

Schwinger model in the light-cone representation

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Abstract. I present a solution to the Schwinger model in the light-cone representation which corrects an error in a previous work. I emphasize the details of the mechanism by which the physical vacuum is different than the perturbative vacuum. I suggest that the method of analyzing vacuum structure presented here may be of use in more complicated theories such as QCD.

1 Introduction

In a previous paper [1] I presented several solutions to the Schwinger model in light-cone gauge, some quantized at equal time, some quantized on the characteristic. The starting point in all cases was the Coulomb gauge solution of Nakawaki [2]. One of the solutions in reference 1 has an inconsistency. In the present paper I shall point out the inconsistency and supply what I believe is a correct construction.

In presenting the solution we shall pay particular attention to the mechanism by which degenerate vacua are possible in the light-cone representation (in spite of formal arguments which suggest that only the perturbative vacuum can be a physical vacuum in that representation). The effect is precisely like an anomaly: one is faced with an ill defined operator product; in giving a precise definition to that product one cannot maintain all properties of the classical product; in this case one must either give up gauge invariance or the kinematical nature of the operator P^+ which forms the basis of the argument that degenerate vacua are not possible in the light-cone representation. Even though other states than the perturbative vacuum do become degenerate with it under the interaction, and even though the operator P^+ does change (slightly) due to the interaction, the degenerate physical vacua of the model are much simpler in the light-cone representation than in the equal-time representation and the operator P^+ is much simpler than the manifestly dynamical operator P^- . The corrections to P^+ are independent of the coupling constant so the effect is nonper-

turbative. If one wishes to find the states degenerate with the vacuum it is only necessary to study P^+ (a full description of the dynamics certainly requires the study of P^-). Such an effect may be present in more complicated theories and the study of a fully gauge invariant and renormalized P^+ may provide a way to study vacuum structure without involving the full dynamics.

In what follows we shall attempt to adhere to the following notation which is consistent with [1] and [2]:

Coordinates:

$$x^0 = t; \quad x^1 = x; \quad g^{00} = -g^{11} = 1; \quad g^{10} = g^{01} = 0$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$x^+ = x^0 + x^1; \quad x^- = x^0 - x^1; \quad g^{+-} = g^{-+} = 2;$$

$$g^{++} = g^{--} = 0$$

$$g_{+-} = g_{-+} = \frac{1}{2}; \quad \gamma^+ = \gamma^0 + \gamma^1; \quad \gamma^- = \gamma^0 - \gamma^1$$

Particulars:

$$m = \frac{e}{\sqrt{\pi}} \quad (e \text{ is the electromagnetic coupling constant})$$

$$k_-(n) = \frac{(n - \frac{1}{2})\pi}{L}; \quad k_+(n) = \frac{(n - \frac{1}{2})\pi}{L}$$

$$p_-(n) = \frac{n\pi}{L}; \quad p_+(n) = \frac{m^2 L}{4n\pi}$$

$$p_+(n) = \frac{1}{2}(p_0(n) + p_1(n)); \quad p_-(n) = \frac{1}{2}(p_0(n) - p_1(n))$$

$$p^-(n) = 2p_+(n); \quad p^+(n) = 2p_-(n)$$

ψ -field:

$$\begin{aligned} \psi_- &\text{-first component of } \psi \\ \psi_+ &\text{-second component of } \psi \end{aligned}$$

2 Solution

The solution in [1] which has the inconsistency is that given by equations (3.89) to (3.96) of that paper. The solution is canonical on the initial value surface and satisfies the equations of motion. Furthermore almost all the operators satisfy the Heisenberg equations; but there are two operators which do not. These are the spurions, σ_+ and σ_- (these operators are defined in [1] and also below). The space-time dependence for the spurions as given in [1] was:

$$\sigma_+(x) = e^{-i\frac{\sqrt{\pi}}{4Lm}((Q-Q_s)x^- + Qx^+)} \times \sigma_+(0) e^{-i\frac{\sqrt{\pi}}{4Lm}((Q-Q_s)x^- + Qx^+)} \quad (2.1)$$

$$\sigma_-(x) = e^{-i\frac{\sqrt{\pi}}{4Lm}((Q+Q_s)x^+ - Q_sx^-)} \times \sigma_-(0) e^{-i\frac{\sqrt{\pi}}{4Lm}((Q+Q_s)x^+ - Q_sx^-)}, \quad (2.2)$$

