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A Method for Exact Calculation of the Stardiscrepancy of Plane Sets Applied to the Sequences of Hammersley

By

L. De Clerck, Bochum

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Abstract. A method is given to calculate exactly the stardiscrepancy of arbitrary finite plane sets. Using this method the stardiscrepancy of the sequences of Hammersley is obtained. The recursive structure of these sets allows for a proof by induction.

I. Introduction

The stardiscrepancy can be seen as a measure for the deviation between the empirical distribution of a given set in the unit hypercube and the uniform distribution. Therefore, it is for instance used by statisticians as a test for uniformity. Moreover, it has been studied extensively in number theory and due to its applications in numerical quadrature it attracted also the attention in numerical analysis. A survey of this theory has been given by NIEDERREITER in [10].

Given a subset S of $[0, 1)^k$ containing N elements and a hyperrectangle $\mathcal{Q} \subset [0, 1]^k$, then the positive rest of $\mathcal{Q}$ versus S is given by

$$E^+(\mathcal{Q}; S) := \frac{A(\mathcal{Q}; S)}{N} - \operatorname{vol}(\mathcal{Q}) \tag{1.1}$$

where $A(\mathcal{Q}; S)$ is the number of elements in $\mathcal{Q} \cap S$ and $\operatorname{vol}(\mathcal{Q})$ is the volume of $\mathcal{Q}$.

$$E^-(\mathcal{Q}; S) := -E^+(\mathcal{Q}; S) \tag{1.2}$$

and

$$E(\mathcal{Q}; S) := |E^+(\mathcal{Q}; S)| \tag{1.3}$$

are respectively the negative rest and the rest of $\mathcal{Q}$ versus S. The stardiscrepancy is given by

$$D_N^*(S) := \sup_{\substack{0 \leqslant a_i \leqslant 1 \\ 1 \leqslant i \leqslant k}} E\Big(\prod_{i=1}^{k} [0, a_i); S\Big) . \tag{1.4}$$

Other notions of discrepancy have been studied in the literature such as the $\mathscr{L}^2$-discrepancy which is the $\mathscr{L}^2$-norm over $[0,1]^k$ of the function $E(\prod_{i=1}^{k}[0,a_i);S)$.

When the set S is clear from the context, we will write $E(\mathscr{Q})$, $E^+(\mathscr{Q})$, $E^-(\mathscr{Q})$ instead of $E(\mathscr{Q};S)$, $E^+(\mathscr{Q};S)$, $E^-(\mathscr{Q};S)$. We note that the positive rest is additive with respect to $\mathscr{Q}$, in that sense that $\mathscr{Q}=\mathscr{Q}_1\cup\mathscr{Q}_2$; $\mathscr{Q}_1\cap\mathscr{Q}_2=\emptyset$ implies that $E^+(\mathscr{Q})=E^+(\mathscr{Q}_1)+E^+(\mathscr{Q}_2)$. The same holds true for the negative rest.

In the past decennia, a lot of research has been done to find sets with the lowest stardiscrepancy. A motivation herefore is the fact that there exists an upper bound for the quadrature error

$$\frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^k} f(\mathbf{t})\,d\mathbf{t}$$

having the form $C(f)\cdot D_N^*(S)$, where $C(f)$ is a constant only depending on f and $S=\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}$, [10].

Sets which often have been studied in this context, are the so-called sequences of Hammersley. In order to define them, the functions $\Phi_r(n)$, $r\geqslant 2$ are introduced: when $n=\sum_{j=0}^{\infty} a_j r^j$, $a_j\in\{0,1,\ldots,r-1\}$ then

$$\Phi_r(n):=\sum_{j=0}^{\infty} a_j r^{-j-1}\,. \tag{1.5}$$

These functions $\Phi_r(n)$ are also called "radical inverse functions." A set

$$\left\{\left(\frac{n}{N},\Phi_{r_1}(n),\ldots,\Phi_{r_{k-1}}(n)\right)\middle|\, 0\leqslant n\leqslant N-1\right\}$$

where $k\geqslant 2$ and the natural numbers r_i, $1\leqslant i\leqslant k-1$ are pairwise coprime, is called a sequence of Hammersley. HALTON proved in [6] that, when S is a sequence of Hammersley in $[0,1)^k$, then

$$D_N^*(S)=O(N^{-1}(\log N)^{k-1})\,. \tag{1.6}$$

The constant of Halton has been successively improved by MEIJER [8] and FAURE [4]. But actually no better order of magnitude has been found for the stardiscrepancy of a finite subset of $[0,1)^k$.

Exact results for the stardiscrepancy are only known in one and two dimensions. In two dimensions, for instance, a more extensive study of the sequences

$$\omega_{r^n} = \left\{\left(\frac{i}{r^n}, \Phi_r(i)\right) \middle| 0 \leqslant i \leqslant r^n - 1\right\} \tag{1.7}$$

has been done. HALTON and ZAREMBA calculated the stardiscrepancy and the $\mathscr{L}^2$-discrepancy of the sets ω_{r^n} for the case $r = 2$, [7]. WHITE studied in [12] the $\mathscr{L}^2$-discrepancy of the sequences ω_{r^n} for arbitrary natural numbers $r \geqslant 2$ and $n \geqslant 1$. How to calculate in arbitrary dimensions the stardiscrepancy of a given finite set S in a finite number of steps, is shown by NIEDERREITER in [9].

In section II, we will derive a formula by which the number of steps to calculate the stardiscrepancy in two dimensions is still reduced. The only restriction we impose on the sets S studied is that no two points of S have the same abscis or ordinate. From the point of view of sets with low stardiscrepancy, this restriction is not important. In section III, we will calculate the stardiscrepancy of the sets ω_{r^n}. Our method is based on our formula for the stardiscrepancy in two dimensions and on the recursive structure of the sets ω_{r^n} which allows for an inductive method to be used.

We use the standard notations: $[x]$ for the largest integer smaller than or equal to x and $\{x\}$ for $x - [x]$, x being a real number.

