigma$, for any $x \in \Omega_i$ we have $(p' = p/p-1)$

$$u(x) \leqslant \frac{1}{t(x)^m \omega_m} \int_{D(t(x))} u\, dy \leqslant \frac{1}{t(x)^m \omega_m} \Big(\int_{D(t(x))} \rho^{-p'/p} dy \Big)^{1/p'} \cdot$$

$$\cdot \Big(\int_{D(t(x))} u^p \rho\, dy \Big)^{1/p} \leqslant \frac{1}{t(x)^{m/p} \omega_m^{1/p}} \cdot \mathcal{K}(\rho^{1/1-p}, t)^{1/p'} \cdot \Big(\int_{\Omega_i^t} u^p \rho\, dy \Big)^{1/p} \leqslant \frac{M^{1/p'}}{\omega_m^{1/p}\, t(x)^{m/p} \rho(x)^{1/p}} \Big(\int_{\Omega_i^t} u^p \rho\, dy \Big)^{1/p}.$$

Raising both sides to the power $q>0$, integrating with respect to the measure $\mu(x)$, and i, summing with respect to all i, we obtain

$$\int |u|^q d\mu \leqslant \frac{M^{q/p'}}{\omega_m^{q/p}} \sum_i d_i^q \Big(\int_{\Omega_i^t} u^p \rho\, dy \Big)^{q/p}.$$

Then, for $1 \leqslant q/p < \infty$ $(\varkappa \leqslant 0)$ we make use of the inverse Minkowski inequality and of the finite multiplicity of the covering $\{\Omega_i^t\}$.

$$\| u \|_{q,\mu}^q \leqslant \frac{M^{q/p'}}{\omega_m^{q/p}} (\sup_i d_i^q) \sum_i \Big(\int_{\Omega_i^t} u^p \rho\, dy \Big)^{q/p} \leqslant$$

$$\leqslant \frac{M^{q/p'}}{\omega_m^{q/p}} d^q \Big(\sum_i \int_{\Omega_i^t} u^p \rho\, dy \Big)^{q/p} \leqslant M^{q/p'} (N/\omega_m)^{q/p} d^q \| u \|_{p,q}^q.$$

If $0 < q/p < 1$ $(\varkappa > 0)$, then by applying Hölder's inequality with the exponents p/q and $p/p \cdot q = \varkappa/q$ to the sum in the right-hand side of (1.2), we obtain the required statement:

$$\|u\|^q_{q,\mu} \leq \frac{M^{q/p'}}{\omega^{q/p}} \Big(\sum_i d_i^{\varkappa}\Big)^{q/\varkappa} \Big(\sum_i \int_{\Omega_i^t} u^p \rho \, dy\Big)^{q/p} \leq M^{q/p'} \big(N/\omega_m\big)^{q/p} d^q \, \|u\|^q_{p,\rho}.$$

2) In the case $p = 1$, $(\rho^{-1}, t) \in \sum_\infty$ and for $x \in \Omega_i$ we have

$$u(x) \leq \mathcal{H}(u, t(x)) \leq M/\omega_m \big(\rho(x) t(x)^m\big)^{-1} \int_{\Omega_i^t} u \rho \, dy.$$

Then, as in case 1), we start with inequality (1.2). The theorem is proved.

Proof of Theorem 1.2. If u is a logarithmically subharmonic function, then for any $x \in \Omega$ and $p > 0$ we have

$$\log u^p(x) \leq \frac{1}{t(x)^m \omega_m} \int_{D(t(x))} \log u^p \, dy. \tag{1.3}$$

In additon, since $(\rho^{-1}, t) \in \sum_{\log}$, there exists a finite number $M \geq 1$ such that

$$\log \rho(x) \leq \frac{1}{t(x)^m \omega_m} \int_{D(t(x))} \log \rho \, dy + \log M. \tag{1.4}$$

Adding (1.3) and (1.4), we obtain

$$\log[u(x)^p \rho(x)] \leq \frac{1}{t(x)^m \omega_m} \int_{D(t(x))} \log [u^p \rho] \, dy + \log M.$$

Making use of the inequality between the geometric and arithmetic means, we arrive at the relation

$$u(x)^p \rho(x) \leq \frac{M}{t(x)^m \omega_m} \int_{D(t(x))} u^p \rho \, dy,$$

which can be rewritten in the form

$$u(x) \leq \big(M/\omega_m\big)^{1/p} \big[\rho(x) t(x)^m\big]^{-1/p} \Big(\int_{\Omega_i^t} u^p \rho \, dy\Big)^{1/p}.$$

Then, one has to proceed as in the proof of Theorem 1.1, starting with inequality (1.2). The theorem is proved.

The assumption about the finite multiplicity of the covering $\{\Omega_i^t\}$ can be replaced by explicit conditions on $t(x)$; then the sets Ω_i are constructed in the canonical manner. In connection with this, we prove a lemma about coverings. We denote by $C_0 = C_0(\Omega)$ the class of continuous functions t such that for all $x, y \in \Omega$, $|t(x) - t(y)| \leq 1/4 \, |x - y|$,

$0 < t(x) \leq 1/4\, d_x$.

Lemma on Coverings. Let Ω be a bounded open set in R^m and $t \in C_0$. Then there exists a sequence of points $x_i \in \Omega$, $i = 1, 2, \dots$, such that 1) $x_i \notin D(t(x_j))$, $i \neq j$; 2) $\bigcup_i D(t(x_i)) = \Omega$; 3) $D^t(t(x_i)) \subset D(3t(x_i)) \Subset \Omega$, $i = 1, 2, \dots$; 4) $\{D(3t(x_i))\}_1^\infty$ is a covering of Ω of finite multiplicity.

