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A Practicable γ_5-scheme in dimensional regularization

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Abstract. We present a new simple γ_5 regularization scheme. We discuss its use in the standard radiative correction calculations including the anomaly contributions. The new scheme features an anticommuting γ_5 which leads to great simplifications in practical calculations. We carefully discuss the underlying mathematics of our γ_5-scheme which is formulated in terms of simple projection operations.

1 Introduction

Local quantum field theories are plagued with infinities. Often all the infinites can be absorbed into the parameters (couplings and masses) of the Lagrangian – we then speak of renormizable theories. Before the process of renormalization can be started the divergent Feynman integrals must be regulated. If possible the regularization should respect all symmetries of the bare theory, such as gauge invariance, Bose symmetry, and Ward identities. An optimal scheme in this respect is regularization through dimensional continuation [1]. For parity conserving amplitudes the scheme is extremely efficient: one evaluates the Feynman graphs in D dimensions only at the end of the calculation. If traces over Dirac matrices are involved D must be even to preserve the usual Clifford algebra of γ-matrices. To fully preserve gauge invariance a physical renormalization scheme should be used, such as on shell renormalization, and infrared divergences must also be evaluated in D dimensions. Then even the wave function renormalization is gauge invariant [2].

This beautiful and practical scheme, however, fails for parity violating amplitudes involving the Dirac matrix γ_5 because one cannot continue to $D \neq 4$ dimensions traces of the form

$$\mathrm{tr}(\gamma_5 \gamma_{\mu_1} \dots \gamma_{\mu_2}) = 4 i \varepsilon_{\mu_1 \dots \mu_4}, \qquad (1)$$

as the totally antisymmetric ε-tensor is a purely 4-dimensional object. To over-come this difficulty one of us proposed to redefine the trace operation using a projection on four dimensional subspace (which agrees with the usual trace operation for $D=4$) [3]. The price to pay for this definition of 'trace' is that cyclicity is no longer valid in γ_5-odd traces. This scheme can be used consistently to regularize UV-divergences. We show in this paper how the scheme may be extended to also regularize infrared (IR) and collinear (M) divergences. Since a practical and consistent regularization scheme of divergences in parity conserving *and* parity violating amplitudes is desirable for the radiative correction calculations in all sections of the Standard Model we present a practical list of rules in Sect. 2, which, when adhered to scrupulously, will guarantee correct results. The rules are simple and can easily be implemented in algebraic programs such as REDUCE. The only other consistent γ_5-regularization schemes, at present, are the schemes favouring a non-vanishing anticommutator $\{\gamma_5, \gamma_\mu\} \neq 0$. This class of schemes is discussed in general in [4] and [5]. An explicit version is the well-known scheme of 't Hooft and Veltman and Breitenlohner and Maison [1, 6]. This scheme distinguishes 4-dimensional and $(D-4)$-dimensional objects, creates spurious anomalies and quite generally constitutes a nightmare to anybody involved in practical Standard Model calculations.

An example in point is the calculation of the one-loop flavour changing neutral current (FCNC) vertices in the Standard Model [7]. Here the BM scheme does not satisfy the naive Ward identities and a FCNC counterterm must be introduced by hand into the Lagrangian.

In the third section we present non-trivial examples of practical calculations and demonstrate how the rules are applied. We recommend these examples (including their evaluation by an algebraic computer program such as REDUCE) as a benchmark for future schemes of dimensional regularization.

In Sect. 4 we discuss the use of our γ_5-scheme in the context of IR/M singularities where we emphasize that one has to choose the same reading point when one evaluates IR/M singular tree and loop graph contributions. We present an explicit sample calculation, namely

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the calculation of the $\mathcal{O}(\alpha_s)$ radiative corrections to the parity-odd asymmetry in e^+e^--annihilation into two massless jets (partons). Section 5, finally, contains our conclusions. In an Appendix we present the theoretical underpinnings of our γ_5-scheme. We discuss in particular the behaviour of an action $S=\int \bar{\psi} \not{D} \psi$ under local infinitesimal chiral gauge transformations $\psi'=(1-\mathrm{i}\Theta(x)\gamma_5)\psi$, $\bar{\psi}'=\bar{\psi}(1-\Theta(x)\gamma_5)$ in order to pinpoint the origin of anomalies in the context of our γ_5-scheme.

2 The rules

For the convenience of the reader we present in this section the rules for handling Dirac matrices in our dimensional regularization scheme:

Rule I. (1) Anticommutation relations

$$\{\gamma_\mu, \gamma_\nu\}=2g_{\mu\nu}, \tag{2}$$

$$\{\gamma_\mu, \gamma_5\}=0. \tag{3}$$

From these follow the usual contraction rules, e.g.

$$\gamma_\mu\gamma_\alpha\gamma^\mu=(2-D)\gamma_\alpha, \tag{4}$$

$$\gamma_\mu\gamma_\alpha\gamma_\beta\gamma^\mu=(D-4)\gamma_\alpha\gamma_\beta+4g_{\alpha\beta}. \tag{5}$$

Rule II. Algebraic relations for traces of strings of Dirac matrices

$$\mathrm{Tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_{2n-1}})=0,$$

$$\mathrm{Tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_{2n}})=4\sum_{\text{perm}}(-1)^{\sigma(\text{perm})}g_{\mu_{i_1}\mu_{j_1}}\ldots g_{\mu_{i_n}\mu_{j_n}}, \tag{6}$$

$1=i_1<\ldots<i_n, \quad i_k<j_k,$
where perm means permutation of $i_1 j_1 \ldots i_n j_n$

and

$$\mathrm{Tr}(\gamma_{\mu 1}\ldots\gamma_{\mu_{2n-1}}\gamma_5)=0, \tag{7}$$

$$\mathrm{Tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_4}\gamma_5)=4\mathrm{i}\varepsilon_{\mu_1\ldots\mu_4} \tag{8}$$

$$\mathrm{Tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_{2n}}\gamma_5)=4\mathrm{i}\sum_{\text{perm}}(-1)^{\sigma(\text{perm})}\varepsilon_{\mu_{i_{n+1}}\mu_{i_{n+2}}\mu_{j_{n+1}}\mu_{j_{n+2}}}\times \times g_{\mu_{i_1}\mu_{j_1}}\ldots g_{\mu_{i_{n+2}}\mu_{j_{n+2}}}, \tag{9}$$

$1=i_1<\ldots<i_{n+2}, \quad i_k<j_k,$
where perm means permutation of $i_1 j_1 \ldots i_{n+2} j_{n+2}$

traces of reversed strings of γ-matrices:

$$\mathrm{Tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_n})=\mathrm{Tr}(\gamma_{\mu_n}\ldots\gamma_{\mu_1}), \tag{10}$$

$$\mathrm{Tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_n}\gamma_5)=\mathrm{Tr}(\gamma_{\mu_n}\ldots\gamma_{\mu_1}\gamma_5), \tag{11}$$

where $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ is the 4-dimensional ε-tensor, in more explicit notation $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}=\varepsilon^{0,1,2,3}_{\mu_1\mu_2\mu_3\mu_4}$.

Rule III. It is forbidden to use cyclicity in traces involving odd number of γ_5 matrices.

Rule IV. If there are several diagrams contributing to a given process all traces must be read starting at the same vertex, called the *reading point*. This rule applies also to traces resulting from squared fermionic amplitudes.

Rule V. In theories with anomalous axial currents the trace of an anomalous graph must be read starting from an axial vector vertex in order to fulfill the usual convention of conserved vector currents. In the case of several axial vector vertices a symmetric choice of the reading prescription must be used.

3 Some comments and examples

In this section we would like to present some comments and examples to the rules given in the previous section.

Rule II. We distinguish two cases:

Case i. traces not including γ_5 or, equivalently, because of anticommutativity (3), an even number of γ_5.