II. The Calculation of the Stardiscrepancy of an Arbitrary Set

When S is an arbitrary subset of $[0, 1)^k$ with N elements, then we define for all j, $0 \leqslant j \leqslant N$, the following sets:

$$O_j := \{\prod_{i=1}^{k} [0, b_i) \mid 0 \leqslant b_i \leqslant 1, A(\prod_{i=1}^{k} [0, b_i); S) = j \text{ and } \forall\, \varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \text{ with } \varepsilon_i \geqslant 0 \text{ and not } \varepsilon \equiv \mathbf{0}\colon A(\prod_{i=1}^{k} [0, b_i + \varepsilon_i); S) > j\} \tag{2.1}$$

$$o_j := \{\prod_{i=1}^{k} [0, b_i] \mid 0 \leqslant b_i \leqslant 1, A(\prod_{i=1}^{k} [0, b_i]; S) = j \text{ and } \forall\, \varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \text{ with } \varepsilon_i \geqslant 0 \text{ and not } \varepsilon \equiv \mathbf{0}\colon A(\prod_{i=1}^{k} [0, b_i - \varepsilon_i]; S) < j\} \tag{2.2}$$

O_j is the set of maximal hyperrectangles containing j points and o_j is the set of minimal hyperrectangles containing j points. With these notations one can formulate the following lemma:

Lemma. *The stardiscrepancy of a set $S \subset [0, 1)^k$ with N elements is given by*

$$D_N^*(S) = \max_{0 \leqslant j \leqslant N} \max \{\max_{O \in O_j} E^-(O), \max_{o \in o_j} E^+(o)\} . \tag{2.3}$$

Proof: We follow Niederreiter's proof of the one dimensional case [9]. One can easily see that

$$D_N^*(S) = \max_{0 \leqslant j \leqslant N} \sup X_j,$$

where

$$X_j = \left\{ \left| \frac{j}{N} - \prod_{i=1}^{k} x_i \right| \,\middle|\, \forall i, 1 \leqslant i \leqslant k \colon 0 \leqslant x_i \leqslant 1 \text{ and } A(\prod_{i=1}^{k} [0, x_i); S) = j \right\}.$$

Since $\left| \frac{j}{N} - x \right|$ is a convex function, one has

$$\sup X_j = \max \left\{ \max_{o \in o_j} \left\{ \left| \frac{j}{N} - \operatorname{vol}(o) \right| \right\}, \max_{O \in O_j} \left\{ \left| \operatorname{vol}(O) - \frac{j}{N} \right| \right\} \right\} .$$

For every maximal hyperrectangle O containing j points, there is a minimal hyperrectangle o containing these points. Since evidently

$$\frac{j}{N} - \operatorname{vol}(O) < \frac{j}{N} - \operatorname{vol}(o)$$

and

$$\operatorname{vol}(o) - \frac{j}{N} < \operatorname{vol}(O) - \frac{j}{N},$$

one only has to consider positive rests of minimal sets and negative rests of maximal sets. □

In the next theorem, we convert formula (2.3) to a more detailed one for sets $S = \{(x_n, y_n) \mid 1 \leqslant n \leqslant N\} \subset [0, 1)^2$ satisfying

$$i < j \rightarrow x_i < x_j \quad \text{and} \quad y_i \neq y_j. \tag{2.4}$$

From the point of view of sets with low stardiscrepancy, this restriction is not important since for each set S' which does not satisfy (2.4) there exists a set S satisfying (2.4) such that $D_N^*(S) \leqslant D_N^*(S')$.

The crucial point in deriving a formula for the stardiscrepancy in $[0, 1]^2$ is the stepwise ordering of the y-coordinates of the points of S. Let the numbers g_{ik}, $1 \leqslant k \leqslant i$, be the y-coordinates y_k, $1 \leqslant k \leqslant i$ arranged from smaller to bigger. Let t_i, $1 \leqslant i \leqslant N$ be the place of y_i between $g_{i-1,k}$, $1 \leqslant k \leqslant i - 1$. This means

$$t_1 = 1 \tag{2.5 a}$$

and for $i \geqslant 2$

$$t_i = 1 \leftrightarrow 0 \leqslant y_i < g_{i-1,1} \tag{2.5b}$$

$$1 < t_i \leqslant i-1 \leftrightarrow g_{i-1,t_i-1} < y_i \leqslant g_{i-1,t_i} \tag{2.5c}$$

$$t_i = 1 \leftrightarrow g_{i-1,i-1} < y_i < 1 \; . \tag{2.5d}$$

In $[0,1]^2$ the maximal and minimal rectangles can be classified as follows. Firstly, we introduce the set

$$\begin{aligned} \mathscr{R}_1(S) &:= \{[0,x_i)\times[0,1) \mid 1 \leqslant i \leqslant N\} \cup \{[0,1)\times[0,y_i) \mid 1 \leqslant i \leqslant N\} \\ &= \{[0,x_i)\times[0,1) \mid 1 \leqslant i \leqslant N\} \cup \{[0,1)\times[0,g_{Ni}) \mid 1 \leqslant i \leqslant N\} \; . \end{aligned} \tag{2.6}$$

According to formula (2.4) and the definition of the numbers g_{Ni}, it is clear that

$$[0,x_i)\times[0,1) \in O_{i-1} \tag{2.7a}$$

and

$$[0,1)\times[0,g_{Ni}) \in O_{i-1} \; . \tag{2.7b}$$

Secondly, we introduce the set

$$\mathscr{R}_2(S) := \{[0,x_i]\times[0,y_i] \mid 1 \leqslant i \leqslant N\} \; . \tag{2.8}$$

From (2.5) one knows that

$$[0,x_i]\times[0,y_i] \in o_{t_i} \; . \tag{2.9}$$

Finally, we introduce the sets

$$\begin{aligned} \mathscr{R}_3(S) &:= \{[0,x_i)\times[0,y_j) \mid 1 \leqslant j < i \leqslant N, y_j > y_i\} \\ &:= \{[0,x_i)\times[0,g_{i-1,k}) \mid 1 \leqslant i \leqslant N, t_i \leqslant k \leqslant i-1\} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \mathscr{R}_4(S) &:= \{[0,x_i]\times[0,y_j] \mid 1 \leqslant j < i \leqslant N, y_j > y_i\} \\ &:= \{[0,x_i]\times[0,g_{i-1,k}] \mid 1 \leqslant i \leqslant N, t_i \leqslant k \leqslant i-1\} \end{aligned} \tag{2.11}$$