Proof. Since t is a continuous bounded function, there exists a point x_1 such that $t(x_1) = \max t(x)$, where $\max$ is taken over all $x \in \Omega$. Let $x_1, x_2, \dots, x_{i-1}$ be the points which have been already chosen for the desired sequence. The point x_i is one of the points $\xi \in \Omega$ for which $t(\xi) = \max t(x)$, the $\max$ being extended over all $x \in \Omega \setminus \bigcup_{k<i} D(t(x_k))$. Continuing this process, we construct the sequence $\{x_i\}$ for which condition 1) holds. Condition 2) follows from the boundedness of Ω and the continuity for $t > 0$. Since $t \in C_0$, for $y \in D(t(x))$ we have $t(x) + t(y) \leq t(x) + [t(x) + 1/4\, |x-y|] \leq 3\, t(x) < d_x$. From here we obtain condition 3). Finally, let us indicate a number N such that each of the spheres $D(3t(x_j))$ intersects at most N spheres of the sequence $\{D(3t(x_j))\}$, i.e., such that condition 4) holds. First we note that, according to condition 1), the spheres $D(1/2\, t(x_i))$, $i = 1, 2, \dots$, do not intersect. On the other hand, if the two spheres $D(3t(x))$ and $D(3t(y))$ do not have common points, then $7^{-1} t(x) \leq t(y) \leq 7 t(x)$, since $|x-y| \leq 3t(x) + 3t(y)$, and $t(y) \leq t(x) + 1/4\, |x-y|$. Therefore, $D(1/2\, t(y)) \subset D(x, 2^5 t(x)) \cap \Omega$ and $1/2\, t(y) > 2^{-4} t(x)$. In the sphere of radius $2^5 t(x)$ nonintersecting spheres of radius $2^{-4} t(x)$ are located, at most 2^{9m} pieces. We now set $N = 2^{9m}$. The lemma is proved.

Remark. If the domain Ω is not assumed to be bounded, then we represent it as the union of the bounded sets $G_0 = \{x \in \Omega, |x| \leq 4\}$, $G_n = \{x \in \Omega,\ 2^{2n} \leq |x| \leq 2^{2(n+1)}\}$, $n = 1, 2, \dots$. For $t \in C_0$ and for any $n \geq 0$ the sets G_n^t and G_{n+2}^t do not intersect. Each set G_n admits a covering by the spheres $\{D_{ni}\}$ such that $\{D_{ni}^t\}$ is a covering of finite multiplicity. Taking the union with respect to n of all such coverings, we obtain a covering of Ω, satisfying the conditions 2)-4) of the Lemma on Coverings.

Every covering, satisfying conditions 2)-4) of the proved lemma for $t \in C_0$, will be called a t *-covering of* Ω.

We reformulate Theorems 1.1 and 1.2. We consider an arbitrary open set Ω, $t \in C_0(\Omega)$ and a sequence of points $x_i \in \Omega$, $i = 1, 2, \dots$, such that the spheres $\{D(t(x_i))\}$ form a t-covering of Ω. In this case for $0 < p \leq q < \infty$ we have

$$d_i^q = \int_{D(t(x_i))} (\rho t^m)^{-q/p} d\mu \asymp t(x_i)^{-mq/p} \int_{D(t(x_i))} \rho^{-q/p} d\mu,$$

$$\alpha = \sup \alpha_i \asymp \sup_{x\in\Omega} t(x)^{-m/p}\Big(\int_{D(t(x))} \rho^{-q/p}\, d\mu\Big)^{1/q} = \beta ,$$

and for $0 < q < p < \infty$ we have

$$\beta = \Big(\sum_{i=1}^{\infty}\Big[t(x_i)^{-mq/p}\int_{D(t(x_i))} \rho^{-q/p}\, d\mu\Big]^{p/(p-q)}\Big)^{\frac{p-q}{pq}} .$$

THEOREM 1.3. Let $0<q<\infty$, $1\leq p<\infty$; $t\in C_0$; $(\rho^{1/(1-p)},t)\in\Sigma$ for and $(\rho^{-1},t)\in\Sigma_\infty$ for $p=1$. If $\beta<\infty$, then for any nonnegative subharmonic function Ω in u we have the inequality

$$\|u\|_{q,\mu} \leq c\beta\,\|u\|_{p,q}, \tag{1.5}$$

where $c<\infty$ and does not depend on u and on μ.

THEOREM 1.4. Let $0<p$, $q<\infty$; $t\in C_0$, $(\rho^{-1},t)\in\Sigma_{\log}$. If $\beta<\infty$, then for any logarithmically subharmonic function u in Ω inequality (1.5) holds.

Remark. Theorems 1.1 and 1.3 (1.2 and 1.4) can be extended in a natural manner to the case of (logarithmically) plurisubharmonic functions in pluricylindric domains.

2. Classes of Harmonic and Analytic Functions

In this section we give applications of the theorems of the preceding sections to the problems of boundedness and compactness of the embedding of different classes of analytic and harmonic functions. We denote by $S_p(\Omega,\rho)$ $(p\geq 1)$ the space of functions harmonic in Ω, for which $\|u\|_{p,\rho}<\infty$. Similarly, for $\Omega\subset\mathbb{C}^m(m\geq 1)$ by $\mathcal{H}_p(\Omega,\rho)$ $(p>0)$ we denote the space of functions analytic in Ω, for which $\|u\|_{p,\rho}<\infty$. Finally, $L_p(\Omega,\mu)$ is the space of measurable functions for which $\|u\|_{p,\mu}<\infty$. For $p\geq 1$, $\mathcal{H}_p$ and S_p are normed spaces (in general, not complete). For $0<p<1$, $\mathcal{H}_p(\Omega,\rho)$ $(L_p(\Omega,\mu))$ is a metric space with the metric $d(u,v)=\|u-v\|^p_{p,\rho}$ $(\|u-v\|^p_{p,\mu}$, respectively). It is known that $|u|^p$ is a subharmonic function if u is harmonic and $p\geq 1$. If u is analytic in $\Omega\subset\mathbb{C}^1$ and $p>0$, then $|u|^p$ is a logarithmically subharmonic function. Finally, if u is analytic in $\Omega\subset\mathbb{C}^m$ $(m>1$, $z=(z_1,\ldots,z_m)$ and $p>0$, then $|u|^p$ is a logarithmically subharmonic function in Ω with respect to the variables z_k $(k=1,2,\ldots,m)$. Thus, Theorems 1.1-1.4 refer in the same degree to harmonic and analytic functions. In terms of the embeddings of the spaces, Theorems 1.3 and 1.4 can be formulated in the following manner.

