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Math. Zeitschr. Bd. 70, S. 372—380 (1959)

Finiteness of the injective hull

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1. Introduction

It is well known that every left module M over a ring R can be embedded in an injective left R-module. In this note, we consider the following more detailed questions: If M has finite length (i.e. has a finite composition series) does there exist an injective module containing M which also has finite length? Even more generally, do there exist any injective R-modules of finite length?

This problem was brought to the fore recently by AZUMAYA [2] who showed that in extending Pontryagin-type duality to modules over a ring with minimum condition, the dual module of M must be defined as $\mathrm{Hom}_R(M, Q)$ where Q is an injective module of finite length containing all simple left R-modules (cf. also [9, § 4] [11, p. 108]).

The answer to the questions posed in the first paragraph is clearly no as long as R is arbitrary (e.g., $R =$ the integers). However, we restrict ourselves to rings with left minimum condition where the problem had its origin. Here the answers are again no, because the condition for embeddability turns out to be equivalent to an additional, nonvacuous finiteness condition on the ***right*** structure of the ring (Theorem 1). If we restrict ourselves further to rings with both minimum conditions, our question becomes equivalent to a generalized version of an old question of ARTIN'S [5, p. 176] and [8, p. 6]: If a division ring has finite left dimension over a division subring, does it also have finite right dimension? The generalized problem is the same with division rings replaced by semisimple rings with minimum condition. We conjecture that the answers to all these questions are still no, but in view of the complications inherent in division rings infinite over their centre, finding a counterexample seems to be quite difficult.

On the other hand, we can show that the answers to our embeddability questions are yes if the ring is sufficiently like a finite dimensional algebra (the extra hypothesis in Theorem 3 if imposed on all the simple homomorphic images is actually equivalent to assuming a polynomial identity). It should also be noted that quasi-Frobenius rings satisfy our embeddability condition.

In 3 we drop all minimum conditions and assume that every simple left R-module is contained in an injective module of finite length. Then the radical turns out to be nil and nearly nilpotent (Theorem 4). If in addition R is commutative our assumption is satisfied if and only if every ring of quo-

[1]) Part of this work was done while the authors were on leave at the Institute for Advanced Study, Princeton, N. J. A. ROSENBERG was there supported by N. S. F. grant G-1974 and D. ZELINSKY was a John Simon Guggenheim Fellow.

tients R_M (M a maximal ideal of R) satisfies the minimum condition (Theorem 5). Examples of rings satisfying the latter hypothesis are furnished by Theorem 6, which is due to KAPLANSKY: R is (von Neumann-) regular if and only if R_M is a field for every M, or equivalently, if and only if each module R/M is injective.

2. Rings with minimum condition

We consider only rings with unit but we begin by imposing no finiteness restrictions on the rings. For such rings it seems more appropriate to consider, not finitely generated modules, but modules of finite length, i.e. modules having finite composition series. Of course, for rings with minimum condition these two classes of modules coincide.

According to ECKMANN and SCHOPF [4] every module A over a ring R has a unique injective, essential extension module[2]), called the *injective hull* of A and written $\hat{A}$. The injective hull of A is contained in every injective extension of A and contains every essential extension of A. Moreover, the following facts follow readily:

a) If A is an essential extension of B, then $\hat{A} = \hat{B}$.

b) If $A = A_1 \oplus \cdots \oplus A_n$ then $\hat{A} = \hat{A}_1 \oplus \cdots \oplus \hat{A}_n$.

c) A module A is embeddable in an injective module of finite length if and only if $\hat{A}$ has finite length.

d) The injective hull of A has finite length only if A has finite length.

e) Let S denote the *socle* of A, i.e. the sum of all simple submodules of A. If A has finite length, A is an essential extension of S and thus $\hat{A} = \hat{S}$ by a). Moreover, if $S = S_1 \oplus \cdots \oplus S_n$ is written as a direct sum of simple modules, $\hat{S} = \hat{S}_1 \oplus \cdots \oplus \hat{S}_n$ by b). Cf. [6, § 2 and § 3].