The terms in the dynamical operators of that solution which do not commute with the spurions, which we shall call P_0^+ and P_0^- , were given as:

$$P_0^+ = \frac{1}{4Lm^2} (Q^2 - 2QQ_s) \quad (2.3)$$

$$P_0^- = \frac{1}{4Lm^2} (Q^2 + 2QQ_s). \quad (2.4)$$

The relevant commutator algebra is:

$$[Q, \sigma_+(0)] = -[Q_s, \sigma_+(0)] = -\sigma_+(0) \quad (2.5)$$

$$[Q, \sigma_-(0)] = [Q_s, \sigma_-(0)] = -\sigma_-(0) \quad (2.6)$$

It is trivial to use these relations to check that the spurions do not satisfy the Heisenberg equations. It might be thought that obvious way to proceed would be to modify the dynamical operators to properly translate the spurions and thus produce a canonical system satisfying the equations of electrodynamics. Unfortunately that does not work for it seems that there are no operators which one can choose for P^+ and P^- that will work. That such a situation is possible may be of interest in itself but I will not further discuss the point here. Rather we shall now give what we believe to be a correct construction.

The classical Lagrangian density is:

$$\mathcal{L} = \frac{1}{2} (i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\partial_\mu\bar{\psi}\gamma^\mu\psi) - \frac{1}{4} F^{\mu\nu}F_{\mu\nu} - A^\mu J'_\mu. \quad (2.7)$$

The need to use J' is discussed in reference 1; the relation of J' to J depends on boundary conditions. For the boundary conditions we will use below (periodic observables along the initial characteristics) it is:

$$J^{+'} = J^+ - \frac{1}{2}J^+(0) - \frac{1}{2}J^-(0) \quad (2.8)$$

$$J^{-'} = J^- - \frac{1}{2}J^+(0) - \frac{1}{2}J^-(0). \quad (2.9)$$

Where

$$J^\mu = : \bar{\psi}\gamma^\mu\psi : \quad (2.10)$$

and $J^+(0)$ and $J^-(0)$ are the zero modes in J^+ and J^- . We initialize ψ_+ on $x^+ = 0$ with antiperiodic boundary conditions:

$$\psi_+(0, x^-) = \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} b(n) e^{-ik_-(n)x^-} + d^*(n) e^{ik_-(n)x^-} \quad (2.11)$$

and ψ_- on $x^- = 0$:

$$\psi_-(x^+, 0) = \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} \beta(n) e^{-ik_+(n)x^+} + \delta^*(n) e^{ik_+(n)x^+}. \quad (2.12)$$

We work in the gauge where A^+ is independent of x^- ; in which case it is independent of space-time [1]. The equations of motion are:

$$\frac{\partial\psi_+}{\partial x^+} + i\frac{1}{4} e(A^-\psi_+ + \psi_+A^-) = 0 \quad (2.13)$$

$$\frac{\partial\psi_-}{\partial x^-} + i\frac{1}{4} e(A^+\psi_- + \psi_-A^+) = 0 \quad (2.14)$$

$$\frac{\partial^2 A^-}{\partial x^{-2}} = -\frac{1}{2} J'^+ \quad (2.15)$$

$$\frac{\partial^2 A^-}{\partial x^+ \partial x^-} = -\frac{1}{2} J'^- \quad (2.16)$$