THEOREM 2.1. Let $\Omega\subset R^m(\Omega\subset\mathbb{C}^1)$ and assume that the conditions of Theorem 1.3(1.4) hold. If $\beta<\infty$, then the embedding operator $S_p(\Omega,\rho)$ [$\mathcal{H}_p(\Omega,\rho)$, respectively] in $L_q(\Omega,\mu)$ is bounded.

We elucidate now the conditions under which the embedding in Theorem 2.1 will be compact. The latter means that the unit sphere in the space $S_p(\mathcal{H}_p)$ is a compact set in

$L_q(\Omega,\mu)$. We introduce the following notations: for $\delta>0$, $\Omega_\delta=\{x\in\Omega:\delta\le d_x\le\delta^{-1}\}$, when $\Omega\ne R^m$ and $(R^m)_\delta=D(0,\delta^{-1})$; $\Omega^\delta=\Omega\setminus\Omega_\delta$; μ_δ is the restriction of the measure μ to the set Ω^δ. According to the definition of the number β, we set for $0<p\le q<\infty$

$$\beta_\delta=\beta(\mu_\delta)=\sup_{x\in\Omega}t(x)^{-m/p}\Big(\int_{D(t(x))\cap\Omega^\delta}\rho^{-q/p}\,d\mu\Big)^{1/q},$$

and for $0<q<p<\infty$

$$\beta_\delta=\Big(\sum_{i=1}^{\infty}\Big[t(x_i)^{-mq/p}\int_{D(t(x_i))\cap\Omega^\delta}\rho^{-q/p}\,d\mu\Big]^{p/p-q}\Big)^{\frac{p-q}{pq}}.$$

Finally, we note that the set $K\subset L_q(\Omega,\mu)\ (q>0)$ is compact in $L_q(\Omega,\mu)$ if for any $\varepsilon>0$ there exists $\delta>0$ such that a) K is compact in $L_q(\Omega_\delta,\mu)$ and b) for all $u\in K$ we have $\|u\|^q_{q,\mu_\delta}\le\varepsilon$. It can be easily seen that from a) and b) there follows the existence of a finite ε-net of the set K for any $\varepsilon>0$.

For the sake of definiteness, we consider the space $\mathcal{H}_p(\Omega,\rho)\ (p>0,\ \Omega\subset\mathbb{C}^1)$. Let $K=\{u\in\mathcal{H}_p(\Omega,\rho):\|u\|_{p,\rho}\le 1\}$. We have to find conditions under which the set K is compact in $L_q(\Omega,\mu)$. Since ρ is a positive continuous function in Ω, for any $\delta>0$ there exists $C(\delta)>0$, such that $\rho(x)\ge C(\delta)>0$ for all $x\in\Omega_{\delta/2}$. Then, by virtue of the subharmonicity of $|u|^p$ $(u\in K)$ for all $x\in\Omega_{\delta'}$ $(4\delta'=3\delta)$ we have the inequality

$$|u(x)|^p\le\frac{4^2}{\pi\delta^2C(\delta)}\int_{D(x,\delta/4)}|u|^p\rho\,d\xi\le\frac{16}{\pi\delta^2C(\delta)},$$

which means that the set K is uniformly bounded on $\Omega_{\delta'}$. Consequently, K is compact in the uniform metric on $\Omega_\delta(\Omega_\delta\Subset\Omega_{\delta'})$. The latter implies the compactness of the set K in $L_q(\Omega_{\delta,\mu})$ since $\mu(\Omega_\delta)\le C(\delta)<\infty$. This means that condition a) holds for any $\delta>0$. Now we consider Ω^δ. Under the conditions of Theorem 1.4, for all $u\in K$ we have the inequality $\|u\|_{q,\mu\delta}\le c\beta_\delta$. Therefore, if $\lim_{\delta\to0}\beta_\delta=0$, then condition b) holds and, consequently, K is a compact subset of $L_q(\Omega,\mu)$, i.e., the embedding of $\mathcal{H}_p(\Omega,\rho)$ into $L_q(\Omega,\mu)$ is compact. We note that for $0<q<p<\infty$ from the condition $\beta<\infty$ there follows $\beta_\delta\to0$, for $\delta\to0$. Thus, we have proved the following theorem

THEOREM 2.2. Let $\Omega\subset\mathbb{C}^1$ and assume that the conditions of Theorem 1.4 hold. If $0<p\le q<\infty$ and $\lim_{\delta\to0}\beta_\delta=0$, or if $0<q<p<\infty$ and $\beta<\infty$, then the embedding of $\mathcal{H}_p(\Omega,\rho)$ into $L_q(\Omega,\mu)$ is compact.

In the same terms one can formulate the conditions for the compactness of the embedding of $S_p(\Omega,\rho)(\Omega\subset R^m)$ or $\mathcal{H}_p(\Omega,\rho)$ (Ω is a pluricylindric domain in $\mathbb{C}^m, m>1$) into $L_q(\Omega,\mu)$.

Remark. It is easy to see that condition $\beta<\infty$ in the Theorems 1.3-1.4, 2.1 and 2.2 can be replaced by an equivalent one: $\beta_\delta<\infty$ for some $\delta>0$. Thus, it is sufficient to verify the conditions on the measure μ only in the neighborhood of the boundary $\partial\Omega$ and for large $|x|$.

3. Examples

In many cases the conditions on the measure μ in the theorems of the previous sections are also necessary for the boundedness (compactness) of the corresponding embedding. For the proof of the necessity one has to construct an extremal function (more precisely, a function close to an extremal one) which depends in an essential manner on the domain and on the weight ρ. We consider some examples: in Subsections 1 and 2 we consider the space of analytic and harmonic functions in a bounded domain with a power weight; in Subsection 4 we consider the entire functions in the complex plane; in Subsection 5 the domain is once again bounded but the weight is exponential. The case $q<p$ is discussed in Subsection 3, in the remaining examples $p\leqslant q$.