Case ii. traces including one γ_5 or, equivalently, an odd number of γ_5.

Using the language of the calculus of forms we expand every string of Dirac matrices in terms of a complete set consisting of the unit matrix and anti-symmetric (wedge) products of Dirac matrices (exterior expansion). As an example consider the exterior expansion for strings of two and four Dirac matrices. One has

$$\gamma_\mu\gamma_\nu=g_{\mu\nu}1+\tfrac{1}{2}[\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu]=g_{\mu\nu}1+\gamma_\mu\wedge\gamma_\nu, \tag{12}$$

and

$$\begin{aligned}\gamma_{\mu_1}\ldots\gamma_{\mu_4}=&(g_{\mu_1\mu_2}g_{\mu_3\mu_4}-g_{\mu_1\mu_3}g_{\mu_2\mu_4}+g_{\mu_1\mu_4}g_{\mu_2\mu_3})1\\&+g_{\mu_3\mu_4}\gamma_{\mu_1}\wedge\gamma_{\mu_2}-g_{\mu_2\mu_4}\gamma_{\mu_1}\wedge\gamma_{\mu_3}\\&+g_{\mu_2\mu_3}\gamma_{\mu_1}\wedge\gamma_{\mu_4}-g_{\mu_1\mu_2}\gamma_{\mu_3}\wedge\gamma_{\mu_4}\\&-g_{\mu_1\mu_3}\gamma_{\mu_2}\wedge\gamma_{\mu_4}-g_{\mu_1\mu_4}\gamma_{\mu_2}\wedge\gamma_{\mu_3}\\&+\gamma_{\mu_1}\wedge\gamma_{\mu_2}\wedge\gamma_{\mu_3}\wedge\gamma_{\mu_4}.\end{aligned} \tag{13}$$

The wedge product $\wedge$ is defined in the Appendix.

Case i. The D-dimensional trace is defined as the projection on the unit matrix times the trace of the unit matrix which can be chosen to be four without loss of generality. This agrees with the usual definition of the trace and cyclicity holds. With the help of this definition one finds immediately for the examples (12) and (13)

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu)=4g_{\mu\nu}, \tag{14}$$

and

$$\mathrm{Tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_4})=4(g_{\mu_1\mu_2}g_{\mu_3\mu_4}-g_{\mu_1\mu_3}g_{\mu_2\mu_4}+g_{\mu_1\mu_4}g_{\mu_2\mu_3}). \tag{15}$$

The trace of an odd number of Dirac matrices is zero because the corresponding string contains no unit matrix in its exterior expansion.

Case ii. To motivate our trace definition in this case let us first discuss the corresponding $D=4$ situation:

In the trace of γ_5 with a string of Dirac matrices expand the string according to the exterior basis to see that the surviving term is not the term ~ 1, but the term of maximal antisymmetry, which is proportional to γ_5. So the string itself contains a γ_5 matrix which, together with the γ_5 in the trace, gives a term with a nonvanishing trace because of $\gamma_5^2=1$. This justifies to regard the trace operation in this case as a projection on the $\gamma_5=-\mathrm{i}\gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3=-\frac{\mathrm{i}}{4!}\varepsilon_{\mu_2\ldots\mu_4}\gamma^{\mu_1}\ldots\gamma^{\mu_4}$ content of the string.

For the example of four γ-matrices one has explicitly

$$\begin{aligned}\mathrm{tr}(\gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}) &= \mathrm{tr}(\gamma_5((g_{\mu_1\mu_2}g_{\mu_3\mu_4}-g_{\mu_1\mu_3}g_{\mu_2\mu_4}\\ &\quad +g_{\mu_1\mu_4}g_{\mu_2\mu_3})1\\ &\quad +g_{\mu_3\mu_4}\gamma_{\mu_1}\wedge\gamma_{\mu_2}-g_{\mu_2\mu_4}\gamma_{\mu_1}\wedge\gamma_{\mu_3}\\ &\quad +g_{\mu_2\mu_3}\gamma_{\mu_1}\wedge\gamma_{\mu_4}+g_{\mu_1\mu_2}\gamma_{\mu_3}\wedge\gamma_{\mu_4}\\ &\quad -g_{\mu_1\mu_3}\gamma_{\mu_2}\wedge\gamma_{\mu_4}+g_{\mu_1\mu_4}\gamma_{\mu_2}\wedge\gamma_{\mu_3}\\ &\quad +\gamma_{\mu_1}\wedge\gamma_{\mu_2}\wedge\gamma_{\mu_3}\wedge\gamma_{\mu_4}))\\ &= \mathrm{tr}(\gamma_5(\gamma_{\mu_1}\wedge\gamma_{\mu_2}\wedge\gamma_{\mu_3}\wedge\gamma_{\mu_4}))\\ &= \mathrm{i}\varepsilon_{\mu_1\ldots\mu_4}\mathrm{tr}(\gamma_5\gamma_5)\\ &= \mathrm{i}\varepsilon_{\mu_1\ldots\mu_4}\mathrm{tr}(1),\end{aligned}$$

where we used the definition of the Levi-Civita tensor $\varepsilon_{\mu_1\ldots\mu_4}$ and the definition of the $\wedge$-product (defined in the Appendix) to make the replacement $\gamma_{\mu_1}\wedge\gamma_{\mu_2}\wedge\gamma_{\mu_3}\wedge\gamma_{\mu_4}=\mathrm{i}\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\gamma_5$.

Turning to $D\neq 4$ we take the same projection as the definition of the D-dimensional trace. We then arrive at Rule II immediately.

Note that (9) can also be written as

$$\mathrm{Tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_{2_n}}\gamma_5)=-4\mathrm{i}\varepsilon^{\mu_{i_{n+1}}\mu_{i_{n+2}}\mu_{j_{n+1}}\mu_{j_{n+2}}}\times \\ \times\mathrm{tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_{2(n+2)}}). \tag{16}$$

This gives the following simple prescription for calculating such traces in a symbolic math program as REDUCE. One only has to contract an ε-tensor with a trace not involving γ_5, which is possible even away from 4 dimensions ('vecdim $=D$'). See the Appendix for further details.

As a consequence of this projection the ε-tensor is a 4-dimensional object which in contraction identities gives only 4-dimensional metric tensors, e.g.

$$\varepsilon_{\alpha_1\ldots\alpha_4}\varepsilon^{\alpha_1\alpha_2}{}_{\beta_3\beta_4}=2(g^{(4)}_{\alpha_3\beta_3}g^{(4)}_{\alpha_4\beta_4}-g^{(4)}_{\alpha_3\beta_4}g^{(4)}_{\alpha_4\beta_3}), \tag{17}$$

(where $g^{(4)}_{\mu\nu}=\mathrm{diag}[1,-1,-1,-1]$) and

$$g^{(4)\beta}_{\alpha}g_{\beta\gamma}=g^{(4)}_{\alpha\gamma},\quad g^{(4)\alpha}_{\alpha}=4. \tag{18}$$

Rule III. According to Rule II the trace of γ_5 multiplied by four γ-matrices is cyclic. The first example where the non-cyclicity of the D-dimensional trace enters is in strings of six γ-matrices and one γ_5,

$$\mathrm{Tr}(\gamma_5\gamma_{\mu_1}\ldots\gamma_{\mu_6})-\mathrm{Tr}(\gamma_{\mu_6}\gamma_5\gamma_{\mu_1}\ldots\gamma_{\mu_5})=:A_{\mu_1\ldots\mu_6}. \tag{19}$$

According to (9)

$$A_{\mu_1\ldots\mu_6}=8\mathrm{i}(g_{\mu_1\mu_6}\varepsilon_{\mu_2\mu_3\mu_4\mu_5}-g_{\mu_2\mu_6}\varepsilon_{\mu_1\mu_3\mu_4\mu_5}+g_{\mu_3\mu_6}\varepsilon_{\mu_1\mu_2\mu_4\mu_5}\\ -g_{\mu_4\mu_6}\varepsilon_{\mu_1\mu_2\mu_3\mu_5}+g_{\mu_5\mu_6}\varepsilon_{\mu_1\mu_2\mu_3\mu_4}). \tag{20}$$

Evidently $A_{\mu_1\ldots\mu_6}$ is a tensor which is antisymmetric in five of its indices and consequently vanishes in four dimensions but not in $D\neq 4$ dimensions. The contraction

$$g^{\mu_6\mu_1}A_{\mu_1\mu_2\ldots\mu_6}=8\mathrm{i}(D-4)\varepsilon_{\mu_2\ldots\mu_5} \tag{21}$$

underscores the above assertion that the D-dimensional trace cannot be taken to be cyclic.