We may, therefore, conclude that a module A of finite length is embeddable in an injective module of finite length if and only if each simple submodule of A has an injective hull of finite length. Thus having reduced the embeddability of arbitrary modules in injectives of finite length to that of simple modules, we proceed to our main device

LEMMA 1. *Let R be a ring, $R = N_0 \supseteqq N_1 \supseteqq \cdots \supseteqq N_k = 0$, a chain of two-sided ideals in R. For every left R-module A define the (Loewy-) series*

$$0 = A_0 \subseteqq A_1 \subseteqq \cdots \subseteqq A_k = A$$

by setting $A_i =$ annihilator of N_i in $A = \{x \in A \mid N_i x = 0\}$. Then there is a natural left R-monomorphism[3])

$$A_{i+1}/A_i \to \mathrm{Hom}_R(N_i/N_{i+1}, A).$$

If A is injective, this is an isomorphism.

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[2]) An extension module of A is essential if every nonzero submodule has a nonzero intersection with A.

[3]) The Hom below is a left R-module by virtue of the right operations of R on N_i/N_{i+1}, viz $(r\varphi)(\bar{n}) = \varphi(\bar{n}r)$, for r in R, $\bar{n}$ in N_i/N_{i+1}, φ in $\mathrm{Hom}_R(N_i/N_{i+1}, A)$.

PROOF. We may identify A_i with $\mathrm{Hom}_R(R/N_i, A)$ by letting correspond to x in A_i the homomorphism sending $1+N_i$ into x. Since

$$0 \to N_i/N_{i+1} \to R/N_{i+1} \to R/N_i \to 0$$

is exact and the mappings are two-sided R homomorphisms we have an exact sequence of left R-modules

$$0 \to \mathrm{Hom}_R(R/N_i, A) \to \mathrm{Hom}_R(R/N_{i+1}, A) \to \mathrm{Hom}_R(N_i/N_{i+1}, A)$$

and, in fact, we can adjoin a zero on the right to get a five term exact sequence in case A is injective [3; II, 4.6]. Making the above identification we obtain the desired monomorphism (isomorphism if A is injective).

LEMMA 2. *With notations as in Lemma 1, suppose that each N_i/N_{i+1} is a semisimple (= completely reducible) left R-module. Let A be a left R-module of finite length with socle S. Then A is embeddable in an injective module of finite length if and only if* $\mathrm{Hom}_R(N_i/N_{i+1}, S)$ *has finite length as a left R-module for* $i=0, 1, \ldots, k-1$.

PROOF. By Lemma 1, $\hat{A}$ is the result of a finite succession of module extensions with factors $\mathrm{Hom}_R(N_i/N_{i+1}, \hat{A})$. Thus $\hat{A}$ has finite length if and only if $\mathrm{Hom}_R(N_i/N_{i+1}, \hat{A})$ has finite length for all i. Now every homomorphism of a semisimple module has a semisimple image so that $\mathrm{Hom}_R(N_i/N_{i+1}, \hat{A}) = \mathrm{Hom}_R(N_i/N_{i+1}, S')$ where S' is the socle of $\hat{A}$. However, $S'=S$ because if T is a complement of S in the semisimple module S', $T \cap A = 0$; but $\hat{A} = \hat{S}$ by e), thus $\hat{A}$ is an essential extension of S and so $T=0$ and $S=S'$, proving the Lemma.

As a direct consequence of Lemma 2 and the fact that every module over a semisimple ring (with minimum condition) is semisimple, we have

THEOREM 1. *Suppose N is a nilpotent ideal in a ring R and R/N is semisimple with minimum condition. Then a left R-module A with socle S is contained in some injective module of finite length if and only if the left R-modules* $\mathrm{Hom}_R(N^i/N^{i+1}, S)$ *have finite length* $(i=0, 1, \ldots)$[4].