1. Let be a bounded domain of the complex plane $\mathbb{C}^1$, $\rho(z)=d_z^\alpha\ (-1<\alpha<\infty)$, $t(z)=1/4\ d_z$. Obviously, $(d_z^{-\alpha},\ 1/4\,d_z)\in\Sigma_\infty\subset\Sigma_{\log}$ and $1/4\,d_z\in C_0$, and so, according to Theorems 2.1 and 2.2, a sufficient condition for the boundedness (compactness) of the embedding of $\mathcal{H}_p(\Omega,d_z^\alpha)$ in $L_q(\Omega,\mu)$ for $0<p\leqslant q<\infty$ will be

$$\sup_{z\in\Omega} d_z^{-(2+\alpha)q/p}\mu(D(1/4\,d_z))<\infty$$

$$\Big(\lim_{\delta\to 0}\ \sup_{d_z<\delta} d_z^{-(2+\alpha)q/p}\mu(D(1/4\,d_z))=0\Big). \tag{3.1}$$

2. Let Ω be a bounded domain in the Euclidean space R^m, $x\in\Omega$, $\rho(x)=d_x^\alpha\ (-1<\alpha<\infty)$, $t(x)=1/4\,d_x$. Then $(d_x^{-\alpha},\ 1/4\,d_x)\in\Sigma_\infty$, $1/4\,d_x\in C_0$; therefore, as in Subsection 1, for the boundedness (compactness) of the embedding of $S_p(\Omega,d_x^\alpha)$ in $L_q(\Omega,\mu)$ $(1\leqslant p\leqslant q<\infty)$ it is sufficient that

$$\sup_{x\in\Omega} d_x^{-(m+\alpha)q/p}\mu(D(1/4\,d_x))<\infty$$

$$\Big(\lim_{\delta\to 0}\ \sup_{d_x<\delta} d_x^{-(m+\alpha)q/p}\mu(D(1/4\,d_x))=0\Big). \tag{3.2}$$

The conditions (3.1) ($p\geqslant 1$) and (3.2) have been obtained in a somewhat different manner and for more general spaces in [6].* In the same paper one has established the neces-

*There is a misprint in the formulation of Theorem 1 in [6]. One has to assume: $(1\leqslant p\leqslant q<\infty)$. In addition, for the proof of the necessity in Theorem 1, the restriction $p\geqslant 1$ can be replaced by $p>0$.

sity of the conditions (3.1) and (3.2) in the case of a bounded domain Ω with a smooth boundary ($\partial\Omega \in C^2$) and $\alpha \in [0,\infty)$.

3. We turn to the case $q<p$. We consider only one example. Let $D = D(0,1)$ be the unit circle of the complex plane ($z = x+iy$), $q=1$, $p=2$, and assume that the carrier of the measure μ is on the ray $y=0$, $0 \leqslant x < 1$. We divide the segment $[0,1]$ into the intervals $D_k = [1-4^{-k+1}, 1-4^{-k})$, $k=1,2,\ldots$. We set $\mu_k = \mu(D_k)$, $t_k = 4^{-k}$, $\beta^2 = \sum_k (\mu_k/t_k)^2$. From Theorem 2.1 there follows that under the condition $\beta<\infty$ there exists a constant C such that for any function $u \in \mathcal{H}_2(D,1)$ there holds the inequality

$$\int_0^1 |u|\,d\mu \leqslant C\Big(\int_D |u|^2\,dx\,dy\Big)^{1/2}. \tag{3.3}$$

We prove that the converse is also true, i.e., that under our assumptions, from inequality (3.3) it follows that $\beta<\infty$. First we assume that the carrier of the measure μ is situated strictly inside the segment (−1, 1) and we consider the function

$$u_0(z) = \int_0^1 \frac{d\mu(\xi)}{(1-z\xi)^2}.$$

We have

$$\int_D |u_0|^2\,dx\,dy = \int_D \int_0^1 \frac{d\mu(\xi)}{(1-z\xi)^2} \int_0^1 \frac{d\mu(\eta)}{(1-\bar z\eta)^2}\,dx\,dy = \pi \int_0^1\int_0^1 \frac{d\mu(\xi)\,d\mu(\eta)}{(1-\xi\eta)^2}, \tag{3.4}$$

$$\int_0^1 |u_0|\,d\mu = \int_0^1\int_0^1 \frac{d\mu(\xi)\,d\mu(\eta)}{(1-\xi\eta)^2}, \tag{3.5}$$

$$\int_0^1\int_0^1 \frac{d\mu(\xi)\,d\mu(\eta)}{(1-\xi\eta)^2} \geqslant \sum_k \int_{D_k}\int_{D_k} \frac{d\mu(\xi)\,d\mu(\eta)}{(1-\xi\eta)^2} \geqslant \frac{1}{4}\sum_k \left(\mu_k/t_k\right)^2. \tag{3.6}$$

From (3.3)-(3.6) we obtain that $\beta \leqslant 2\sqrt{\pi}\,C < \infty$. From this follows at once the assertion $\beta<\infty$ for any measure μ.