Contracting the above tensor A with a pair of indices which does not involve μ_6 gives a vanishing result.

Incidentally these arguments show that in a n-dimensional field theory (n even) the first anomalous diagram must involve γ_5 and a string of $(n+2)$ γ-matrices. This implies that the $\left(\frac{n}{2}+1\right)$-point function is the first candidate for an anomalous Greens function.

Rule IV. This rule is contingent to the generation of anomaly structures with respect to the convention of conserved vector currents. Starting the trace at the axial-vector current allocates the anomaly at the axial-vector current and conserves vector currents. The non-locality of the anomaly finds it counterpart in other choices of the reading point. This ambiguity is a typical property of anomalies and is (and must be) present in all regularization prescriptions. A unique result for anomalous graphs can only be obtained by physical conditions (e.g. vector-current conservation) and not by the chosen regularization [8].

The AVV anomaly of the triangle graph $I_{\varrho\mu\nu}$ of Fig. 1 is given (for a single flavor) by

$$(p+q)_\varrho I^{\varrho\mu\nu}=\int\frac{\mathrm{d}^n l}{(2\pi)^4}\frac{\mathrm{Tr}(\gamma_5(\not p+\not q)\not l\gamma_\nu(\not l+\not q)\gamma_\mu(\not l+\not p+\not q))}{l^2(l+q)^2(l+p+q)^2}\\ +[\mu\leftrightarrow\nu,\ p\leftrightarrow q], \tag{22}$$

where the notation is explained in the figure. Note that if one erroneously assumes cyclicity of the trace and an anticommuting γ_5 the trace and thereby the anomaly in (22) obviously vanishes. Without cyclicity as in our scheme the trace can be evaluated directly by Rule II above, yielding

$$(p+q)_\varrho I^{\varrho\mu\nu}=-16A_{\mu\nu\varrho\sigma\alpha\beta}p^\varrho q^\sigma\times\\ \times\int\frac{\mathrm{d}^n l}{(2\pi)^4}\frac{l^\alpha l^\beta}{l^2(p+l)^2(l+p+q)^2}\\ +[\mu\leftrightarrow\nu,\ p\leftrightarrow q] \tag{23}$$

where the six-component tensor $A_{\mu\nu\varrho\sigma\alpha\beta}$ is defined in (19).

Only the coefficient of $g_{\alpha\beta}$ in the above integral is UV divergent and with the help of this and (19) we obtain the well known result [9]

$$(p+q)_\varrho I^{\varrho\mu\nu}=\frac{\mathrm{i}}{2\pi^2}\varepsilon_{\mu\nu\varrho\sigma}p^\varrho q^\sigma. \tag{24}$$

Choosing one of the vector vertices as the reading point the anomaly can be shifted to that vertex. Rule IV (reading a trace starting with the axial vertex as the reading point) can be ignored in the standard model because of anomaly cancellation after fermion summation. Rule III (reading all diagrams starting at the same

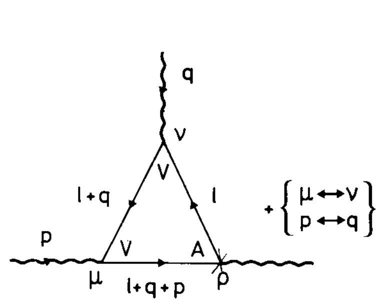

Fig. 1. The triangle VVA anomaly graph. The reading point where the trace reading is started is denoted by a cross

vertex) must be followed scrupulously, however. Otherwise wrong results may be obtained in the full $SU(3)\times SU(2)\times U(1)$ sector. An example in case is the decay $Z_0 \to GG\gamma$ (Fig. 2). The axial vector contribution should vanish because of Bose symmetry. Starting the trace at different vertices in the various graphs results in general in a violation of Bose symmetry and leads to a false anomaly (which would destroy renormalizability) even after fermion summation.

Rule V. As an example for the case of several axial vector vertices let us discuss the AAA triangle anomaly. In the following we indicate the reading point by typing an underlined letter for the corresponding vertex. Then a Bose symmetric reading prescription is given by $\frac{1}{3}(\underline{A}AA + A\underline{A}A + AA\underline{A})$ and gives the well-known result for this anomaly ($\frac{1}{3}$ of the AVV anomaly).

To see this note that each of the readings involves the A-tensor. The three reading possibilities differ by permutations of this tensor. Only one of the permutations does not vanish after contracting with the metrical tensor coming from integration (see the comments to Rule III above).

Calculating the AAA anomaly just by choosing one of its vertices as the reading point, for example $\underline{A}AA$, breaks Bose symmetry and gives three times the usual result for the anomaly at the reading point vertex and no anomaly at the other two vertices.

The Rules IV and V are ad hoc rules. This is a consequence of the fact that anomalous Green functions cannot be uniquely obtained by a regularization procedure. So some physical input is unavoidably necessary [8]. In the case of several axial vector vertices the choice of one reading point may break a permutation symmetry between these axial vector vertices (see the above example). But a symmetrized combination of reading points defines a unique reading prescription respecting this symmetry. This gives Rule V for the case of several axial vertices.

Finally we would like to point out that in practical Feynman diagram calculations one can expand the amplitude (even if divergent) into the usual set of covariants and project out the invariants as usual. This is guaranteed by the orthogonality of the four dimensional subspace and its complement, see the appendix.

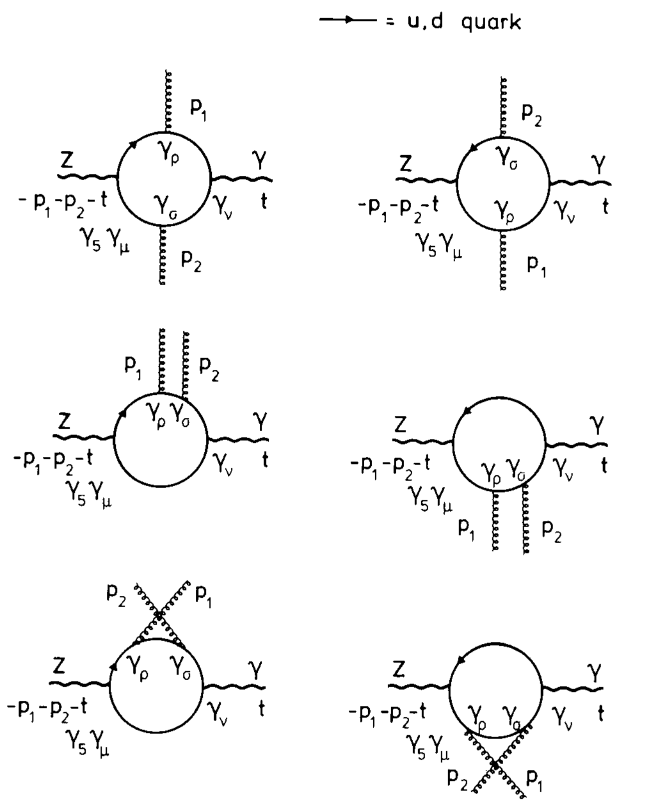

Fig. 2. Anomaly graphs contributing to the decay $Z_0 \to \mathcal{G}\mathcal{G}\gamma$. Gluons are the curly lines, $\rightarrow$ = u, d quark

4 An application to IR/M singularities

Most of the previous discussions on the γ_5-problem in dimensional regularization in the literature have been concerned with the UV divergent sector. It goes without saying that it is of immense practical interest* to give a consistent γ_5-prescription also in the IR/M sector when doing radiative correction calculations.