REMARK 1. An equally good necessary and sufficient condition is: the (R/N)-modules $\mathrm{Hom}_{R/N}(N^i/N^{i+1}, S)$ have finite length (equivalently, are finitely generated) for $i=0, 1, \ldots$.

REMARK 2. The condition that $\mathrm{Hom}_R(N^i/N^{i+1}, S)$ has finite length for *every* simple left R-module S (equivalently, for every semisimple S of finite length) and for $i=1, 2, \ldots$ is equivalent to the same condition postulated only for $i=1$. To prove this by induction on i, consider the natural two-sided R-epimorphism $N^{i-1} \otimes_R N \to N^i$ determined by $m \otimes n \to mn$ for $m \in N^{i-1}$ and $n \in N$. This induces an epimorphism $(N^{i-1}/N^i) \otimes_R (N/N^2) \to N^i/N^{i+1}$, which in turn gives an R-monomorphism

$$\mathrm{Hom}_R(N^i/N^{i+1}, S) \to \mathrm{Hom}_R(N^{i-1}/N^i \otimes_R N/N^2, S) =$$
$$\mathrm{Hom}_R(N^{i-1}/N^i, \mathrm{Hom}_R(N/N^2, S))$$

[4]) Here we set $N^0 = R$.

(cf. [*3*, II, 5.2′]). But $S' = \mathrm{Hom}_R(N/N^2, S)$ has finite length by hypothesis and is semisimple because $NS' = 0$, and so $\mathrm{Hom}_R(N^{i-1}/N^i, S')$ also has finite length by the induction hypothesis. So also do all its submodules, completing the induction.

Suppose now that R is a ring with left minimum condition, N its radical, A a simple left R-module, and V_i the corresponding homogeneous component of the left module N^i/N^{i+1}. It is well known that V_i is a two sided R-module so that if E_i denotes the endomorphism ring of V_i as left R-module, E_i contains as a subring R_i, the mappings induced by the right operations of R on V_i. Moreover, since V_i is a finite direct sum of copies of A, the ring E_i is just a matrix ring over the division ring of R-endomorphisms of A, i.e. E_i is simple with minimum condition. Further, since $V_i N \subseteqq (N^i/N^{i+1})N = 0$, the ring R_i is a homomorphic image of R/N and so is a semisimple subring of E_i. With these notations we then have

Lemma 3. *A is embeddable in an injective module of finite length if and only if E_i is a finitely generated left R_i module for $i = 0, 1, 2, \ldots$.*

Proof. Since V_i is a finite direct sum of copies of A, $E_i = \mathrm{Hom}_R(V_i, V_i)$ is a finite direct sum of copies of $\mathrm{Hom}_R(V_i, A) = \mathrm{Hom}_R(N^i/N^{i+1}, A)$, and this decomposition is a left R-module decomposition. Thus E_i has finite length as a left R_i-module if and only if $\mathrm{Hom}_R(N^i/N^{i+1}, A)$ has. This together with Theorem 1, the subsequent remark, and the fact that left action of R on E_i is exactly the left action on E_i of the subring R_i proves the Lemma.

That the condition of Lemma 3 is not always fulfilled is shown by the following example:

Let K be any field with an isomorphism σ into itself such that $[K:\sigma K] = \infty$; e.g., $K = F(x_1, x_2, \ldots)$ with F a field, $\sigma(x_i) = x_{i+1}$ and $\sigma =$ the identity on F. Define a two-sided K-module N as follows: as a left K-module N is isomorphic to K; on the right $nk = (\sigma k)n$ for $k \in K$, $n \in N$. If we set $N^2 = 0$ the two-sided K-module direct sum $R = K + N$ is a ring with left minimum condition and with only one simple module, $A = R/N$. Here $E_1 = \mathrm{Hom}_R(N, N) =$ the ring of K-endomorphisms of K, which is the set of left multiplications $l_k: n \to kn$, R_1 is the set of right operators on N which by definition is the set of $l_{\sigma(k)}$. Thus E_1 as a left R_1-module is isomorphic to K as a σK-module. Since it is not finitely generated R admits no injective modules of finite length.