$\mathscr{R}_3(S)$ is a set of maximal rectangles, $\mathscr{R}_4(S)$ contains minimal rectangles. It is clear that

$$A([0,x_i]\times[0,g_{i-1,k}];S) = A([0,x_i)\times[0,g_{i-1,k});S) + 2,$$

and it follows from (2.5) and the definition of the numbers g_{ik} that for all i,k, $1 \leqslant i \leqslant N$ and $t_i \leqslant k \leqslant i-1$:

$$[0,x_i)\times[0,g_{i-1,k}) \in O_{k-1} \tag{2.12a}$$

and thus

$$[0,x_i]\times[0,g_{i-1,k}] \in o_{k+1} \; . \tag{2.12b}$$

From the formulas (2.7), (2.9) and (2.12), we obtain the following theorem:

Theorem. *If* $S = \{(x_n, y_n) \mid 1 \leqslant n \leqslant N\} \subset [0,1)^2$ *satisfies condition* (2.4), *then*

$$D_N^*(S) = \max_{1 \leqslant i \leqslant N} \left\{ x_i - \frac{i-1}{N}, g_{Ni} - \frac{i-1}{N}, \frac{t_i}{N} - x_i y_i, \right.$$
$$\left. \max_{t_i \leqslant k \leqslant i-1} \max \left\{ x_i g_{i-1,k} - \frac{k-1}{N}, \frac{k+1}{N} - x_i g_{i-1,k} \right\} \right\},$$

where the numbers t_i *and* $g_{i-1,k}$, $1 \leqslant i \leqslant N$ *and* $t_i \leqslant k \leqslant i-1$ *are defined as above.*

III. The Exact Calculation of the Stardiscrepancy of the Sequences of Hammersley in Two Dimensions

In [2] FAURE proved that for all i, k; $0 \leqslant i \leqslant r^{n-1} - 1$ and $0 \leqslant k \leqslant r-1$:

$$\Phi_r(i + k r^{n-1}) = \Phi_r(i) + \frac{k}{r^n}. \tag{3.1}$$

Therefore, owing to (1.7),

$$\omega_{r^n} = \left\{ \left(\frac{i}{r^n} + \frac{k}{r}, \Phi_r(i) + \frac{k}{r^n} \right) \middle| \; 0 \leqslant i \leqslant r^{n-1} - 1 \text{ and } 0 \leqslant k \leqslant r-1 \right\}. \tag{3.2}$$

This recursion property is fundamental in the exact calculation of the stardiscrepancy of the sequences of Hammersley. The following lemma is from FAURE [3].

Lemma 1. $\forall n, r, n \geqslant 1$ *and* $r \geqslant 2$:

$$E\left(\prod_{k=1}^{2} \left[\frac{u_k}{r^{p_k}}, \frac{u_k + 1}{r^{p_k}} \right) \right) = 0 \tag{3.3}$$

with $u_k, p_k \in \mathbb{N}$ *for* $k = 1, 2$;

$$u_k < r^{p_k} \quad \textit{and} \quad p_1 + p_2 = n. \tag{3.4}$$

It can be seen from (3.2) that one also has

$$E\left(\left[\frac{u_1}{r^{p_1}} + a, \frac{u_1+1}{r^{p_1}} + a\right) \times \left[\frac{u_2}{r^{p_2}}, \frac{u_2+1}{r^{p_2}}\right)\right) = \\ = E\left(\left(\frac{u_1}{r^{p_1}} + a, \frac{u_1+1}{r^{p_1}} + a\right] \times \left[\frac{u_2}{r^{p_2}}, \frac{u_2+1}{r^{p_2}}\right)\right) = 0, \tag{3.5}$$

Where $0 \leqslant a < 1 - \frac{u_1+1}{r^{p_1}}$ and u_k and p_k, $k = 1,2$ satisfy (3.4).

In [5] GABAI mentioned the following lemma, which can also be proved using recursion relation (3.2).

Lemma 2. *For all points* $(x,y) \in [0,1]^2$, *one has*

$$E^-([0,x)\times[0,y);\omega_{r^n}) \leqslant 0\ . \tag{3.6}$$

From this lemma, it follows that it is not necessary to consider the negative rests of the elements of $\mathscr{R}_1(\omega_{r^n}) \cup \mathscr{R}_3(\omega_{r^n})$ to calculate $D^*_{r^n}(\omega_{r^n})$. By the following lemma which is due to PEART [11] the number of positive rests of the elements of $\mathscr{R}_2(\omega_{r^n}) \cup \mathscr{R}_4(\omega_{r^n})$ to be studied is reduced by the half.

Lemma 3. *The sets* ω_{r^n} *are symmetric with respect to the main diagonal of the unit square, for all natural numbers* $n \geqslant 1$ *and* $r \geqslant 2$.

Our aim is now to find the rectangle belonging to $\mathscr{R}_2(\omega_{r^n}) \cup \mathscr{R}_4(\omega_{r^n})$ such that its positive rest is equal to $D^*_{r^n}(\omega_{r^n})$. We will therefore compare the rests of these rectangles mutually.