In fact, it is well known that the γ_5-anomalies that appear in connection with UV singularities have their direct analogues in the appearance of γ_5-anomalies in the IR/M sector [10, 11]. The spurious IR/M anomalies are expected to cancel among loop and tree contributions just as the true IR/M singularities do. For this cancellation to be effected the tree and loop contributions have to be read starting from the same vertex in the fermion loops.

It is the purpose of this section to illustrate the use of our γ_5-scheme in a simple example involving the $\mathcal{O}(\alpha_s)$ radiative corrections to the parity-odd asymmetry in e^+e^- annihilation into 2 massless jets (or partons). This simple radiative correction calculation serves to exemplify what happens when IR/M singularities appear in conjunction with D-dimensional γ_5 manipulations.**

The relevant parity violating (p.v.) hadron tensor $\mathscr{H}^{\text{p.v.}}_{\mu\nu}$ is defined as

$$\mathscr{H}^{\text{p.v.}}_{\mu\nu} = \tfrac{1}{2}(\mathscr{H}^{AV}_{\mu\nu} + \mathscr{H}^{VA}_{\mu\nu}), \tag{25}$$

where

$$\mathscr{H}^{VA}_{\mu\nu} = \sum_{\text{spins}} \langle f | J^V_\mu(0) | 0\rangle \langle f | J^A_\nu(0) | 0\rangle^*, \tag{26}$$

and correspondingly for $\mathscr{H}^{AV}_{\mu\nu}$.

Due to using an anticommuting γ_5 the p.v. hadron tensor simplifies to ($\mathscr{H}^{AV}_{\mu\nu} = \mathscr{H}^{VA}_{\mu\nu}$)

$$\mathscr{H}^{\text{p.v.}}_{\mu\nu} = \mathscr{H}^{VA}_{\mu\nu}. \tag{27}$$

Let us first list the $e^+e^- \to q(p_1)\bar{q}(p_2)g(p_3)$ four point tree graph contribution that enters the phase space integration in the $\mathcal{O}(\alpha_s)$ radiative correction. The Feynman diagrams are read starting with the left vertex as indicated in Fig. 3

* e.g. to supply masses to IR-divergent propagators is full of difficulties beyond the one-loop level

** It is well-known that the sum of the $\mathcal{O}(\alpha_s)$ one-loop contributions are UV convergent

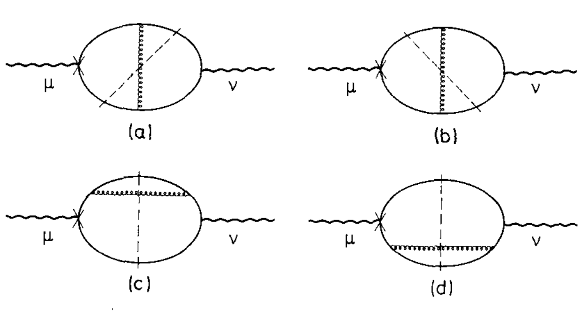

Fig. 3a–d. $\mathcal{O}(\alpha_s)$ tree graphs contributing to $e^+e^- \to q\bar{q}g$. The reading point of the trace is indicated by a cross

by a cross. One finds (in units of $g^2 N_C C_F$)

$$\mathcal{H}_{\mu\nu}^{\text{p.v.}} = \mathrm{Tr}\left(\gamma_\mu \not{p}_2 \left(\gamma_\nu \gamma_5 \frac{\not{p}_1 + \not{p}_3}{s_{13}} \gamma_\beta - \gamma_\beta \frac{\not{p}_2 + \not{p}_3}{s_{23}} \gamma_\nu \gamma_5\right) \times \times \not{p}_1 \gamma_\beta \frac{\not{p}_1 + \not{p}_3}{s_{13}}\right) + \mathrm{Tr}\left(\gamma_\mu \frac{\not{p}_2 + \not{p}_3}{s_{23}} \gamma_\beta \not{p}_2 \times \times \left(-\gamma_\nu \gamma_5 \frac{\not{p}_1 + \not{p}_3}{s_{13}} \gamma_\beta + \gamma_\beta \frac{\not{p}_2 + \not{p}_3}{s_{23}} \gamma_\nu \gamma_5\right) \not{p}_1\right), \tag{28}$$

where we have defined $s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j$.

Using our trace rules (no cyclicity!) one finds

$$\mathcal{H}_{\mu\nu}^{\text{p.v.}} = \frac{-8i}{q^2}\left(\frac{x_1}{(1-x_1)(1-x_2)} \varepsilon(\mu\nu q p_1) - (1 \leftrightarrow 2) - \frac{(4-D)}{2}\left[\left(\frac{1}{1-x_1} + \frac{1}{1-x_2}\right)(2\varepsilon(\mu\nu p_1 p_2) + \varepsilon(\mu\nu q p_1) - \varepsilon(\mu\nu q p_2)) - \frac{1}{q^2} \frac{1}{(1-x_1)(1-x_2)} \varepsilon(\mu q p_1 p_2) 4 p_{3\nu}\right], \tag{29}$$

where $q^2 x_i = 2p_i \cdot q$, $q = p_1 + p_2 + p_3$ and $\varepsilon(\mu\nu p_i p_j) \equiv \varepsilon_{\mu\nu\alpha\beta} p_i^\alpha p_j^\beta$ etc..

Note that the 4-dimensional contribution in (29) obeys current conservation $q^\mu \mathcal{H}_{\mu\nu}^{\text{p.v.}} = q^\nu \mathcal{H}_{\mu\nu}^{\text{p.v.}} = 0$. Also one has the charge conjugation relations $\mathcal{H}_{\mu\nu}^{\text{p.v.}} = -\mathcal{H}_{\nu\mu}^{\text{p.v.}}$ and $\mathcal{H}_{\mu\nu}^{\text{p.v.}}(p_1, p_2) = -\mathcal{H}_{\mu\nu}^{\text{p.v.}}(p_2, p_1)$ for the 4-dimensional piece. The terms proportional to $(D-4)$ show an anomalous behaviour in so far as $q^\nu \mathcal{H}_{\mu\nu}^{\text{p.v.}} \neq 0$ and there is no antisymmetry under $\mu \leftrightarrow \nu$. Depending on where the trace reading is started anomalous features can show up also on the other current index ν or in the violation of the $p_1 \leftrightarrow p_2$ antisymmetry.

Let us now integrate the tree graph contribution over the complete x_1, x_2 phase space region. The relevant D-dimensional integration measure can be found in [12]. One has

$$\mathcal{H}_{\mu\nu}^{\text{p.v.}}(\text{tree}, P_1, P_2) = \frac{q^2}{16\pi^2}\left(\frac{4\pi\mu^2}{q^2}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \times [(1-x_1)(1-x_2)(1-x_3)]^{-\varepsilon} \times \times \mathcal{H}_{\mu\nu}^{\text{p.v.}}(p_1, p_2, p_3), \tag{30}$$

where P_1 and P_2 $(P_1 + P_2 = q)$ are the quark and antiquark momenta for the three point process $e^+e^- \to q(P_1)\bar{q}(P_2)$ and $\varepsilon = (D-4)/2$.