The operator solution is most easily written in terms of the fusion operators which we take to be:

$$i\sqrt{n}C(n) = \sum_{l=1}^n d(l)b(n-l+1) + \sum_{l=1}^{\infty} b^*(l)b(l+n) - d^*(l)d(l+n) \quad (2.17)$$

$$\sqrt{n}D(n) = \sum_{l=1}^n \delta(l)\beta(n-l+1) + \sum_{l=1}^+ \beta^*(l)\beta(l+n) - \delta^*(l)\delta(l+n). \quad (2.18)$$

The reason for the change in phase between the fusion operators associated with the ψ_+ and ψ_- fields is to produce agreement with the notation of [1, 2]. To define the spurion operators we first define the set of states $|M, N\rangle$:

$$|M, N\rangle = \delta^*(M) \dots \delta^*(1) d^*(N) \dots d^*(1) |0\rangle$$

$$(M > 0, N > 0)$$

$$|M, N\rangle = \beta^*(M) \dots \beta^*(1) d^*(N) \dots d^*(1) |0\rangle$$

$$(M < 0, N > 0)$$

$$|M, N\rangle = \delta^*(M) \dots \delta^*(1) b^*(N) \dots b^*(1) |0\rangle$$

$$(M > 0, N < 0)$$

$$|M, N\rangle = \beta^*(M) \dots \beta^*(1) b^*(N) \dots b^*(1) |0\rangle$$

$$(M < 0, N < 0).$$

(2.19)

The spurions can then be defined as:

$$\begin{aligned} [\sigma_+, D(n)] &= [\sigma_+, D^*(n)] = [\sigma_+, C(n)] \\ &= [\sigma_+, C^*(n)] = 0 \end{aligned} \quad (2.20)$$

$$\begin{aligned} [\sigma_-, D(n)] &= [\sigma_-, D^*(n)] = [\sigma_-, C(n)] \\ &= [\sigma_-, C^*(n)] = 0 \end{aligned} \quad (2.21)$$

$$(\sigma_+)^I |M, N\rangle = |M, N+I\rangle \quad (2.22)$$

$$(\sigma_-)^I |M, N\rangle = |M+I, N\rangle. \quad (2.23)$$

We need the charges Q_+ and Q_- :

$$Q_+ = 2e \sum_{n=1}^{\infty} b^*(n)b(n) - d^*(n)d(n) \quad (2.24)$$

$$Q_- = 2e \sum_{n=1}^{\infty} \beta^*(n)\beta(n) - \delta^*(n)\delta(n). \quad (2.25)$$

These are related to the charge and pseudocharge by:

$$\begin{aligned} Q &= \frac{1}{2}(Q_+ + Q_-) \\ Q_5 &= \frac{1}{2}(Q_- - Q_+), \end{aligned} \quad (2.26)$$

Central to the issue we most want to discuss is the definition of Fermi products. We take:

$$\begin{aligned} &:\psi_+^*(x)\psi_+(x): \equiv \\ &\lim_{\varepsilon^- \rightarrow 0} (e^{-ie \int_x^{x+\varepsilon^-} A_-^{(-)} dx^-} \psi_+^*(x+\varepsilon^-)\psi_+(x) \\ &\quad \times e^{-ie \int_x^{x+\varepsilon^-} A_-^{(+)} dx^-} - VEV) \end{aligned} \quad (2.27)$$

$$\begin{aligned} &:\psi_-^*(x)\psi_-(x): \equiv \\ &\lim_{\varepsilon^+ \rightarrow 0} (e^{-ie \int_x^{x+\varepsilon^+} A_+^{(-)} dx^+} \psi_-^*(x+\varepsilon^+)\psi_-(x) \\ &\quad \times e^{-e \int_x^{x+\varepsilon^+} A_+^{(+)} dx^+} - VEV). \end{aligned} \quad (2.28)$$

We shall understand the product defining the coupling term in the Lagrangian as:

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon^2 < 0}} \frac{1}{2} [A_\mu(x+\varepsilon)J'^\mu(x) + J'^\mu(x)A_\mu(x-\varepsilon)]. \quad (2.29)$$