Due to formula (3.2) and the definitions (2.8) and (2.11) of the sets $\mathscr{R}_2(S)$ and $\mathscr{R}_4(S)$, one has

$$\mathscr{R}_2(\omega_{r^n}) := \left\{\left[0, \frac{i}{r^n} + \frac{k}{r}\right] \times \left[0, \Phi_r(i) + \frac{k}{r^n}\right] \middle| \\ 0 \leqslant i \leqslant r^{n-1} - 1;\ 0 \leqslant k \leqslant r-1\right\}. \tag{3.7}$$

We split $\mathscr{R}_4(\omega_{r^n})$ into two parts:

$$\mathscr{R}_4(\omega_{r^n}) := \mathscr{R}_{41}(\omega_{r^n}) \cup \mathscr{R}_{42}(\omega_{r^n}),$$

where

$$\mathscr{R}_{41}(\omega_{r^n}) := \left\{\left[0, \frac{i}{r^n} + \frac{k}{r}\right] \times \left[0, \Phi_r(j) + \frac{k}{r^n}\right] \middle| \\ 0 \leqslant j < i \leqslant r^{n-1} - 1;\ \Phi_r(j) > \Phi_r(i);\ 0 \leqslant k \leqslant r-1\right\} \tag{3.8}$$

and
$$\mathscr{R}_{42}(\omega_{r^n}) := \left\{ \left[0, \frac{i}{r^n} + \frac{k}{r}\right] \times \left[0, \Phi_r(j) + \frac{l}{r^n}\right] \middle| \right.$$
$$\left. 0 \leqslant j, i \leqslant r^{n-1} - 1;\ \Phi_r(j) > \Phi_r(i);\ 0 \leqslant l < k \leqslant r - 1 \right\}. \tag{3.9}$$

We will show now that it suffices to consider the rectangles of $\mathscr{R}_{42}(\omega_{r^n})$ satisfying $l = k - 1$. For this purpose, we calculate for every i, j, k and l satisfying the conditions of $\mathscr{R}_{42}(\omega_{r^n})$ the positive rest of the rectangle

$$\left[0, \frac{i}{r^n} + \frac{k}{r}\right] \times \left[0, \Phi_r(j) + \frac{k-1}{r^n}\right] \Big\backslash \left[0, \frac{i}{r^n} + \frac{k}{r}\right] \times \left[0, \Phi_r(j) + \frac{l}{r^n}\right] =$$
$$= \left[0, \frac{i}{r^n} + \frac{k}{r}\right] \times \left(\Phi_r(j) + \frac{l}{r^n}, \Phi_r(j) + \frac{k-1}{r^n}\right]. \tag{3.10}$$

From formula (3.2) one can see that this rectangle contains the $k - l - 1$ points with ordinates:

$$\Phi_r(j) + \frac{l+1}{r^n}, \ldots, \Phi_r(j) + \frac{k-1}{r^n}.$$

As the volume of the rectangle (3.10) is smaller than $(k - l - 1)/r^n$ it follows that its positive rest is positive.

We may thus restrict ourselves to the rectangles $\mathscr{R}_{42}(\omega_{r^n})$ satisfying $l = k - 1$. Therefore, we only have to consider the positive rests of the elements of

$$\mathscr{R}_2'(\omega_{r^n}) := \left\{ \left[0, \frac{i}{r^n} + \frac{k}{r}\right] \times \left[0, \Phi_r(j) + \frac{k}{r^n}\right] \middle| \right.$$
$$0 \leqslant j < i \leqslant r^{n-1} - 1;\ \Phi_r(j) > \Phi_r(i)$$
$$\left. \text{or} \quad 0 \leqslant i \leqslant r^{n-1} - 1;\ j = i \quad \text{and} \quad 0 \leqslant k \leqslant r - 1 \right\} = \tag{3.11}$$
$$= \mathscr{R}_2(\omega_{r^n}) \cup \mathscr{R}_{41}(\omega_{r^n})$$

and

$$\mathscr{R}_4'(\omega_{r^n}) := \left\{ \left[0, \frac{i}{r^n} + \frac{k}{r}\right] \times \left[0, \Phi_r(j) + \frac{1}{r^{n-1}} + \frac{k-1}{r^n}\right] \middle| \right.$$
$$0 \leqslant i \leqslant r^{n-1} - 1;\ 0 \leqslant j \leqslant r^{n-1} - 2;\ \Phi_r(j) \geqslant \Phi_r(\mathrm{i}); \tag{3.12}$$
$$\left. 0 \leqslant k \leqslant r - 1 \right\}.$$

This last set is empty for $n = 1$. We look now for the values of k for which the positive rests of the elements of $\mathscr{R}_2'(\omega_{r^n})$ and $\mathscr{R}_4'(\omega_{r^n})$ are the largest. To do so, we elaborate on the following quantities:

$$\varepsilon^n_{i+kr^{n-1},j,k} - \varepsilon^n_{i,j,0} \tag{3.13}$$

where i,j and k satisfy the conditions of $\mathscr{R}_2'(\omega_{r^n})$ and

$$\varepsilon^n_{i+kr^{n-1},j,k+(r-1)} - \varepsilon^n_{i,j,0} \tag{3.14}$$

where i,j and k satisfy the conditions of $\mathscr{R}_4'(\omega_{r^n})$. We used here as notation:

$$\varepsilon^n_{I,J,K} := E^+\left(\left[0, \frac{I}{r^n}\right] \times \left[0, \Phi_r(J) + \frac{K}{r^n}\right]\right). \tag{3.15}$$

We detail the calculations for the most difficult case, namely (3.14). It follows from the formulas (3.1) and (3.15) that (3.14) is equal to the positive rest of

$$\left(\frac{i}{r^n}, \frac{i}{r^n} + \frac{k}{r}\right] \times \left[0, \Phi_r(j)\right) \tag{3.16}$$

$$\bigcup \left(\frac{i}{r^n}, \frac{i}{r^n} + \frac{k}{r}\right] \times \left[\Phi_r(j), \Phi_r(j) + \frac{1}{r^{n-1}} + \frac{k-1}{r^n}\right] \tag{3.17}$$

$$\bigcup \left[0, \frac{i}{r^n}\right] \times \left(\Phi_r(j), \Phi_r(j) + \frac{1}{r^{n-1}} + \frac{k-1}{r^n}\right]. \tag{3.18}$$

Now (3.16) is equal to

$$\bigcup_{K=1}^{k} \left(\frac{i}{r^n} + \frac{K-1}{r}, \frac{i}{r^n} + \frac{K}{r}\right] \times [0, \Phi_r(j)) .$$

It follows therefore from (3.5) and the additivity of the positive rest that the positive rest of the rectangles (3.16) is zero, such that we may restrict ourselves to the rectangles (3.17) and (3.18). If $j \leqslant i$, then the union of (3.17) and (3.18) contains the points with ordinates

$$\Phi_r(j) + \frac{1}{r^n}, \ldots, \Phi_r(j) + \frac{k}{r^n} \tag{3.19}$$

and

$$\Phi_r(j) + \frac{1}{r^{n-1}}, \ldots, \Phi_r(j) + \frac{1}{r^{n-1}} + \frac{k-1}{r^n}. \tag{3.20}$$

In the case $j > i$, this union contains the points with ordinates

$$\Phi_r(j), \ldots, \Phi_r(j) + \frac{k-1}{r^n} \tag{3.21}$$

and the points with ordinates (3.20). In both cases, we conclude that the union of the rectangles (3.17) and (3.18) contains $2k$ points.