The integrations in (30) are performed by first extracting the ε-tensor from the integrand. Then one integrates the remaining tensor integrand by standard D-dimensional techniques. By proceeding in this manner one never has to pay attention to the four-dimensional projection implied by the $\varepsilon_{\alpha\ldots\delta}$-contraction in intermediate steps of the calculation.

The nested integration in (30) can be factored into two pieces by the substitution $(1-x_2) = v x_1$. One can orient $\mathbf{p}_1$ along $\mathbf{P}_1$ without loss of generality such that $p_1 = x_1 P_1$. The first contribution in (29) can then easily be integrated to give

$$\varepsilon_{\mu\nu\alpha\beta} q^\alpha P_1^\beta \frac{1}{\Gamma(1-\varepsilon)} \int_0^1 dx_1 \int_0^1 dv x_1^{1-2\varepsilon} \times \times (1-x_1)^{-\varepsilon} v^{-\varepsilon} (1-v)^{-\varepsilon} \frac{x_1}{(1-x_1)v} = = \frac{1}{\Gamma(1-\varepsilon)}\left(\frac{1}{\varepsilon^2} + \frac{3}{2} \cdot \frac{1}{\varepsilon} + \frac{17}{4} - \frac{\pi^2}{2}\right) \varepsilon(\mu\nu q P_1). \tag{31}$$

For the second contribution in (29) one has

$$-\varepsilon_{\mu\nu\alpha\beta} q^\alpha \frac{1}{\Gamma(1-\varepsilon)} \int_0^1 dx_1 \int_0^1 dv x_1^{1-2\varepsilon} (1-x_1)^{-\varepsilon} v^{-\varepsilon} (1-v)^{-\varepsilon} \times \frac{1-v x_1}{(1-x_1) v x_1} p_2^\beta = \frac{1}{\Gamma(1-\varepsilon)}\left(\frac{1}{\varepsilon^2} + \frac{3}{2} \cdot \frac{1}{\varepsilon} + \frac{11}{4} - \frac{\pi^2}{2}\right) \times \times \varepsilon(\mu\nu q P_1) \tag{32}$$

The tensor integral in (32) has been done by expanding the integral as usual along the outer momentum P_1 and q, i.e..

$$\frac{1}{\Gamma(1-\varepsilon)} \int_0^1 dx_1 \int_0^1 dv x_1^{1-2\varepsilon} (1-x_1)^{-\varepsilon} v^{-\varepsilon} \times \times (1-v)^{-\varepsilon} \frac{1-v x_1}{(1-x_1) v x_1} p_2^\beta = A P_1^\beta + B_q^\beta. \tag{33}$$

For our purpose the coefficient A is of interest. It can be projected out by contraction with $2(q_\beta - 2P_{1\beta})/q^2$. The integrand is thereby scalarized. One obtains the scalar products $p_2 \cdot q = x_2 q^2/2$ and $p_2 \cdot P_1 = x_2(1-\cos\Theta_{12}) \cdot q^2/4 = -(1 - x_1 + v x_2 - 1) q^2/2x_1$ where $\cos\Theta_{12}$ is the polar angle between $\mathbf{p}_1$ and $\mathbf{p}_2$.

The remaining integrals in (29) can be done along similar lines. Note that the last term in (29) involving a second rank tensor integrand has no finite contribution to $\mathcal{H}_{\mu\nu}^{\text{p.v.}}(\text{tree}, P_1 P_2)$. Adding all contributions one has (in units of $g^2 C_F N_C$)*

$$\mathcal{H}_{\mu\nu}^{\text{p.v.}}(\text{tree}, P_1, P_2) = \frac{-1}{2\pi^2}\left(\frac{4\pi\mu^2}{q^2}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \times \times \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8 - \pi^2\right) \varepsilon(\mu\nu q P_1). \tag{34}$$

* If the trace (28) would have been started at one of the outer fermion lines the resulting integrated tree graph contribution would be the same as in (34) except for the replacement $8 \to 7$. The loop result with the corresponding trace reading is obtained from (35) by the replacement $-8 \to -7$. Such a trace reading cannot be advocated, though, as it entails an unnecessary complication in the calculation of the fermionic self energy parts in the loop contributions because these loop contributions would become 'cut open'

One notes that the anomalous terms in (29) have vanished after integration. The result (34) is $\mu \leftrightarrow \nu$ and quark $\leftrightarrow$ antiquark antisymmetric as well as conserved in both current indices μ and ν.

Note though, that the vanishing of the anomalous pieces after IR/M integration is specific to this simple example. The γ_5-odd tree graph contributions to higher n-point functions retain an anomaly structure even after IR/M integration as will be discussed later on.

The $\mathcal{O}(\alpha_s)$ one loop contribution to the three point function $e^+e^- \to q(P_1)\bar{q}(P_2)$ can be done using standard loop integrals (see e.g. [11]). Note that the loop contribution corresponding to cutting Fig. 3 to the right of the gluon line may not be simply obtained from the well-known three point loop amplitude as the fermion string is 'cut open' at the vertex by the given trace reading prescripion as Fig. 4a shows. One finally obtains*

$$\mathscr{H}^{\text{p.v.}}_{\mu\nu}(\text{loop}, P_1, P_2) = \frac{1}{2\pi^2}\left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon}\frac{1}{\Gamma(1-\varepsilon)} \times \left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+8-\pi^2\right)\varepsilon(\mu\nu q P_1). \quad (35)$$

Thus the sum of the one-loop and tree level radiative corrections (34) and (35) exactly cancel. This proves the absence of $\mathcal{O}(\alpha_s)$ radiative corrections to the p.v. asymmetry in $e^+e^- \to q\bar{q}$ for mass zero quarks. This result was derived before in [11] using BM γ_5-scheme. The same result can be found in [13, 14] although the authors of [13, 14] never specify their γ_5 rules despite of using dimensional regularization. An incorrect result in quoted in [15].

The tree-level $\mathcal{O}(\alpha_s)$ contribution can be written in a more symmetric way by taking the mean of the two trace results starting the reading of the trace at the left (μ) and right (ν) vertex as drawn in Fig. 5. In this way the corresponding loop contributions can be taken directly from the well-known loop amplitudes.

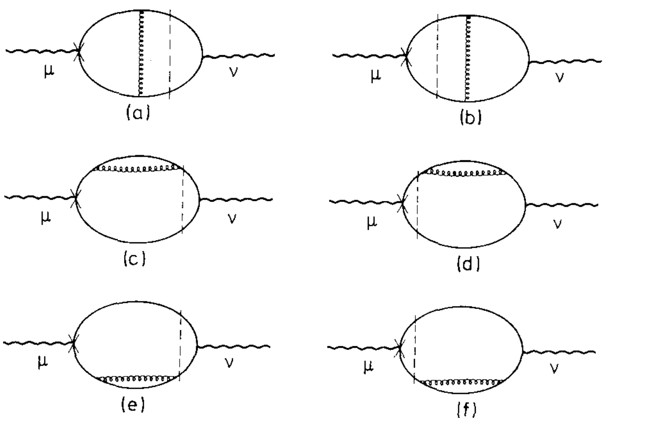

Fig. 4a–f. $\mathcal{O}(\alpha_s)$ loop contributions to $e^+e^- \to q\bar{q}g$. The reading point of the trace is indicated by a cross

It is not difficult to see that the γ_5-odd radiative correction calculation done in [11] within the BM γ_5-scheme corresponds exactly to latter symmetrized version of our reading prescritption. The authors of [10] calculated the $\mathcal{O}(\alpha_s^2)$ radiative corrections to the p.v. measures in the four point process $e^+e^- \to 3$ jets (or partons). Differing from the above example the resulting p.v. tree graph and one-loop 3-jet hadron tensors turn out to have an anomaly structure even after IR/M integration (at $\mathcal{O}(1/\varepsilon)$ *and* in the finite contributions!). The anomalous pieces do, however, cancel among loop and tree contributions.