Although the ring in this example has left minimum condition it does not satisfy the right minimum condition. The next Lemma shows that the condition of Lemma 3 is closely analogous (but presumably not equivalent) to the right minimum condition.

Lemma 4. *With the same notations as in Lemma 3, suppose that R also satisfies the right minimum condition. Then for each $i = 0, 1, 2, \ldots$, E_i is a finitely generated right R_i-module.*

Proof. It is well known [*1*, 5.7A] that as a right E_i-module V_i is the direct sum of a finite number of copies of the unique simple right E_i-module, W_i. Furthermore E_i as a right E_i-module is also a finite direct sum of copies of

W_i. Now since R satisfies the minimum condition the following right R-modules are finitely generated:

$$N^i, N^i/N^{i+1}, V_i \subseteqq (N^i/N^{i+1}), W_i, \text{ and so finally } E_i.$$

Note that if conversely we let $i = 0, 1, 2, \ldots$ and let A range over all the simple left modules of R, the resulting set of conditions, E_i finitely generated as a right R_i-module, imply that R satisfies the right minimum condition.

For rings satisfying both minimum conditions the only information on the embeddability problem we have is given in

THEOREM 2. *The following statements are equivalent:*

(i) *There exists a ring R satisfying both left and right minimum conditions but having a simple left-module not embeddable in a finitely generated injective module.*

(ii) *There exists a simple ring with minimum condition E and a semisimple subring T with minimum condition and containing the unit of E such that E is finitely generated as a right T-module but not as a left T-module.*

PROOF. That (i) implies (ii) is a consequence of Lemmas 3 and 4.

Conversely, given E and T as in (ii) we construct R as follows: Let N be a simple right E-module and $D = \mathrm{Hom}_E(N, N)$ so that N is a finite dimensional left vector space over the division ring D and $T \subseteqq E = \mathrm{Hom}_D(N, N)$. Consider N as a left module over the ring $D \oplus T$ by setting $TN = 0$. Right operation of T on N is already defined since $T \subseteqq E$; if we then set $ND = 0$ we have made N into a two sided $D \oplus T$ module.

Next, define a multiplication in N by setting all products equal to zero. Then $R = (D \oplus T) \dot{+} N$[5]) is a ring. Since $TN = NN = 0$ and $[N:D]_l < \infty$, the left R-module N is of finite length. Thus R is a left R-module of finite length, i.e. R satisfies left minimum condition. As for right minimum condition, we are given that E is a right T-module of finite length. But N is an E-direct summand of E so that N is also a right T-module of finite length. Since $ND = NN = 0$, N is also a right R-module of finite length and so R satisfies the right minimum condition.

If we now choose $A = R/(T + N) = D$, then $E_1 = E$, $R_1 = T$ and so Lemma 3 asserts that A is not embeddable in a finitely generated injective.

REMARK. If there is a simple ring E with N and D as above and if there is a ring isomorphism of E onto a proper subring T with $[E:T]_l = \infty$ but $[E:T]_r < \infty$, we can construct a ring $R^* = D + N$ much as above which satisfies both minimum conditions and which is also completely primary and so has no finitely generated injectives at all.