Once this is known, one can derive by simple calculations that (3.14) is equal to

$$\frac{2k}{r^n} - \frac{ik}{r^{2n}} - \frac{k^2}{r^{n+1}} - \frac{i(r-1)}{r^{2n}} - \frac{k(r-1)}{r^{n+1}} := f_2^n(i,k)\,. \tag{3.22}$$

In the same way, it can be seen that (3.13) is equal to

$$\frac{k}{r^n} - \frac{ik}{r^{2n}} - \frac{k^2}{r^{n+1}} := f_1^n(i,k)\,. \tag{3.23}$$

With these notations one has

$$D^*_{r^n}(\omega_{r^n}) = \max A \cup B \tag{3.24}$$

where

$$\begin{aligned} A := \{\varepsilon^n_{i,j,0} + \max_{0\leqslant k\leqslant r-1} \{f_1^n(i,k)\} \mid 0 \leqslant j < i \leqslant r^{n-1} - 1;\\ \Phi_r(j) > \Phi_r(i) \quad \text{or} \quad 0 \leqslant i \leqslant r^{n-1} - 1; j = i\} \end{aligned} \tag{3.25}$$

and for $n > 1$

$$\begin{aligned} B := \{\varepsilon^n_{i,j,0} + \max_{0\leqslant k\leqslant r-1} \{f_2^n(i,k)\} \mid\\ 0 \leqslant i \leqslant r^{n-1} - 1; 0 \leqslant j \leqslant r^{n-1} - 2; \Phi_r(j) \geqslant \Phi_r(i)\}\,. \end{aligned} \tag{3.26}$$

while for $n = 1$ B is empty.

One has

$$\max_{0\leqslant k\leqslant r-1} \{f_1^n(i,k)\} = f_1^n\left(i, \left[\frac{r}{2}\right]\right) \tag{3.27}$$

and

$$\max_{0\leqslant k\leqslant r-1} \{f_2^n(i,k)\} = f_2^n\left(i, \left[\frac{r+1}{2}\right]\right). \tag{3.28}$$

Using (3.27) and (3.28) we show now that for $n > 1$

$$\max B \geqslant \max A\,. \tag{3.29}$$

Comparing the functions $f_2^n\left(i, \left[\frac{r+1}{2}\right]\right)$ and $f_1^n\left(i, \left[\frac{r}{2}\right]\right)$ one has that for $0 \leqslant i \leqslant K$:

$$f_2^n\left(i,\left[\frac{r+1}{2}\right]\right) \geqslant f_1^n\left(i,\left[\frac{r}{2}\right]\right) \tag{3.30}$$

where

$$K = \begin{cases} \left[\dfrac{r^{n-1}}{2} + \dfrac{r^{n-2}}{2}\right] & \text{for } r \text{ odd} \\ \left[\dfrac{r}{2(r-1)} r^{n-1}\right] & \text{for } r \text{ even.} \end{cases} \tag{3.31}$$

For the case $K < i \leqslant r^{n-1} - 1$, we compare each positive rest (We remark that this case does not occur for $r = 2$.)

$$E^+\left(\left[0, \frac{i}{r^n} + \frac{k}{r}\right] \times \left[0, \Phi_r(j) + \frac{k}{r^n}\right]\right), \tag{3.32}$$

belonging to A, with the positive rest

$$E^+\left(\left[0, \frac{i - r^{n-2}}{r^n} + \frac{k+1}{r}\right] \times \left[0, \Phi_r(j) + \frac{k}{r^n}\right]\right). \tag{3.33}$$

From $i > K$ and (3.31), we can conclude that $(i - r^{n-2})/r^n > 0$. Therefore, we know that this last rest belongs to B for the case that r is odd and is smaller than a rest belonging to B for the case that r is even (see formula (3.28)).

From (3.25) we know that

$$0 \leqslant j < i \leqslant r^{n-1} - 1;\ \Phi_r(j) > \Phi_r(i) \tag{3.34a}$$

or

$$0 \leqslant i \leqslant r^{n-1} - 1;\ j = i \tag{3.34b}$$

and

$$k = \left[\frac{r}{2}\right]. \tag{3.34c}$$

We compare now the rests (3.32) and (3.33). From (3.34) and the definition of the functions Φ_r, it follows that there exists a J, $0 \leqslant J \leqslant r^{n-1} - 1$ such that

$$\Phi_r(j) = \frac{J}{r^{n-1}}. \tag{3.35}$$

We conclude from (3.35) that the difference between (3.33) and (3.32) is equal to the positive rest of

$$\left(\frac{i}{r^n} + \frac{k}{r}, \frac{i - r^{n-2}}{r^n} + \frac{k+1}{r}\right] \times \left[0, \frac{J}{r^{n-1}} + \frac{k}{r^n}\right]. \tag{3.36}$$