5 Conclusions

The γ_5-scheme presented here belongs to the class of *non-cyclicity*-schemes. With respect to recent criticism [16] let us make some clarifying remarks.

- It is possible to derive all our results for traces without using cyclicity just by *Clifford algebra* rules. In particular the traces not involving a γ_5 turn out to be cyclic by this derivation. Non-cyclic traces always appear together with calculations involving anomalies in the wider sense, that is including infrared anomalies as discussed in the previous section. Calculations avoiding such anomalous terms can make use of the 'naive' γ_5 as in four dimensions. This applies to most of the Standard Model calculations.
- Our scheme does not need infinite-dimensional representations for γ-matrices but allows a straightforward generalization to this case without any modifications. This may be regarded as a conceptual advantage.
- Bose symmetry for non-anomalous graphs is obvious. Bose symmetry in anomalous graphs is not violated in our scheme as long as the reading prescription is chosen not to break this symmetry. For instance in the usual *AVV* triangle anomaly the two vector vertices must be treated symmetrically. The reading Rule IV is chosen to respect Bose symmetry.
- Concerning UV-singularities the question arises if dimensional regularization with our γ_5 prescription is in agreement with the requirements of renormalization. For

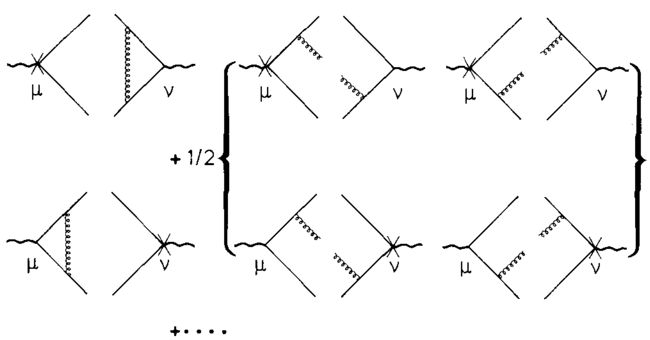

Fig. 5. Symmetrical choice of the reading point for $\mathcal{O}(\alpha_s)$ radiative correction calculations. The fermion string of the 3-point loop amplitudes is not 'cut open'. The symmetrized graphs corresponding to Figs. 3c, 3d and 4c, 4f are not shown as they are trivially symmetric

* A different result is obtained for the p.v. one-loop contribution in the BM γ_5 regularization scheme. In fact the vector current and axial vector current one-loop amplitudes are not simply related in the BM γ_5-scheme and one has to introduce an explicit counterterm to restore the chiral invariance of the theory [11]

example, in dimensional regularization together with a BM-scheme this is discussed in [6].

In a renormalizable (by power-counting) theory this question can be traced back to the question of possible violations of Ward-Identities.

It is clear that for the case of closed fermion-loops (on the amplitude level) the BM-scheme and our scheme give identical results on the one-loop level (see the Appendix). This agrees with a comparison of the anomaly calculation done in [17] for the chiral and Bardeen anomalies. It is only a matter of patience to check that the traces giving rise to anomalous contributions that originate from a non-vanishing anticommutator $\{\gamma_\mu, \gamma_5\} \neq 0$ correspond to the non-cyclicity of the graphs contributing to the involved anomalous 3-, 4- and 5-point Greens functions.

Thus the only difference may come from contributions originating from the non-vanishing anti-commutator $\{\gamma_5, \gamma_\mu\} \neq 0$ outside of traces. Such contributions identically vanish in our scheme but are commonly regarded as spurious anomalies in the context of BM-like schemes [16]. As a consequence they have to be compensated by appropriate (finite) counterterms. Similar problems arise on the multi-loop level in BM-like schemes. That the Ward-Identities may be restored in any loop-order by appropriate counterterms is an assumption which has to be proven [18].

In our scheme also on the multi-loop level there is no possible graph violating the Ward-Identities in an anomaly-free theory. This follows from a simple power-counting argument (see the Appendix).

So for the UV sector our scheme might be regarded as equivalent to a BM-scheme with the additional advantage of suppressing spurious anomalies.

- *KLN*-type cancellations of IR and collinear divergences between virtual and bremsstrahlung diagrams will always take place in our scheme for the following reason. Every loop contribution has its appropiate counterpart in the squared (tree) amplitudes which can be seen just by cutting the loop (see Fig. 3 for an example). Both the loop terms and the terms from squared amplitudes have the structure *trace (algebraic part)* $\otimes$ *infrared divergent integrals (analytic part)*. In both cases the first factor is a polynomial in $(D-4)$, the second factor a Laurent series in $(D-4)$. The cancellations will not be destroyed in any order $(D-4)$ as long as a γ_5-scheme treats the algebraic part for loops and squared amplitudes in the same manner. This is one of the reasons for rule IV) in our scheme.
- In our scheme every trace evaluation results in a sum of products of metrical tensors $g_{\mu\nu}$ and Levi-Civita tensors $\varepsilon_{\mu\nu\varrho\sigma}$. Higher antisymmetric tensors $\varepsilon_{\mu_1 \ldots \mu_k}$, $k>4$ (evanescent operators) which belong also to the expansion of big enough strings of Dirac-matrices vanish by the projection included in our D-dimensional trace. That it is possible to have a γ_5-scheme avoiding these evanescent operators supports the result in [19] that only four-dimensional covariants are necessary in the expansion of products of bilinear spinor densities. But this evanescent content may still have an implicit effect as discussed just there. In our scheme this reflects the fact that the contraction of two evanescent operators, each one vanishing in four dimensions may not vanish in four dimensions and so survives our trace projection, e.g. $\varepsilon^{\mu_1 \cdots \mu_6} \varepsilon_{\mu_1 \ldots \mu_6} \sim 1$.

In conclusion, we have developed an unambigous and viable γ_5-scheme within dimensional regularized quantum field theory. This γ_5-scheme is both simple and mathematically rigorous. Bearing practical applications in mind we have provided a comprehensive discussion of how our γ_5-scheme applies to the standard pertubative settings involving both the UV *and* IR/M sectors.

Our scheme features an anticommuting γ_5 which leads to great calculational simplifications. In addition it avoids the tedium of having to remove spurious anomalous contributions which arise in other γ_5-schemes. The price to pay for an anticommuting γ_5 is that one has to give up cyclicity of γ_5-odd traces in divergent contributions. However, we have demonstrated that the computational complications introduced by the noncyclicity of γ_5-odd traces is minimal.

The anticommuting γ_5 and the result that the anomalies are the only consequence of the γ_5-problem confirms the results of the scheme presented in [20]. The authors of [20] use the anomaly to 'normalize' their scheme. They do so because they do not give a mathematical rigorous definition of their scheme. In doing so we *end* with the correct anomaly and a well-defined prescription for handling γ_5 in dimensional regularization.

Our γ_5-scheme should alleviate the bad conscience of the practitioners of radiative corrections in the electroweak sector of the Standard Model who have traditionally employed an anticommuting γ_5 to dimensionally regularize UV singularities.

Acknowledgement. We would like to thank G. Thompson for very helpful discussions.

Appendix

It is a common property of different realizations of dimensional regularization schemes to use higher dimensional finite or infinite Clifford Algebras. The γ_5-scheme presented in this paper agrees with the one presented in [3] which was formulated for the case of infinite dimensional Clifford Algebras. In our approach we need not specify the dimension of the algebra. In the following n denotes the dimension of such an algebra where n is not necessarily finite but chosen to be even.