The question naturally arises as to the independence of the two minimum conditions on a ring R and the condition that all the simple left modules have injective hulls of finite length. The examples we have constructed so far consist in producing a left D-, right T-module N and asking about the finiteness of $[N:D]$ (left minimum condition in R), of $[N:T]$ (right minimum condition)

[5]) $\dot{+}$ means two sided $(D \oplus T)$-module direct sum.

and of $[N^*:T]$ (finite length of injective hulls) where $N^* = \mathrm{Hom}_R(N, A) = \mathrm{Hom}_D(N, D)$ [since $\mathrm{Hom}_R(N, T) = 0$, it is only the injective hull of the simple module D that need be considered]. It is standard, of course, that $[N^*:D]$ is finite if and only if $[N:D]$ is. Thus if we assume the existence of asymmetric ring extensions as in Theorem 2, and build a ring with radical N^* rather than N, an anti-isomorphism of this ring would give an example showing that the right minimum condition is independent of the other two conditions. However, it is plausible that the left minimum condition is implied by the other two; in the type of example analyzed above, if T is a simple ring then the finiteness $[N:T]$ and of $[N^*:T]$ imply that N and N^* have the same cardinality, which implies at least that $[N:D]$ is less than the cardinal of D.

It is well-known that if R is a finite dimensional algebra over a field K, then every simple left module $(R/N)\bar{e}$ is embeddable in the injective module $\mathrm{Hom}_K(eR, K)$ which has finite length (even as a K-module) (cf. [7, Lemma 5]). We proceed to generalize this fact, using the techniques above.

Lemma 5. *Let E be an algebra over a commutative ring Z and finitely generated as a Z-module. Let R be a right Noetherian subring of E (having the same unit as E) and suppose that E is finitely generated as a right R-module. Th n E is finitely generated as a left R-module, too.*

Proof. Since $ZR \subset E$, the right R-module ZR is also finitely generated and so we may write $ZR = z_1R + z_2R + \cdots + z_nR = Rz_1 + Rz_2 + \cdots + Rz_n$ for z_i in Z, $i = 1, 2, \ldots, n$. Hence ZR is also a finitely generated left R-module. But then any left ZR-module which is finitely generated will also be finitely generated as a left R-module. Now E is a finitely generated left Z module and so a finitely generated left ZR-module, proving the theorem.

Theorem 3. *Let R be a ring satisfying both left and right minimum conditions and A a simple left R-module. Let U be the annihilator of A in R and suppose that the simple ring $R/U = R'$ is finite over its center. Then $\hat{A}$ is of finite length as a left R-module.*

Proof. The module V_i as defined before Lemma 3 is a left R'-module and $E_i = \mathrm{Hom}_R(V_i, V_i) = \mathrm{Hom}_{R'}(V_i, V_i)$ is finite over its center if R' is. But Lemma 4 implies that E_i is a finitely generated right R_i-module whence by Lemma 5 E_i is also finitely generated as a left R_i-module. Thus Lemma 3 concludes the proof.

3. General rings

In this section we study the effect on the structure of a ring R of assuming that all the injective hulls of its simple left modules have finite length. Most of our results are based on a general lemma which is a kind of analogue of the usual subdirect sum decomposition of a ring with zero radical:

Lemma 6. *Let R be any ring with unit and $\{\hat{A}\}$ the set of injective hulls of simple left R-modules. Then R is faithfully represented by its action on the direct sum of the $\hat{A}$. That is, if $x \in R$ and $x\hat{A} = 0$ for every $\hat{A}$ then $x = 0$. More* *specifically, if M is any maximal left ideal in R containing the left annihilator $l(x)$ of x, and if $\hat{A}$ is the injective hull of R/M, then $x\hat{A} \neq 0$.*

PROOF. The cyclic module Rx is isomorphic to $R/l(x)$, which can be mapped onto R/M. This gives a nonzero mapping of Rx into the injective hull $\hat{A}$ of R/M, which must be of the form $rx \to rxa$ for some fixed a in $\hat{A}$. Hence $xa \neq 0$.

THEOREM 4. *Let R be a ring with unit and with* (JACOBSON) *radical N. Suppose that the injective hull of every simple left R-module has finite length. Then N is a nilideal and $\bigcap_{i=1}^{\infty} N^i = 0$. Furthermore, if R has only a finite number of isomorphism classes of simple modules, then N is nilpotent.*

PROOF. First we remark that, since N annihilates every simple module, NB has smaller length than B for every module B of finite length. Thus $N^k B = 0$ for some k.