One has that the number of points in the intersection of (3.36) with ω_{r^n} is equal to

$$A\left(\left(\frac{i}{r^n}+\frac{k}{r},\frac{i}{r^n}+\frac{k+1}{r}\right]\times\left[0,\frac{J}{r^{n-1}}+\frac{k}{r^n}\right];\ \omega_{r^n}\right) \tag{3.37}$$

minus

$$A\left(\left(\frac{i-r^{n-2}}{r^n}+\frac{k+1}{r},\frac{i}{r^n}+\frac{k+1}{r}\right]\times\left[0,\frac{J}{r^{n-1}}+\frac{k}{r^n}\right];\ \omega_{r^n}\right). \tag{3.38}$$

and as $j<i$ one has that (3.37) and (3.38) are respectively equal to

$$A\left(\left(\frac{i}{r^n}+\frac{k}{r},\frac{i}{r^n}+\frac{k+1}{r}\right]\times\left[0,\frac{J}{r^{n-1}}\right);\ \omega_{r^n}\right) \tag{3.39}$$

and

$$A\left(\left(\frac{i-r^{n-2}}{r^n}+\frac{k+1}{r},\frac{i}{r^n}+\frac{k+1}{r}\right]\times\left[0,\frac{[J/r]}{r^{n-2}}\right);\ \omega_{r^n}\right). \tag{3.40}$$

It follows then from (3.5) that (3.39) is equal to J and (3.40) to $[J/r]$. Hence it follows that the positive rest of (3.37) is equal to

$$\frac{J-[J/r]}{r^n}-\left(\frac{J}{r^{n-1}}+\frac{k}{r^n}\right)\times\left(\frac{1}{r}-\frac{1}{r^2}\right)=\frac{1}{r^n}\times\left(\left\{\frac{J}{r}\right\}-\frac{k}{r}+\frac{k}{r^2}\right). \tag{3.41}$$

Due to (3.34), one has that $k=[r/2]$. Therefore, (3.41) is positive

$$\left\{\frac{J}{r}\right\}>\left[\frac{r}{2}\right]\left(\frac{1}{r}-\frac{1}{r^2}\right). \tag{3.42}$$

If the rest (3.32) does not satisfy this condition, it can be seen that there exist values i' and j' satisfying (3.34) and such that the rest

$$E^+\left(\left[0,\frac{i'}{r^n}+\frac{k}{r}\right]\times\left[0,\Phi_r(j')+\frac{k}{r^n}\right]\right)$$

is larger than (3.32). Therefore, we conclude from (3.30) that (3.29) is indeed satisfied. Using (3.24), (3.26), (3.28), (3.29) and the symmetry of the sets ω_{r^n} with respect to the main diagonal, the results obtained until now can be summarized in the following lemma.

Lemma 4. *$\forall\, r,n\in\mathbb{N}_0$ with r and $n\geqslant 2$ one has*

$$D^*_{r^n}(\omega_{r^n})=\max\{\mathscr{F}^n_{i,j}\mid 0\leqslant i,j\leqslant r^{n-2}-1\}, \tag{3.43}$$

where

$$\mathscr{F}_{i,j}^{n} := E^{+}\left(\left[0,\left[\frac{r+1}{2}\right]/r+\left[\frac{r-1}{2}\right]/r^{n}+i/r^{n-1}\right]\times\right.$$
$$\left.\times\left[0,\left[\frac{r+1}{2}\right]/r+\left[\frac{r-1}{2}\right]/r^{n}+j/r^{n-1}\right]\right). \tag{3.44}$$

Using this lemma we will derive at the end of this section an expression for $D_{r^2}^{*}(\omega_{r^2})$. By an analogous reasoning as the one leading to (3.24) we may conclude from Lemma 4 that for the case $n \geqslant 3$

$$D_{r^n}^{*}(\omega_{r^n}) = \max\{\mathscr{F}_{i,1+Kr}^{n} + \max_{0\leqslant L\leqslant r-1}\{g_1^n(L,i), g_2^n(L,i)\} \mid$$
$$1 \leqslant i \leqslant r^{n-3},\ 0 \leqslant K \leqslant r^{n-3}-1 \quad \text{and} \tag{3.45}$$
$$1 \leqslant i + Lr^{n-3},\ 1+Kr+L \leqslant r^{n-2}-1\}.$$

where

$$g_1^n(L,i) := \mathscr{F}_{i+Lr^{n-3},1+Kr+L}^{n} - \mathscr{F}_{i,1+Kr}^{n} = \tag{3.46}$$
$$= \frac{1}{r^{2n}}\left(-iLr^2 + L\left[\frac{r-1}{2}\right](-r^{n-2}-r) + L(r^n - r^{n-1}) - L^2 r^{n-1}\right) \tag{3.47}$$

and

$$g_2^n(L,i) := \mathscr{F}_{i+Lr^{n-3},1+Kr+L+(r-1)}^{n} - \mathscr{F}_{i,1+Kr}^{n} = \tag{3.48}$$
$$= \frac{1}{r^{2n}}\left(-i(L+r-1)r^2 + L\left[\frac{r-1}{2}\right](-r^{n-2}-r) + Lr^n - \right.$$
$$\left. - \left[\frac{r-1}{2}\right]r(r-1) - L^2 r^{n-1}\right). \tag{3.49}$$

The expressions (3.47) and (3.49) can be derived in the same way as the expressions (3.23) and (3.22) for $f_1^n(i,k)$ and $f_2^n(i,k)$.

Continuing with the case r odd we introduce the function

$$h^n(i) := \mathscr{F}_{i,1+Kr}^{n} - \frac{1}{r}\mathscr{F}_{i,1+K}^{n-1} \tag{3.50}$$

which makes the link between $D_{r^n}^{*}(\omega_{r^n})$ and $D_{r^{n-1}}^{*}(\omega_{r^{n-1}})$. One can prove that this function is equal to

$$\frac{1}{r^{2n}}\left(\frac{r-1}{2} + ri\right)\left(\frac{3r^2}{2} - 2r + \frac{1}{2}\right). \tag{3.51}$$

The following lemma is fundamental for the derivation of $D_{r^n}^{*}(\omega_{r^n})$ for $n \geqslant 3$.