In order to distinguish between our trace functional and the conventional trace definition we use the notation $\mathrm{Tr}(\ldots)$ for our trace functional and $\mathrm{tr}(\ldots)$ for the conventional trace.

Consider a n-dimensional complexified Clifford algebra $\mathscr{G}^c(1, n-1)$ with the defining relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}. \tag{36}$$

The algebra contains an element γ_5 with the property

$$\{\gamma_5, \gamma_\mu\} = 0 \quad (\mu = 0, \ldots, n-1). \tag{37}$$

The existence of such a fully anticommuting element is guaranteed by the theory of Clifford algebras. We keep the notation γ_5 for this element for arbitrary n.

Every element of the Clifford algebra $\mathscr{G}^c(1,n-1)$ can be expanded according to a exterior (Grassmann) basis [21]

$$A \in \mathscr{G}^c(1,n-1) \Rightarrow A = a_0 1 + a_1^\mu \gamma_\mu + a_2^{\mu\nu,\mu<\nu}\gamma_\mu \wedge \gamma_\nu \dots \\ \dots a_n \gamma_0 \wedge \dots \wedge \gamma_{n-1}. \quad (38)$$

The wedge product $\wedge$ is defined by

$$\gamma_{i_1} \wedge \gamma_{i_2} \dots \wedge \gamma_{i_k} = \frac{1}{k!}\sum_{\text{perm}} \text{sign(perm)}\gamma_{\text{perm}(i_1)} \dots \gamma_{\text{perm}(i_k)}, \quad (39)$$

where the sum runs over all permutations from $i_1, \dots, i_k$. For example $\gamma_{\mu_1} \wedge \gamma_{\mu_2} = \frac{1}{2!}(\gamma_{\mu_1}\gamma_{\mu_2} - \gamma_{\mu_2}\gamma_{\mu_1})$.

For an anticommuting γ_5 one has to distinguish between two cases of traces of strings of γ-matrices

Case i. traces not including γ_5.

Case ii. traces including one γ_5.

In both cases the trace in *four dimension* can be regarded as a projection on one of the terms in the expansion (38).

In Case i all terms in the expansion are traceless except the first one, so the trace acts as a projection on the first term ~ 1

$$\text{tr}(A) = a_0 \text{tr}(1) = 4a_0. \quad (40)$$

For Case ii the last term (in four dimensions) in the expansion (38) is $a_4\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = a_4\gamma_5$. Expanding

$$\gamma_{\mu_1} \dots \gamma_{\mu_k} = (a_0)_{\mu_1 \dots \mu_k} 1 + \dots (a_4)_{\mu_1 \dots \mu_k}\gamma_5, \quad (41)$$

one finds

$$\text{tr}(\gamma_5\gamma_{\mu_1} \dots \gamma_{\mu_k}) = \text{tr}(\gamma_5 (a_4)_{\mu_1 \dots \mu_k}\gamma_5) \\ = (a_4)_{\mu_1 \dots \mu_k}\text{tr}(\gamma_5\gamma_5) \\ = (a_4)_{\mu_1 \dots \mu_k}\text{tr}(1). \quad (42)$$

Thus the trace acts as a projection on the $\gamma_5 = \frac{-i}{4!}\varepsilon^{0,1,2,3}_{\mu_1 \dots \mu_4} \cdot \gamma^{\mu_1} \dots \gamma^{\mu_4}$ element.

This gives a very simple solution to the γ_5-problem in dimensional regularization. Instead of taking the conventional trace in $n>4$ dimensions and operating with a purely four-dimensional, i.e. not fully anticommuting γ_5^{BM}-as in BM-type schemes-take the fully anticommuting γ_5 but replace the trace by the functional defined with the help of the above mentioned projections for the Cases i and ii. In four dimensions this *is* identical to taking the trace. However, for arbitrary n this is simply a linear functional Tr(. . .) acting on strings of γ-matrices (expanded according to (38):

$$\text{Tr}(a_0 1 + a_1^\mu\gamma_\mu + \dots) := 4a_0, \quad (43)$$

$$\text{Tr}\left(\gamma_5\left(a_0 1 + a_1^\mu\gamma_\mu + \dots + a_4^{0123}\frac{i}{4!}\varepsilon_{\mu_1 \dots \mu_k}\gamma^{\mu_1} \dots \gamma^{\mu_4} + \dots\right)\right) := 4a_4^{0123}. \quad (44)$$

Rule II follows immediately. Note that the expansion (38) involves only a finite number of terms for a string containing only a finite number of γ-matrices. Equation (6) is obviously the usual result. Equation (9) is most easily obtained by projecting out the coefficient a_4^{0123} by multiplying the string with $\varepsilon^{\mu_1 \dots \mu_4}\gamma_{\mu_1} \dots \gamma_{\mu_4}$.

These projection properties of Tr(. . .) can be easily defined also with the help of a projection operator $\mathscr{P}$ such that

$$\text{Tr}(\gamma_5\gamma_{\mu_1} \dots \gamma_{\mu_{2k}}) =: \text{tr}(\mathscr{P}(\gamma_5\gamma_{\mu_1} \dots \gamma_{\mu_{2k}})), \quad (45)$$

where $\mathscr{P}(\gamma_5) = \frac{-i}{4!}\varepsilon^{0,1,2,3}_{\mu_1 \dots \mu_4}\gamma^{\mu_1} \dots \gamma^{\mu_4} =: \gamma_5^4$ and this definition (45) makes the action of $\mathscr{P}$ unique.

As a consequence for any element $M \in \mathscr{G}^c(1,n-1)$ one has

$$\text{Tr}(\{\gamma_\mu, \mathscr{P}\gamma_5\}M) = \text{tr}(\gamma_\mu\mathscr{P}\gamma_5 M) + \text{tr}(\mathscr{P}\gamma_5\gamma_\mu M) \\ = \text{tr}(\mathscr{P}\gamma_5 M\gamma_\mu) + \text{tr}(\mathscr{P}\gamma_5\gamma_\mu M) \\ = \text{tr}(\gamma_5^4 M\gamma_\mu) + \text{tr}(\gamma_5^4\gamma_\mu M) \\ = \text{tr}(\gamma_\mu\gamma_5^4 M) + \text{tr}(\gamma_5^4\gamma_\mu M) \\ = \text{tr}(\{\gamma_\mu, \gamma_5^4\}M), \quad (46)$$

where cyclicity of the trace tr(. . .) and (45) have been used.

This shows the equivalence of the BM scheme and our scheme for the case of closed fermion loops. For example by applying the above calculation to an action:

$$S_{\text{classical}} = \int d^4x \bar{\psi}_a (D^\mu\gamma_\mu)_{ab}\psi_b = \int d^4x (D^\mu\gamma_\mu)_{ab}(\psi \otimes \bar{\psi})_{ba} \\ = \int d^4x \,\text{tr}((D^\mu\gamma_\mu)(\psi \otimes \bar{\psi})) \\ \to S_{\text{regularized}} = \int d^D x \,\text{Tr}((D^\mu\gamma_\mu)(\psi \otimes \bar{\psi})), \quad (47)$$

(where $(\psi \otimes \bar{\psi}) \in \mathscr{G}^c(1,n-1)$ by its very definition of being a tensor product of spinors) one finds in both schemes the same breaking of the chiral invariance under infinitesimal chiral gauge transformations of the kind

$$\psi \to \psi' = (1 - i\Theta(x)\gamma_5)\psi, \\ \psi - \bar{\psi} = \bar{\psi}(1 - i\Theta(x)\gamma_5).$$

Also one has immediately

$$\text{Tr}(\gamma_5) = \text{Tr}(\gamma_5\gamma_\mu) = \text{Tr}(\gamma_5\gamma_\mu\gamma_\nu) = \text{Tr}(\gamma_5\gamma_\mu\gamma_n\gamma_r) = 0, \quad (48)$$

because the strings $1, \gamma_\mu, \gamma_\mu\gamma_\nu, \gamma_\mu\gamma_\nu\gamma_\varrho$ do not contain terms $\sim \varepsilon^{0,1,2,3}_{\mu_1 \dots \mu_4}\gamma_{\mu_1} \dots \gamma_{\mu_4}$ in the expansion according to (38). Checking the consequences of this trace definition one finds that the first strings that really violate cyclicity are strings of γ_5 with six or more γ-matrices.