Now let $n \in N$. If there is a single maximal left ideal M containing all the left annihilators $l(n^i)$, $i = 1, 2, \ldots$, then by Lemma 6, the injective hull $\hat{A}$ of R/M satisfies $n^i \hat{A} \neq 0$ for all i, which contradicts the fact that $N^k \hat{A} = 0$ for some k. Thus $\bigcup_{i=1}^{\infty} l(n_i) = R$, so that 1 annihilates some n^i, proving that N is nil.

Next, if $n \in \bigcap_i N^i$, then n annihilates every module of finite length. Under the hypotheses of our Theorem, we may apply Lemma 6 to conclude $n = 0$.

Finally, if there is only a finite number of nonisomorphic simple modules, there is only a finite number of nonisomorphic injective hulls $\hat{A}$. Then a single power of N annihilates them all, and, by Lemma 6, this power of N is 0.

REMARK. It is not true that N is nilpotent if we assume only that the injective hulls of the simple modules have finite lengths. For example, let $C_i = K[x]/x^i K[x]$ where K is a field and x an indeterminate. Let R be the K-algebra obtained by adjoining a unit (algebra style) to the direct sum of the C_i. Then the radical of R is nil but not nilpotent, while the injective hull of every simple R-module has finite length.

From now on we suppose R is a commutative ring with unit. For any maximal ideal M of R there is associated a ring of quotients R_M and a ring-homomorphism $R \to R_M$ such that every element of R which is not in M maps into a unit in R_M, and R_M is universal with respect to this property; that is, any ring-homomorphism $R \to S$ which associates units to all elements outside M is factorable thus: $R \to R_M \to S$. In particular, if A is any R-module with the property that every element of R not in M induces an automorphism of A, then A is an R_M-module in a natural way (take $S = \mathrm{Hom}_R(A, A)$). We should also note that R_M has a unique maximal ideal [*10*, Chapter IV, §§ 9—11].

We can now localize our basic hypothesis as follows:

LEMMA 7. *Let R be a commutative ring with unit, M a maximal ideal in R, $A = R/M$ and $\hat{A}$ the injective hull of A. Then A and $\hat{A}$ are R_M-modules. As an*

R_M-module, A is still simple and $\hat{A}$ is its injective hull. Furthermore, $\hat{A}$ has finite length as an R-module if and only if it has finite length as an R_M-module.

PROOF. Let $t \in R$, $t \notin M$. Then t acts on A as a nonzero element of the field R/M; thus t acts as an automorphism of A. It follows that A is an R_M-module. Every R_M-submodule of A will be an R-submodule, so A is a simple R_M-module.

Now consider the action of t on $\hat{A}$. The kernel of this operation is a submodule of $\hat{A}$ intersecting A in 0. Since $\hat{A}$ is an essential extension of A, this kernel is 0. It follows that $t\hat{A} \cong \hat{A}$ so that $t\hat{A}$ is an injective submodule of $\hat{A}$ containing A (we proved above that $tA = A$). Thus $t\hat{A} = \hat{A}$, showing that t induces an automorphism of $\hat{A}$[6]). Once again this implies that $\hat{A}$ is an R_M-module. Clearly, if $\hat{A}$ satisfies chain conditions on R-submodules, it will also satisfy chain conditions on R_M-submodules. Conversely, if $\hat{A}$ has an R_M-composition series, the factors will be isomorphic to R/M (the only simple R_M-module) which is also a simple R-module; thus this same series will be an R-composition series.