With these definitions we can now give the operator solution as:

$$\psi_+ = \frac{1}{\sqrt{2L}} e^{\lambda_+^{(-)}(x)} \sigma_+(x) e^{\lambda_+^{(+)}(x)} \quad (2.30)$$

where

$$\begin{aligned} \lambda_+(x) &= i \sqrt{\frac{\pi}{L}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{p_-(n)}} \\ &\quad \times (C(n)e^{-ip(n)x} + C^*(n)e^{ip(n)x}) \end{aligned} \quad (2.31)$$

and

$$\sigma_+(x) = e^{-i\frac{\sqrt{\pi}}{4Lm}(Q_+(x^-+x^+))} \sigma_+(0) e^{-i\frac{\sqrt{\pi}}{4Lm}(Q_+(x^-+x^+))} \quad (2.32)$$

$$\psi_- = \frac{1}{\sqrt{2L}} e^{-\lambda_-^*(x^+)} \sigma_-(x) e^{\lambda_-(x^+)} \quad (2.33)$$

where

$$\lambda_D(x^+) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} D(n) e^{-ik_+(n)x^+} \quad (2.34)$$

and

$$\sigma_-(x) = e^{-i\frac{\sqrt{\pi}}{4Lm}(Q_-(x^+-x^-))} \sigma_-(0) e^{-i\frac{\sqrt{\pi}}{4Lm}(Q_-(x^+-x^-))} \quad (2.35)$$

$$\begin{aligned} A^- &= -\frac{i}{\sqrt{Lm}} \sum_{n=1}^{\infty} \frac{p_-(n)}{\sqrt{p_-(n)}} (C(n)e^{-ip(n)x} - C^*(n)e^{ip(n)x}) \\ &\quad - \frac{1}{Lm^2} Q_+ \end{aligned} \quad (2.36)$$

$$A^+ = -\frac{1}{Lm^2} Q_-. \quad (2.37)$$

The dynamical operators are:

$$\begin{aligned} P^- &= \frac{1}{4Lm^2} (Q_-^2 - Q_+^2) + \sum_{n=1}^{\infty} p_-(n) C^*(n) C(n) \\ &\quad + \sum_{n=1}^{\infty} 2k_+(n) D^*(n) D(n). \end{aligned} \quad (2.38)$$

$$P^+ = \frac{1}{4Lm^2} (Q_+^2 - Q_-^2) + 2 \sum_{n=1}^{\infty} p_-(n) C^*(n) C(n). \quad (2.39)$$

These relations can be used to calculate the currents which are:

$$\begin{aligned} J^+ &= \frac{im}{\sqrt{L}} \sum_{n=1}^{\infty} \sqrt{p_-(n)} \\ &\quad \times (C^*(n)e^{ip(n)x} - C(n)e^{-ip(n)x}) + \frac{1}{L} Q \end{aligned} \quad (2.40)$$

$$\begin{aligned} J^- &= \frac{im}{\sqrt{L}} \sum_{n=1}^{\infty} \frac{p_+(n)}{\sqrt{p_-(n)}} \\ &\quad \times (C(n)e^{-ip(n)x} - C^*(n)e^{ip(n)x}) + \frac{e}{L} \sum_{n=1}^{\infty} \sqrt{n} \\ &\quad \times (D^*(n)e^{ik_+(n)x^+} + D(n)e^{-ik_+(n)x^+}) + \frac{1}{L} Q. \end{aligned} \quad (2.41)$$