The reversal symmetries (10, 11) follow easily from the fact that our trace functional still commutes with transposition as the four-dimensional trace does.

Our trace functional respects the inner orthogonality of four-dimensional covariants which means that the sixteen matrices $1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_5\gamma_\mu, \gamma_5$ which are orthogonal with respect to tr(. . .) are also orthogonal with respect to Tr(. . .). All other covariants belong to its kernel. This justifies Rule V.

Using the equivalence between non-cyclic effects and effects coming from a non-vanishing anti-commutator $\{\gamma_5, \gamma_\mu\} \neq 0$ in traces one can translate both schemes as in the following example.

in our scheme

$$\mathrm{Tr}(\gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_k})-\mathrm{Tr}(\gamma_{\mu_k}\gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_{k-1}})$$
$$=\mathrm{Tr}(\gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_k})-\mathrm{Tr}(\gamma_5\gamma_{\mu_k}\gamma_{\mu_1}\dots\gamma_{\mu_{k-1}})$$
$$=2\sum_{i=1}^{k-1}(-1)^i g_{\mu_i\mu_k}\mathrm{Tr}(\gamma_5\gamma_{\mu_1}\dots\widehat{\gamma_{\mu_i}}\dots\gamma_{\mu_{k-1}}),$$

(where the hat symbol ^ above a γ-matrix means that this matrix has to be omitted)

while in BM-type schemes

$$\mathrm{tr}(\gamma_5^{\mathrm{BM}}\gamma_{\mu_1}\dots\gamma_{\mu_k})+\mathrm{tr}(\gamma_5^{\mathrm{BM}}\gamma_{\mu_k}\gamma_{\mu_1}\dots\gamma_{\mu_{k-1}})$$
$$=\mathrm{tr}(\{\gamma_5^{\mathrm{BM}},\gamma_{\mu_k}\}\gamma_{\mu_1}\dots\gamma_{\mu_{k-1}}) \tag{49}$$

In this way any non-cyclic trace can be translated to the corresponding terms in a BM scheme.

There is the question of renormalizability. In our scheme it is the question of the construction of a graph which violates its Ward-Identities. The only candidates for such graphs are closed fermion loops including an odd number of γ_5's. For an anomaly free theory the only case which has to be considered is the case of an arbitrary n-point Green's-function ($n>4$) with j exterior vertices involving γ_5 (j odd) and k exterior parity even vertices ($n=k+j$). Such a Green's-function is finite at the one-loop level by power-counting. What has to be discussed is such a n-point fermion-loop with arbitrary sub-divergencies. To renormalize this Green's-function we have to add the corresponding counterterm graphs of all 1PI sub-divergencies, that is, we need the lower-rank Z-factors. The sub-divergencies defining these Z-factors are graphs with open fermion lines. As a consequence they can be defined without problems in our scheme.

Do these Z-factors compensate the sub-divergencies of the full Green's function? The problem lies in our reading-prescription. There may be sub-divergencies in the graph which are sensible to the reading-prescription. We have to prove that they are still countertermed by the unique Z-factors.

Consider the fermion-loop (Fig. 6). To construct divergent sub-graphs which are sensible to reading prescriptions we have to construct divergent vertex corrections, that is subgraphs of the type of Fig. 7. Note that additional loops overlapping these subgraphs do not generate further problems because they are finite by power-counting. Now there are reading-prescriptions which renormalize the Green's function. Any reading point choosen to be

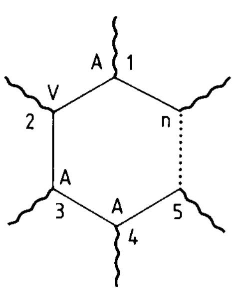

Fig. 6. A n-point Green's function with parity even (V) and parity odd (A) vertices

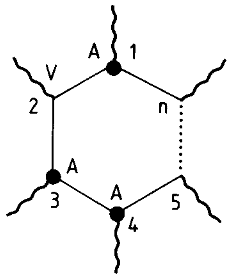

Fig. 7. The n-point Green's function with big black bubbles representing divergent vertex-subgraphs sensible to reading prescriptions

outside the divergent subgraphs will do. There is no overall degree of divergence in our n-point Green's function ($n>4$). So with one of the above mentioned reading prescriptions we have a finite result after counterterming. Let us denote traces calculated with a reading prescription starting outside divergent sub-graphs by $\mathrm{Tr}_{\mathrm{out}}(\dots)$ and the other traces calculated with a reading prescription starting inside divergent subgraphs by $\mathrm{Tr}_{\mathrm{in}}(\dots)$. Let us further denote the full Green's function by G and the set of all graphs containing relevant counterterms by C. Then we have

$$\mathrm{Tr}_{\mathrm{out}}(G)-\mathrm{Tr}_{\mathrm{out}}(C)=\text{finite}.$$

For the 'in'-prescription we have

$$\mathrm{Tr}_{\mathrm{in}}(G)-\mathrm{Tr}_{\mathrm{in}}(C)=\mathrm{Tr}_{\mathrm{out}}(G)-\mathrm{Tr}_{\mathrm{out}}(C)+\mathcal{O}(D-4).$$

The only difference might come from the different way of evaluating the trace but this is at least an operator of order $D-4$. There is no overall divergence so this operator does not give any contribution. For the case of several axial vector vertices the same argument goes through with a symmetrized reading-prescription for the graph and its counterterm-graphs.

The above argument can be summarized as that: there is no problem with divergent subgraphs because they all represent graphs with open fermion lines. This is also equivalent to the statement that our scheme fulfills Ward-Identities (apply again the open fermion line argument). So we do not have a chance to violate Ward-Identities in our scheme and as a consequence there is no problem with renormalization.

For the case of anomalies we have fermion loops with a non-vanishing degree of divergence by itself. But still the anomaly is a one-loop affair in our scheme for the following reasons. The anomaly can be written as an alternating sum of current divergencies [8]. This gives to the anomaly an 'effective superficial degree of divergence' vanishing beyond the one-loop level. After counterterming one has a finite expression from the integrals. The trace evaluation gives an $\mathcal{O}(d-4)$ operator. Without superficial degree of divergence we have a vanishing result. Equivalently one can state that (again the open fermion line argument) coupling constant renormalization (which is vertex renormalization) is not affected in our scheme. So the renormalization group argument of Zee [22, 23] applies without modification.

All the above arguments use the fact that we do not have spurious anomalies for open fermion lines. So the situation changes drastically when we go to schemes having spurious anomalies in this sector, e.g. BM-like schemes. Nevertheless there are indications that one can restore the Ward-Identities in BM-like schemes with appropriate counterterming [24].

What happens when an anomalous fermion loop appears itself as a subdivergence in a bigger graph? To answer this question note that there are only two possibilities. Either the fermion loop is part of a subdivergence as a whole. In that case the result for the fermion loop appears as a factor in the whole result. No matter what the (possible divergent) value of this factor is the coefficients of such terms must be zero in an anomaly-free theory. Or the fermion-loop is 'cut open' by a subdivergence. Than the open fermion line argument applies.

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