4. Let Ω be an arbitrary open set of the complex place C^1. We consider the space $\mathcal{H}_p(\Omega, \rho(z))$. It is convenient to set $\rho(z) = e^{-\varphi(z)}$; then the condition $(\rho^{-1}, t) \in \sum_{\log}$ (see Theorem 1.4) can be written in the following form: there exists a number M such that for all $z\in\Omega$ we have

$$\frac{1}{\pi t(z)^2}\int_{|z-\zeta|\leqslant t(z)} \varphi(\zeta)\,d\xi\,d\eta \leqslant \varphi(z) + M. \tag{3.7}$$

In order to satisfy condition (3.7) it is sufficient that the function φ be twice continuously differentiable and that there should exist constants C_1 and C_2 such that for $z \in \Omega$ (Δ is the Laplace operator)

$$\begin{aligned} &1)\ t(z)^2 \Delta \varphi(z) \leq C_1, \\ &2)\ \Delta\varphi(\zeta) \leq C_2 \Delta\varphi(z), \text{ when } |z-\zeta| \leq t(z). \end{aligned} \tag{3.8}$$

Indeed, let $z=0$; integrating the identity

$$\frac{1}{2\pi}\int_{|\zeta|=r} \varphi(\zeta)ds = r\varphi(0) + \frac{r}{2\pi}\int_{|\zeta|\leq r} \Delta\varphi(\zeta) \log\frac{r}{|\zeta|}\, d\xi\, d\eta$$

with respect to the variable r between zero and $t=t(0)$ and making use of the conditions 1) and 2), we obtain

$$\frac{1}{\pi t^2}\int_{|\zeta|\leq t} \varphi(\zeta)d\xi\, d\eta \leq \varphi(0) + \frac{C_2\Delta\varphi(0)}{\pi t^2}\int_0^t r\,dr\int_{|\zeta|\leq r} \log\frac{r}{|\zeta|}\, d\xi\, d\eta \leq \varphi(0) + 1/8\, C_2 t^2 \Delta\varphi(0) \leq \varphi(0) + C_1 C_2/8 .$$

From here we obtain (3.7) with $M = C_1C_2/8$.

THEOREM 3.1. Assume $0 < p \leq q < \infty$, φ is a twice continuously differentiable function in the domain $\Omega \subset \mathbb{C}^1$, $\Delta\varphi > 0$, $\tau_\varphi(z) = [\Delta\varphi(z)]^{-1/2}$. Assume, in addition, that there exist constants C_1 and C_2 such that the following conditions hold:

a) $|\tau_\varphi(z) - \tau_\varphi(\zeta)| \leq C_1 |z-\zeta|$,

b) $\tau_\varphi(z) \leq C_2 \operatorname{dist}(z,\partial\Omega)$ ($\tau_\varphi(z) \leq C_2|z|$ if $\Omega = \mathbb{C}^1$). We set $4\gamma = \min(C_1^{-1}, C_2^{-1})$. Then, under the condition

$$\sup_{z\in\Omega} (\Delta\varphi(z))^{q/p} \int_{|z-\zeta|\leq \gamma\tau_\varphi(z)} e^{q/p\,\varphi(\zeta)} d\mu(\zeta) = K < \infty \tag{3.9}$$

for any analytic function u in Ω we have the inequality:

$$\left(\int |u|^q d\mu\right)^{1/q} \leq CK\left(\int |u|^p e^{-\varphi} dx\,dy\right)^{1/p},$$

where the constant C does not depend on u and on μ .

Proof. We verify the conditions of Theorem 1.4. We set $t(z) = \gamma\tau_\varphi(z)$, and then $|t(z)-t(\zeta)| \leq 1/4\,|z-\zeta|$ and $t(z) \leq 1/4\, d_z$, i.e., $t(z) \in C_0(\Omega)$. In addition, $t(z)^2\Delta\varphi(z) = \gamma^2$ and for $|z-\zeta| \leq t(z)$ we have $(\Delta\varphi(\zeta))^{1/2} = (\tau_\varphi(z)-\tau_\varphi(\zeta))(\Delta\varphi(z)\cdot\Delta\varphi(\zeta))^{1/2} + (\Delta\varphi(\zeta))^{1/2}$ $\leq C_1\gamma(\Delta\varphi(\zeta))^{1/2} + (\Delta\varphi(z))^{1/2} \leq 1/4\,(\Delta\varphi(\zeta))^{1/2} + (\Delta\varphi(z))^{1/2}$, from which we obtain (3.8) and thus also (3.7). Thus $(e^{\varphi(z)}, t(z)) \in \Sigma_{\log}$. In order to conclude the proof of the theorem it remains to make use of Theorem 1.4 for $0 < p \leq q < \infty$. The theorem is proved.

Now we prove the necessity of condition (3.9) for the case $\Omega=\mathbb{C}^1$ and $\varphi(z)=\varphi(|z|)=|z|^{2\alpha}\ell(|z|^2)$ $(\alpha>1/2)$, where ℓ is an analytic, slowly increasing function, i.e., a function regular and continuous in the plane with the cut $(-\infty,-1)$, $|\arg\zeta|\leq\pi$, $(\ell(|\zeta|)>0$, for which we have $\lim_{\zeta\to\infty}\frac{\zeta\ell'(\zeta)}{\ell(\zeta)}=0$. We need the following properties of the function ℓ (see [7, 1.2]):

1) $\lim_{\zeta\to\infty}\frac{\zeta^k\ell^{(k)}(\zeta)}{\ell(\zeta)}=0 \quad (k=1,2,\ldots);$

2) for any $\beta>0$ there exists $a\geq 0$ such that $|\zeta|^\beta\ell(|\zeta|)\to\infty$ monotonically $(|\zeta|\to\infty$, $|\zeta|>a)$:

3) for any $z\neq 0$ we have

$$\ell(z\zeta)=\ell(\zeta)(1+0(1))\ (\zeta\to\infty,\ |\arg\zeta|\leq\pi-|\arg z|).$$

The estimate is uniform with respect to z if $0<a\leq|z|\leq A<\infty$,

$$|\arg z|\leq\psi<\pi,\quad |\arg\zeta|\leq\pi-\psi.$$

THEOREM 3.2. Let ℓ be an analytic, slowly increasing function, $\varphi(\xi)=\xi^{2\alpha}\ell(\xi^2)$, $\alpha>1/2$, $0<p\leq q<\infty$. In order that for any entire function $u\in\mathcal{H}_p(\mathbb{C}^1,e^{-\varphi(|z|)})$ we should have the inequality

$$\Big(\int_{\mathbb{C}^1}|u|^q d\mu\Big)^{1/q}\leq C\Big(\int_{\mathbb{C}^1}|u|^p e^{-\varphi(|z|)}\,dx\,dy\Big)^{1/p} \tag{3.10}$$

it is necessary and sufficient to have the following condition: for some $\tau_0>0$

$$\sup_{|z|\geq\tau_0} t(|z|)^{-2q/p}\int_{|z-\zeta|\leq t(|z|)} e^{q/p\,\varphi(|\zeta|)}\,d\mu(\zeta)<\infty, \tag{3.11}$$

where $t(\xi)=\xi^{1-\alpha}\ell(\xi^2)^{-1/2}$.