To complete the specification of the solution we must also define a physical subspace. It is usual to define the physical subspace for the Schwinger model to be the charge zero sector and we must do that here; if $|p\rangle$ is in the physical subspace then:

$$Q|p\rangle = 0. \quad (2.42)$$

Additionally, we must impose:

$$D(n)|p\rangle = 0. \quad (2.43)$$

States in the set (2.19) which have the form $|M, -M\rangle$ are in the physical subspace and we can choose any linear combination of these states for the vacuum then generate the entire representation space by applying polynomials in the C^* and D^* to that vacuum; if we wish to impose cluster decomposition we must choose a θ -state for the

vacuum [3]. It is easy to check that the Heisenberg equations are all satisfied and that (2.13)–(2.15) are satisfied. Equation (2.16) is not satisfied. To examine that fact in more detail we write out the left hand side and right hand side of (2.16) explicitly:

$$\begin{aligned} & \frac{ie}{2\sqrt{\pi L}} \sum_{n=1}^{\infty} \frac{p_+(n)}{\sqrt{p_-(n)}} (C(n)e^{-ip(n)x} - C_{(n)}^* e^{ip(n)x}) \\ & \neq \frac{ie}{2\sqrt{\pi L}} \sum_{n=1}^{\infty} \frac{p_+(n)}{\sqrt{p_-(n)}} (C(n)e^{-ip(n)x} - C_{(n)}^* e^{ip(n)x}) \\ & + \sum_{n=1}^{\infty} (D^*(n)e^{ik_-(n)x^+} + D(n)e^{-ik_-(n)x^+}). \end{aligned} \tag{2.44}$$

The difficulty is seen to be the last sum on the right hand side, which involves the D 's and D^* 's. That inequality is the reason we must make the specification of (2.43). With that specification of the physical subspace (2.16) is satisfied in matrix elements between physical states.

The only two complicated issues associated with finding the solution (2.30)–(2.39) are finding the zero modes in the A -fields and calculating the dynamical operators; we shall now discuss these two points. The boundary conditions we choose require that A^- be periodic along x^- . That is only possible if (2.15) and (2.16) have no zero mode. Those are the conditions which determine the zero modes in A^+ and A^- . Using (2.10), (2.8), (2.9), (2.27) and (2.28) we find that the requirement that (2.15) and (2.16) have no zero modes gives the relations:

$$A_0^- + \frac{\pi}{Le^2} Q_+ = 0 \tag{2.45}$$

$$A^+ + \frac{\pi}{Le^2} Q_- = 0 \tag{2.46}$$

where A_0^- is the zero mode of A^- .

We now turn to the problem of calculating the dynamical operators P^- and P^+ . That these are the correct operators is shown by the consistency of the solution. The problem with calculating them occurs any time different surfaces are used to initialize different fields. Since the initialization was done on both $x^+ = 0$ and $x^- = 0$ we expect to have to integrate some density over each surface to calculate the dynamical operators. If we work out the formal densities we find (see below) that we need to integrate functionals of the fields over regions which are not the initial value surfaces for those fields. If that were true we would be in the position of having to solve the problem before we could formulate it. Generally one would not expect that situation to arise: if we have properly chosen a set of degrees of freedom we should be able to calculate the momenta and energy from the given information. In all cases where the answer is known [4, 5, 1, 6] the following rule applies: work out the densities as usual but integrate over a given initial value surface only those parts of these densities which involve fields initialized on the integration surface; for gauge theories there is a further consideration which we shall come to below. We shall now illustrate this rule and show that it works in the

present case. Using:

$$T^{\mu\nu} = \sum_{\phi} \frac{\partial\phi}{\partial x_{\mu}} \frac{\partial\mathcal{L}}{\partial(\partial_{\nu}\phi)} - g^{\mu\nu} \mathcal{L} \tag{2.47}$$

we calculate:

$$T^{++} = 2i(\psi_{\dagger}^* \partial_- \psi_+ - \partial_- \psi_{\dagger}^* \psi_+): \tag{2.48}$$

$$\begin{aligned} T^{+-} = & 2i(\psi_{\dagger}^* \partial_+ \psi_+ - \partial_+ \psi_{\dagger}^* \psi_+): \\ & + :(\partial_- A^-)^2: + 2J'^{\mu} A_{\mu} \end{aligned} \tag{2.49}$$

$$\begin{aligned} T^{-+} = & 2i(\psi^* \partial_- \psi_- - \partial_- \psi^* \psi_-): \\ & - :(\partial_- A^-)^2: + 2J''^{\mu} A_{\mu} \end{aligned} \tag{2.50}$$

$$\begin{aligned} T^{--} = & 2i(\psi^* \partial_+ \psi_- - \partial_+ \psi^* \psi_-): \\ & + 2\partial_- A^- \partial_+ A^-: \end{aligned} \tag{2.51}$$