Lemma 5. *When n and k satisfy $n \geqslant k+2$, $k \geqslant 1$ then*

$$D^*_{r^n}(\omega_{r^n}) = \max_{0 \leqslant K \leqslant r^{n-(k+2)}-1} \left\{ \left\{ \frac{1}{r^{k-1}} \mathscr{F}^{n-(k-1)}_{i,1+Kr} + A_{n,k}(i) \,\middle|\, 1 \leqslant i \leqslant \left[\frac{r^{n-(k+2)}}{2} - \frac{r-1}{2r} \right] \right\} \right.$$
$$\cup \left\{ \frac{1}{r^{k-1}} \mathscr{F}^{n-(k-1)}_{i,1+Kr} + B_{n,k}(i) \,\middle|\, \left[\frac{r^{n-(k+2)}}{2} - \frac{r-1}{2r} \right] < i \leqslant \left[r^{n-(k+3)} \left(\frac{r+1}{2} \right) - \frac{r-1}{2r} \right] \right\} \tag{3.52}$$
$$\left. \cup \left\{ \frac{1}{r^{k-1}} \mathscr{F}^{n-(k-1)}_{i,1+Kr} + C_{n,k}(i) \,\middle|\, \left[r^{n-(k+3)} \left(\frac{r+1}{2} \right) - \frac{r-1}{2r} \right] < i \leqslant r^{n-(k+2)} \right\} \right\}$$

where

$$A_{n,k}(i) := \sum_{j=0}^{k-2} \frac{1}{r^j} \left(h^{n-j} \left(i + \left(\frac{r-1}{2} \right) \sum_{h=j+4}^{k+2} r^{n-h} \right) + \right.$$
$$\left. + g_2^{n-j} \left(\frac{r-1}{2}, i + \left(\frac{r-1}{2} \right) \sum_{h=j+4}^{k+2} r^{n-h} \right) \right) + \tag{3.53}$$
$$+ \frac{1}{r^{k-1}} g_2^{n-(k-1)} \left(\frac{r-1}{2}, i \right);$$

$$B_{n,k}(i) := \sum_{j=0}^{k-2} \frac{1}{r^j} \left(h^{n-j} \left(i + \left(\frac{r-1}{2} \right) \sum_{h=j+4}^{k+2} r^{n-h} \right) + \right.$$
$$\left. + g_1^{n-j} \left(\frac{r-1}{2}, i + \left(\frac{r-1}{2} \right) \sum_{h=j+4}^{k+2} r^{n-h} \right) \right) + \tag{3.54}$$
$$+ \frac{1}{r^{k-1}} g_1^{n-(k-1)} \left(\frac{r-1}{2}, i \right)$$

and

$$C_{n,k}(i) := \sum_{j=0}^{k-2} \frac{1}{r^j} \left(h^{n-j} \left(i + \left(\frac{r-1}{2} \right) \sum_{h=j+4}^{k+1} r^{n-h} + \left(\frac{r-3}{2} \right) r^{n-(k+2)} \right) + \right.$$

$$+ g_2^{n-j}\left(\frac{r-1}{2}, i + \left(\frac{r-1}{2}\right)\sum_{h=j+4}^{k+1} r^{n-h} + \left(\frac{r-3}{2}\right) r^{n-(k+2)}\right)\Bigg) +$$

$$+ \frac{1}{r^{k-1}} g_1^{n-(k-1)}\left(\frac{r-3}{2}, i\right). \tag{3.55}$$

Proof: We prove this lemma by induction.

1) For the case $k = 1$ the proof follows from formula (3.45). One can prove that

$$\max\{g_1^n(L, i) \,|\, 0 \leqslant L \leqslant r-1\} =$$

$$= \begin{cases} g_1^n\left(\frac{r-1}{2}, i\right), & \text{when } i \leqslant \left[r^{n-4}\left(\frac{r+1}{2}\right) - \frac{r-1}{2r}\right] \\ g_1^n\left(\frac{r-3}{2}, i\right), & \text{when } i > \left[r^{n-4}\left(\frac{r+1}{2}\right) - \frac{r-1}{2r}\right] \end{cases}$$

and on the other hand that

$$\max\{g_2^n(L, i) \,|\, 0 \leqslant L \leqslant r-1\} = g_2^n\left(\frac{r-1}{2}, i\right), \quad \text{when } 1 \leqslant i \leqslant r^{n-3}.$$

Formula (3.52) follows from the comparison of the functions

$$g_1^n\left(\frac{r-1}{2}, i\right), g_1^n\left(\frac{r-3}{2}, i\right) \text{ and } g_2^n\left(\frac{r-1}{2}, i\right).$$

2) We suppose now that we proved formula (3.52) for k. Then it follows from (3.46), (3.48), (3.50) and the fact that

$$\frac{r^{n-(k+2)}}{2} = \frac{r-1}{2} r^{n-(k+3)} + \frac{r^{n-(k+3)}}{2},$$

that

$$D_{r^n}^*(\omega_{r^n}) = \max_{0 \leqslant K \leqslant r^{n-(k+3)}-1} \Bigg\{ \frac{1}{r^k}\mathscr{F}_{i,1+Kr}^{n-1} + \frac{1}{r^k}\max\{g_1^{n-k}(L, i), g_2^{n-k}(L, i)\} +$$

$$+ h^n(i + L r^{n-(k+3)}) + A_{n,k}(i + L r^{n-(k+3)}) \Bigg|$$

$$1 \leqslant i \leqslant r^{n-(k+3)},\ 0 \leqslant L \leqslant \frac{r-3}{2} \text{ or}$$

$$1 \leqslant i \leqslant \left[\frac{r^{n-(k+3)}}{2} - \frac{r-1}{2r}\right],\ L = \frac{r-1}{2}\Bigg\}$$

$$\bigcup\left\{\frac{1}{r^k}\mathscr{F}^{n-k}_{i,1+Kr}+\frac{1}{r^k}\max\{g_1^{n-k}(L,i),g_2^{n-k}(L,i)\}+\right. \tag{3.56}$$

$$+h^n(i+Lr^{n-(k+3)})+B_{n,k}(i+Lr^{n-(k+3)})\,\Big|$$

$$\left.\left[\frac{r^{n-(k+3)}}{2}-\frac{r-1}{2r}\right]<i\leqslant r^{n-(k+3)},\ L=\frac{r-1}{2}\right\}$$

$$\bigcup\left\{\frac{1}{r_k}\mathscr{F}^{n-k}_{i,1+Kr}+\frac{1}{r^k}\max\{g_1^{n-k}(L,i),g_2^{n-k}(L,i)\}+\right.$$

$$+h^n(i+Lr^{n-(k+3)})+C_{n,k}(i+Lr^{n-(k+3)})\,\Big|$$

$$\left.1\leqslant i\leqslant r^{n-(k+3)},\ \frac{r+1}{2}\leqslant L\leqslant r-1\right\}$$