Proof. First of all we note that by virtue of property 1) of the function ℓ for $\zeta\to\infty$, $\Delta\varphi(|z|)=4\alpha^2|z|^{2\alpha-2}\ell(|z|^2)(1+0(1))$ and, in addition, $[\Delta\varphi(|z|)]^{-1/2}=0(|z|)$. Therefore, according to Theorem 3.1, condition (3.11) is sufficient for inequality (3.10) to hold.

The necessity will be proved in the following manner. We construct an entire function F_τ $(\tau>\tau_0>0)$ for which

$$\int|F_\tau(z)|^p e^{-\varphi(|z|)}dx\,dy\leq c_1 t(\tau)^2 \tag{3.12}$$

$$|F_\tau(z)|^p\geq c_2 e^{\varphi(|z|)},\quad |\tau-z|\leq t(\tau). \tag{3.13}$$

Here the constants C_1 and C_2 do not depend on τ. Then, inserting the function F_τ into the inequality (3.10) and taking into account (3.12) and (3.13), we obtain condition (3.11) on the real axis $(z=\tau)$, and consequently also for all z.

I. We construct F_τ. Following [7, 1, p.54], we set for $|z|<R$

$$F(z)=\frac{1}{2\pi i}\int_{L_R}\frac{e^{2/\rho\,\zeta^{\alpha}\ell(\zeta)}}{\zeta-z}\,d\zeta,$$

where L_R consists of two rays $(Re^{i\pi/2\alpha_1}, \infty e^{i\pi/2\alpha_1})$, $(\infty e^{i\pi/2\alpha_1}, Re^{-i\pi/2\alpha_1})$ and the arc of a circle: $|\zeta|=R$, $|\arg\zeta|\leqslant\pi/2\alpha_1$, $1/2<\alpha_1<\alpha$. Outside the angle $|\arg z|\leqslant\pi/2\alpha_1$ the function F is bounded. For $|z|<R<R_1$, by Cauchy's theorem we have

$$\frac{1}{2\pi i}\int_{L_R}\frac{e^{2/\rho\,\zeta^{\alpha}\ell(\zeta)}}{\zeta-z}\,d\zeta=\frac{1}{2\pi i}\int_{L_{R_1}}\frac{e^{2/\rho\,\zeta^{\alpha}\ell(\zeta)}}{\zeta-z}\,d\zeta.$$

Therefore, F can be continued analytically into the entire plane and is an entire function. If $|z|>R$, $|\arg z|<\pi/2\alpha_1$, then by the residue theorem we have

$$F(z)=e^{2/\rho\,z^{\alpha}\ell(z)}+\frac{1}{2\pi i}\int_{L_R}\frac{e^{2/\rho\,\zeta^{\alpha}\ell(\zeta)}}{\zeta-z}\,d\zeta. \tag{3.14}$$

We note that the integral in (3.14) is bounded. Now we set $F_\tau(z)=e^{-1/\rho\,\varphi(\tau)}F(\tau z)$. Then the following estimates hold for F_τ, uniformly with respect to $(\tau\to\infty)$:

$$e^{1/\rho\,\varphi(\tau)}|F_\tau(z)|=O(1),\quad |\arg z|>\pi/2\alpha_1 \tag{3.15}$$

$$e^{1/\rho\,\varphi(\tau)}\,|F_\tau(z)|=e^{2/\rho\,\tau^{\alpha}\mathrm{Re}(z^{\alpha}\ell(\tau z))}+O(1),\quad |\arg z|\leqslant\frac{\pi}{2\alpha_1}. \tag{3.16}$$

II. We estimate from above the integral

$$I(\tau)=\int|F_\tau(z)|^{\rho}e^{-\varphi(|z|)}dx\,dy=\sum_{j=0}^{3}I_j(\tau).$$

a) $I_0(\tau)$ is an integral over the domain $|\arg z|>\pi/2\alpha_1$. From (3.15) it follows at once that $I_0(\tau)\leqslant ce^{-\varphi(\tau)}=o(t(\tau)^2)$, $\tau\to\infty$.

b) $I_1(\tau)$ is an integral over the domain $|z|\leqslant\tau/4$, $|\arg z|\leqslant\pi/2\alpha_1$. By virtue of the properties 2) and 3) of the function $\ell(\zeta)$, for any $\varepsilon>0$ and some $\tau_0=\tau_0(\varepsilon)>0$ for $|\zeta|\leqslant|\zeta_1|\leqslant|\zeta_2|$, $|\arg\zeta|\leqslant\pi/2\alpha_1$, $|\zeta_1|\geqslant\tau_0$ we have the inequalities: $|\zeta^{\alpha}\ell(\zeta)|\leqslant(1+\varepsilon)|\zeta_1|^{\alpha}\ell(|\zeta_1|)\leqslant(1+\varepsilon)|\zeta_2|^{\alpha}\ell(|\zeta_2|)$. Therefore, $\mathrm{Re}[(\tau z)^{\alpha}\ell(\tau z)]\leqslant|(\tau z)^{\alpha}\ell(\tau z)|\leqslant\tau^{2\alpha}/4^{\alpha}\ell(\tau^2/4)(1+\varepsilon)\leqslant\tau^{2\alpha}/4^{\alpha}\ell(\tau^2)(1+\varepsilon)^2$, when $|z|\leqslant\tau/4$, $|\arg z|\leqslant\pi/2\alpha_1$, $\tau>4\tau_0$. We choose ε from the condition $1-2^{1-2\alpha}(1+\varepsilon)^2\equiv\gamma>0$; then $2\,\mathrm{Re}[(\tau z)^{\alpha}\ell(\tau z)]-\tau^{2\alpha}\ell(\tau^2)\leqslant-\gamma\varphi(\tau)$. From here, taking into account estimate (3.16), we obtain $I_1(\tau)\leqslant ce^{-\gamma\varphi(\tau)}=o(t(\tau)^2)$, $\tau\to\infty$.