Using the rule stated above we calculate the dynamical operators as:

$$\begin{aligned} P^- = & \frac{1}{2} \int_{-L}^L - :(\partial_- A^-)^2: + 2J'^{\mu} A_{\mu} dx^- \\ & + \frac{1}{2} \int_{-L}^L :2i(\psi^* \partial_+ \psi_- - \partial_+ \psi^* \psi_-): dx^+ \\ P^+ = & \frac{1}{2} \int_{-L}^L :2i(\psi_{\dagger}^* \partial_- \psi_+ - \partial_- \psi_{\dagger}^* \psi_+): dx^- \end{aligned}$$

There is a further point associated with the fact we are dealing with a gauge theory: the fermi products in the above expressions need to be gauge corrected and for the case of P^- that again leads us to the need for a field, A^- along $x^- = 0$, where we do not know it; we again use only what we know and gauge correct only with the (space-time independent) zero mode of A^- . We thus calculate the dynamical operators as given by (2.38), (2.39).

3 Discussion

Equation (2.43) removes most of the states associated with the ψ_- field from the physical subspace. Indeed the only operators from that field which remain physical under the interaction are the spurion, σ_- , and the charge, Q_- . The D -field is an auxiliary field. While this may seem strange, the same conclusion can be reached by gauge transforming the Coulomb gauge solution [2] to light-cone gauge [1]. In the Coulomb gauge solution only physical degrees of freedom are present; under the gauge transformation only σ_- and Q_- survive from ψ_- . The D -field must be added to represent the operator solution. The need for auxiliary fields seems to be ubiquitous in light-cone gauge. Even free Maxwell theory requires one [7, 6]. The present situation is somewhat different however. Usually the additional fields are ghosts and their unphysical nature is manifest in the commutation relations; here, the D -field—which is a perfectly physical field in free theory—satisfies normal commutation relations and the only ways I know to find its unphysical nature are to gauge transform the Coulomb gauge solution or examine the equation of motion (2.16).

Finally we review the way in which states different from the perturbative vacuum come to mix with it. The usual argument that that cannot happen is that the operator, P^+ for the interacting theory is just the P^+ , P^+ for free theory. Since the physical vacuum must be an eigenstate of P^+ with eigenvalue 0, and since there is only one such state in free theory, that must be the vacuum state. Indeed the density, (2.48), we used to calculate P^+ has the same form as the free density, but the need to gauge correct the products, (2.27), introduces a modification. For free theory:

$$P^+ = \frac{1}{4Lm^2} Q_+^2 + 2 \sum_{n=1}^{\infty} p_-(n) C^*(n) C(n).$$

For the interacting theory (see (2.39)):

$$P^+ = \frac{1}{4Lm^2} (Q_+^2 - Q_-^2) + 2 \sum_{n=1}^{\infty} p_-(n) C^*(n) C(n).$$

The extra term in P^+ which allows the degeneracy of the states, $|M, N\rangle$, with the perturbative vacuum is independent of the coupling constant. Note that the additional term in P^+ is composed of operators associated with the field initialized on the surface $x^- = 0$. It seems that that must be the case since for all the degrees of freedom initialized on $x^+ = 0$ translations by P^+ move within the initial value surface and are thus set by the initial condi-

tions. For the Schwinger model one can find all the states which are candidates to mix with the vacuum by studying the operator P^+ , which, although not exactly the free P^+ is much simpler than P^- . To find the vacuum one would have to apply P^- to the candidate states but that is a much simpler task than finding the eigenvectors of P^- . The procedure may provide a useful way to study vacuum structure even for more complicated theories like QCD. I hope to report more on that possibility in the future.

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