To prove formula (3.52) for $k+1$ it suffices to calculate the maxima of the functions

$$\frac{1}{r^k}g_j^{n-k}(L,i)+h^n(i+Lr^{n-(k+3)})+H_{n,k}(i+Lr^{n-(k+3)})\,,$$

where $j=1,2$ and $H_{n,k}$ can be $A_{n,k}$, $B_{n,k}$ or $C_{n,k}$, for the values of (i,L) indicated in (3.56). (3.52) follows from a comparison of these maxima. □

Using this lemma we prove now the following theorem, stating the exact result for the stardiscrepancy of the sequence ω_{r^n} in the case r odd.

Theorem 1. *If r is odd then one has for every natural number $n\geqslant 2$ that*

$$D^*_{r^n}(\omega_{r^n})=\frac{r-1}{4r^n}n+\frac{1}{r^n}\left(\frac{5}{4}+\frac{1}{r}\right)-\frac{1}{4r^{2n}} \tag{3.57}$$

and this value is equal to the positive rest of the square

$$\left[0,\frac{1}{2}+\frac{1}{r}-\frac{1}{2r^n}\right]\times\left[0,\frac{1}{2}+\frac{1}{r}-\frac{1}{2r^n}\right]. \tag{3.58}$$

Proof. We treat first the case $n\geqslant 3$. It follows from (3.52) that

$$D^*_{r^n}(\omega_{r^n})=\frac{1}{r^{n-3}}\mathscr{F}^3_{1,1}+C_{n,n-2}(1)\,. \tag{3.59}$$

It can be verified that

$$\mathscr{F}^3_{1,1} = \frac{1}{r^3} - \left(\frac{r-1}{2r^3} + \frac{1}{r^2}\right)^2 + 2\left(\frac{r+1}{2r^3} - \frac{r+1}{2r}\cdot\frac{r-1}{2r^3}\right). \quad (3.60)$$

Formula (3.57) follows then from (3.59) and (3.60) by substituting the expressions for the functions h^n, g^n_1 and g^n_2 in (3.55). Formula (3.58) can be derived from the arguments of the function g^n_2 in $C_{n,n-2}(1)$. The case $n = 2$ is a consequence of Lemma 4. □

For r odd we did essentially the following. Starting from Lemma 4 we reduced the number of arguments to be considered for the calculation of $D^*_{r^n}(\omega_{r^n})$ by a factor r and using the function h^n we made the connection between the remaining arguments for $D^*_{r^n}(\omega_{r^n})$ and the arguments for $D^*_{r^{n-1}}(\omega_{r^{n-1}})$, and so on. In the case r is even, one has to make a distinction between even powers of r and odd powers of r. Therefore one has to reduce the number of arguments to be considered for the calculation of $D^*_{r^n}(\omega_{r^n})$ by a factor r^2 and make the connection between the remaining arguments for $D^*_{r^n}(\omega_{r^n})$ and the arguments for $D^*_{r^{n-2}}(\omega_{r^{n-2}})$. Using this procedure one can prove by induction for the case r even a lemma similar to Lemma 5, which leads to the following theorem.

Theorem 2. *For r even and for all even natural numbers $n \geqslant 2$ one has*

$$D^*_{r^n}(\omega_{r^n}) = \frac{r^2 n}{4r^n(r+1)} + \frac{1}{r^n}\left(\frac{5}{4} + \frac{2r+3}{4(r+1)^2}\right) - \frac{1}{4r^{2n}}\left(1 + \frac{2r+3}{(r+1)^2}\right) =$$

$$= E^+\left(\left[0, \frac{r+2}{2(r+1)} - \frac{r+2}{2r^n(r+1)}\right] \times \left[0, \frac{r+2}{2(r+1)} - \frac{r+2}{2r^n(r+1)}\right]\right).$$

and for r even and for all odd natural numbers $n \geqslant 3$ one has

$$D^*_{r^n}(\omega_{r^n}) = \frac{r^2 n}{4r^n(r+1)} + \frac{1}{r^n}\left(\frac{5}{4} + \frac{5r+4}{4r(r+1)^2}\right) +$$

$$+ \frac{1}{r^{2n}}\left(\frac{r}{2} - \frac{1}{4} - \frac{1}{r} + \frac{5}{4r^2} - \frac{6r+5}{4r^2(r+1)^2}\right) =$$

$$= E^+\left(\left[0, \frac{r^2+2r+2}{2r(r+1)} - \frac{r+2}{2r^n(r+1)}\right] \times \left[0, \frac{r+2}{2(r+1)} + \frac{2r^2-r-2}{2r^n(r+1)}\right]\right).$$

This theorem generalizes the formulas for $D^*_{2^n}(\omega_{2^n})$ obtained by HALTON and ZAREMBA [7]. Finally we treat in the last theorem the case $n = 1$ for all values of r.

Theorem 3. *For all r one has*

$$D_r^*(\omega_r) = \frac{[r/2]+1}{r} - \frac{[r/2]^2}{r^2} = \tag{3.61}$$

$$= E^+\left(\left[0, \left[\frac{r}{2}\right]/r\right] \times \left[0, \left[\frac{r}{2}\right]/r\right]\right) \tag{3.62}$$

Proof. Formulas (3.61) and (3.62) follow from the formulas (3.23), (3.24), (3.25) and (3.26).

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L. DE CLERCK
Instituut voor Theoretische Fysica
Katholieke Universiteit Leuven
Celestijnenlaan 200 d
B-3030 Leuven (Heverlee), Belgium