c) $I_2(\tau)$ is an integral over the domain $|\arg z| \leqslant \pi/2\alpha_1$, $\tau/4 \leqslant |z| \leqslant 4\tau$. As it follows from (3.16), in order to estimate $I_2(\tau)$ it is sufficient to estimate the function $h(\tau, z) = \varphi(\tau) - 2\tau^{\alpha}\operatorname{Re}(z^{\alpha}\ell(\tau z)) + \varphi|z|)$. We consider the identities:

$$\ell(|z|^2) = \ell(\tau^2) + \ell'(\tau^2)(|z|^2 - \tau^2) + \int_{\tau^2}^{|z|^2} \ell''(\xi)(|z|^2 - \xi)\, d\xi, \tag{3.17}$$

$$\ell(\tau z) = \ell(\tau^2) + \ell'(\tau^2)(\tau z - \tau^2) + \int_{\tau^2}^{\tau z} \ell''(\zeta)(\tau z - \zeta)\, d\zeta. \tag{3.18}$$

We note that by virtue of the properties 1) and 3) of the function ℓ for $\tau^2/4 \leqslant |\zeta| \leqslant 4\tau^2$,

$$\ell''(\zeta) = o(\zeta^{-2}\ell(\zeta)) = o(\tau^{-4}\ell(\tau^2)), \quad \tau \to \infty. \tag{3.19}$$

Inserting (3.17) and (3.18), taking into account (3.19), into the expression for $h(\tau, z)$, we obtain:

$$h(\tau, z) = \ell(\tau^2)|z^{\alpha} - \tau^{\alpha}|^2 + \ell'(\tau^2)\left(|z^{\alpha+1} - \tau^{\alpha+1}|^2 - \right.$$

$$\left. -\tau^2|z^{\alpha} - \tau^{\alpha}|^2\right) + \ell(\tau^2)\tau^{2(\alpha-1)}|z - \tau|^2\, o(1) \asymp \tau^{2(\alpha-1)}\ell(\tau^2)|z - \tau|^2 = t(\tau)^{-2}|z - \tau|^2. \tag{3.20}$$

The two-sided estimate (3.18) is uniform with respect to z in the domain under consideration and with respect to τ, $\tau > \tau_0$, for some τ_0. Finally, we have $I_2(\tau) \leqslant c\int_0^{\infty} e^{-t(\tau)^{-2}\tau^2}\, \tau\, d\tau \leqslant c\,t(\tau)^2$. The integral $I_2(\tau)$ gives the main contribution in $I(\tau)$.

d) $I_3(\tau)$ is an integral over the domain $|\arg z| \leqslant \pi/2\alpha_1$, $|z| \geqslant 4\tau$. In the same way as for the estimation of the integral $I_1(\tau)$, in this case we have $(\tau \leqslant |z|/4,\ \varepsilon > 0,\ \gamma(\varepsilon) > 0,\ \tau > \tau_0(\varepsilon))$:

$$\operatorname{Re}[(\tau z)^{\alpha}\ell(\tau z)] \leqslant |z|^{2\alpha}/4^{\alpha}\,\ell(|z|^2)(1+\varepsilon)^2,$$

$$2\operatorname{Re}[(\tau z)^{\alpha}\ell(\tau z)] - |z|^{2\alpha}\ell(|z|^2) \leqslant -\gamma\varphi(|z|).$$

Consequently, $I_3(\tau) \leqslant ce^{-\varphi(\tau)} = o(t(\tau)^2)$, $\tau \to \infty$.

Finally, from a)-d) the estimate (3.12) follows for all $\tau > \tau_0$ for some $\tau_0 > 0$.

III. We estimate from below the integral with respect to the measure μ. Under the condition $|t - \tau| \leqslant t(\tau)$ the estimate (3.20) gives: $h(\tau, z) \leqslant C$, where the constant C does not depend on z and on $\tau > \tau_0$. Therefore, taking into account (3.16), we obtain

$$\int_{\mathbb{C}^1} |F_\tau|^q\, d\mu \geqslant \int_{|z-\tau| \leqslant t(\tau)} |F_\tau|^q\, d\mu \geqslant c\int_{|z-\tau| \leqslant t(\tau)} e^{q/p(\varphi(|z|) - h(\tau, z))}\, d\mu \geqslant c\int_{|z-\tau| \leqslant t(\tau)} e^{q/p\,\varphi(|z|)}\, d\mu.$$

The theorem is proved.

Remark. Theorem 3.2 holds also for $0 < \alpha \leqslant 1/2$. In this case the function F_τ will have several directions of largest increase of the modulus (the indicatrix is proportional to $\cos \alpha\theta$, $|\theta| \leq \pi$). However, for each α these directions form a finite set. Therefore, the entire plane can be split into a finite number of angles and in each angle one can carry out the same estimates as in the proof of Theorem 3.2.

Remark. Theorem 3.2 can be extended in a natural manner to the space of entire functions of several complex variables.