In the same vein, if $\hat{A}$ is an essential extension of A as an R-module, it will be likewise as R_M-module. It remains to prove $\hat{A}$ is R_M-injective. If X is any R_M-module, it is an easy consequence of the definition of R_M that the ring of operators on X induced by R_M is generated by the operators induced by R and the inverses of the operations by elements of R not in M. It follows that $\mathrm{Hom}_R(X, \hat{A}) = \mathrm{Hom}_{R_M}(X, \hat{A})$. Since the first of these Hom's is an exact functor of X, so is the second, proving $\hat{A}$ is R_M-injective.

THEOREM 5. *Let R be a commutative ring with unit. Then the injective hull of a simple R-module R/M has finite length if and only if R_M is a ring with minimum condition.*

PROOF. If R_M satisfies the minimum condition, the R_M-injective hull of R/M has finite length as an R_M-module by Theorem 3. Lemma 7 then completes the proof. Conversely if the injective hull of R/M has finite length, the radical N of R_M must be nilpotent by Theorem 4 and the modules $\mathrm{Hom}_{R_M}(N^i/N^{i+1}, R/M)$ have finite length by Lemma 2. Since R_M is commutative, this simply means that N^i/N^{i+1} is a finite dimensional vector space over the field R/M, so that R_M does indeed satisfy the minimum condition.

COROLLARY. *If R is an integral domain and some simple R-module has an injective hull of finite length, then R is a field.*

PROOF. Suppose the injective hull of R/M has finite length. Then R_M is an integral domain with minimum condition by Theorem 5. Thus R_M must

[6]) Even for noncommutative R and for nonsimple modules A, this proof shows that if u is an element of $S = \mathrm{Hom}_R(\hat{A}, \hat{A})$ which vanishes on A, then $1 + u$ is an automorphism of $\hat{A}$. Hence, these elements u form a radical left ideal U of S. If, besides, A is a characteristic submodule of $\hat{A}$ (e.g., if A is semisimple) then U is a two-sided ideal and, from the injectivity of $\hat{A}$, it is trivial to prove that $S/U = \mathrm{Hom}_R(A, A)$ (cf. [6, Theorem 4.2]).

be a field. But this means $MR_M=0$, $M=0$ [*10*, Chapter IV, Theorem 19] so that R is a field.

LEMMA 8. *Let R be a commutative ring and M a maximal ideal in R. Then R/M is itself an injective R-module if and only if R_M is a field.*

PROOF. If R/M is R-injective, then it is R_M-injective by Lemma 7, and the radical of R_M is zero by Lemma 6. Thus R_M is a field. Conversely, if R_M is a field, R/M is R_M-injective, hence R-injective by Lemma 7.

The next result was first found by I. KAPLANSKY, who proved it by direct methods.

THEOREM 6. *A commutative ring R (with unit) is (von Neumann-) regular if and only if every simple R-module is injective.*

PROOF. We shall show that R is regular if and only if every R_M is a field. If R is regular, consider its image R' under the mapping $R\to R_M$. Since R' is also regular, every element not a zero divisor is a unit. The images of elements not in M are not zero divisors even in R_M, so they have inverses in R'. Thus $R_M=R'$ is a regular ring with a unique maximal ideal. Hence R_M is a field.

For the converse, we remark that the natural identification $R_M\to R_M\otimes_R R$ induces an identification of the mapping $R\to R_M$ with the mapping $a\to 1\otimes a$ for $a\in R$. If R_M is a field, the ideals generated by $1\otimes a$ and by $1\otimes a^2$ are equal. Since R_M is R-flat [*3*, VII, Ex. 10] these ideals are $R_M\otimes Ra$ and $R_M\otimes Ra^2$, respectively. Thus $(R_M\otimes Ra)/(R_M\otimes Ra^2)=R_M\otimes(Ra/Ra^2)=0$ for all $a\in R$ and all M. By [*3*, VII, Ex. 11] this implies $Ra/Ra^2=0$ for all a; that is, $a=a^2x$ for some x in R. Thus R is regular.

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(Eingegangen am 1. Juni 1958)