5. Let Ω be a bounded domain of the complex plane $\mathbb{C}^1$ $\rho(z) = \exp(-d_z^{-\alpha})$, where d_z is the distance from the point $z \in \Omega$ to the boundary $\partial\Omega$. Assume, in addition, that $\alpha > 1$ and that the domain Ω is such that

$$\left.\begin{array}{l} \text{for some } \varepsilon > 0 \text{, for each point } z \in \Omega,\ d_z \leq \varepsilon \\ \text{there exists only one point } \hat{z} \in \partial\Omega \text{ nearest } z \text{, i.e., } d_z = |z - \hat{z}|. \end{array}\right\} (A)$$

In this case one can make use of Theorem 3.1 and 3.2 in order to prove criteria for the boundedness of the embedding of $\mathcal{H}_p(\Omega, \rho(z))$ in $L_q(\Omega, \mu)$. We start with two remarks:

1) We consider the unit circle $D_0 = \{ z \in \mathbb{C}^1,\ 0 < |z| < 1 \}$. As usual, $\mathcal{H}_p(D_0, \exp(-|z|^{-\alpha}))$ is the class of functions analytic in D_0 , p is the power whose modulus is summable over the area with the weight function $\exp(-|z|^{-\alpha})$. Performing the substitution $z = w^{-1}$, we arrive at $\mathcal{H}_p(\mathbb{C}^1 \setminus D_0, |w|^{-4} \exp(-|w|^{\alpha}))$. In this case $\alpha > 1$, $|w|^{-4} \exp(-|w|^{\alpha})$, $= \exp[-|w|^{\alpha}(1 + 2|w|^{-\alpha} \ln |w|^2)] = \exp(-|w|^{\alpha} \ell(|w|^2))$, where $\ell(\zeta) = 1 + 2\zeta^{-\alpha/2} \ln \zeta$ is an analytic, slowly increasing function $(|\zeta| \geq 1)$. Consequently, one can apply Theorem 3.2. Returning to the variable z , we obtain a necessary and sufficient condition on the measure μ in the neighborhood of the point $z = 0$. It appears in the following manner:

$$\sup_{|z| < \eta} t(|z|)^{-2q/p} \int_{|z - \zeta| < t(|z|)} \exp\left(q/p\, |\zeta|^{-\alpha} \right) d\mu(\zeta) < \infty , \tag{3.21}$$

where $t(\xi) = \xi^{1+\alpha/2}$, $\eta > 0$.

2) Let $D = \{ z \in \mathbb{C}^1,\ |z| < 1 \}$. Since $\left[\Delta (1 - |z|)^{-\alpha} \right]^{-1/2} \asymp (1 - |z|)^{1+\alpha/2}$, $|z| < 1$, the pair (see Theorem 3.1) $(\exp(1 - |z|)^{-\alpha},\ (1 - |z|)^{1+\alpha/2})$ belongs to $\sum_{\log}(D)$.

THEOREM 3.3. Let Ω be a bounded domain of the complex plane, satisfying condition (A), $\alpha > 1$, $0 < p \leq q < \infty$. In order that for any function $u \in \mathcal{H}_p(\Omega, \exp(-d_z^{-\alpha}))$ we should have the inequality

$$\left(\int_\Omega |u|^q d\mu \right)^{1/q} \leq c \left(\int_\Omega |u|^p \exp(-d_z^{-\alpha})\, dx\, dy \right)^{1/p} ,$$

it is necessary and sufficient that the following condition be satisfied:

$$\sup_{z \in \Omega} t(z)^{-2q/p} \int_{|z - \zeta| < t(z)} \exp\left(q/p\, d_\zeta^{-\alpha} \right) d\mu(\zeta) < \infty , \tag{3.22}$$

where $t(z) = d_z^{1+\alpha/2}$.

Proof. We fix $z_0 \in \Omega$, $d_{z_0} \leq \varepsilon$; let $\hat{z}_0$ be the nearest point of the boundary z_0 to the point $\partial\Omega$. We consider through the point $\hat{z}_0$ a circumference C_ε of radius ε so that the points z_0 and $\hat{z}_0$ should lie on the same radius. We denote by D_ε the circle with boundary C_ε. By virtue of the conditions (A), the circle D_ε is entirely situated in $\bar{\Omega}$ and $\hat{z}_0$ is the unique common point of C_ε and $\partial\Omega$. We also construct the tangent ℓ to the circumference C_ε at the point $\hat{z}_0$. We set $\hat{d}_\zeta = \operatorname{dist}(\zeta, C_\varepsilon)$ and $\tilde{d}_\zeta = \operatorname{dist}(\zeta, \ell)$. It is easy to see that for $|z_0 - \zeta| \leq t(z_0)$, $d_{z_0} \leq \varepsilon$ we have the chain of inequalities*

$$|\hat{z}_0 - \zeta|^{-\alpha} \leq d_\zeta^{-\alpha} \leq \hat{d}_\zeta^{-\alpha} \leq \tilde{d}_\zeta^{-\alpha} + c_1 \leq |\hat{z}_0 - \zeta|^{-\alpha} + c_2,$$

where the constants c_1 and c_2 depend only on ε and α. Here the first two inequalities are obvious, while for the proof of the last two one essentially makes use of the fact that that $t(z) = d_z^{1+\alpha/2}$.

By virtue of the inequalities $|\hat{z}_0 - \zeta|^{-\alpha} \leq d_\zeta^{-\alpha} \leq |z_0 - \zeta|^{-\alpha} + c_2$, the necessity of condition (3.22) follows at once from (3.21).

Sufficiency. By virtue of Remark 2), applied to D_ε instead of D, and by virtue of the definition of the class $\Sigma_{\log}$, we have

$$\frac{1}{\pi d_{z_0}^{2+\alpha}} \int_{|z_0 - \zeta| \leq d_{z_0}^{1+\alpha/2}} \hat{d}_\zeta^{-\alpha}\, dx\, dy \leq d_{z_0}^{-\alpha} + M.$$

Here M does not depend on $z_0 \in \Omega$. Therefore from the inequality $d_\zeta^{-\alpha} \leq \hat{d}_\zeta^{-\alpha}$ it follows that the pair $(\exp d_z^{-\alpha},\ d_z^{1+\alpha/2})$ belongs to $\Sigma_{\log}(\Omega)$. Thus, the sufficiency of condition (3.22) is obtained at once from Theorem 2.1. The theorem is proved.

Remark. The compactness condition in Theorems 3.2 and 3.3 can be written in the standard manner [see (3.1) and (3.2)].

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*For $\zeta = z_0$, $|\hat{z}_0 - z_0| = d_{z_0} = \hat{d}_{z_0} = \tilde{d}_{z_0}$.