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NATURAL FREQUENCIES OF A NON HOMOGENEOUS ISOTROPIC ELASTIC INFINITE PLATE OF VARIABLE THICKNESS RESTING ON ELASTIC FOUNDATION

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SOMMARIO. Si sono studiate le oscillazioni libere di una piastra elastica infinita, non omogenea, isotropa, di spessore parabolicamente variabile, poggiata su un suolo elastico. Applicando il metodo di Frobenius per la soluzione della equazione differenziale del moto si sono calcolate le frequenze, le deformate ed i momenti corrispondenti ai primi cinque modi di vibrazione per due combinazioni di condizioni al contorno, incastro-incastro ed incastro-appoggio e diversi valori della rastremazione, del parametro di non omogeneità e del modulo del suolo.

SUMMARY. The dynamic free response of a nonhomogeneous isotropic elastic infinite plate of parabolically varying thickness resting on an elastic foundation has been studied. The frequencies, deflections and moments corresponding to the first five modes of vibration have been computed for the two combinations of boundary conditions, clamped-clamped (C-C) and clamped-simply supported (C-SS) and various values of taper constant, nonhomogeneity parameter and foundation modulus by applying the method of Frobenius for the solution of the governing differential equation of motion.

1. INTRODUCTION

The nonhomogeneous structures are of great practical and academic interest. The nonhomogeneity in real bodies arises due to inclusion of a foreign material, imperfections or as a result of being composite material. Therefore, in such elastic bodies, the material properties are not constant but vary with position in a random manner. Plywood, timber, deltawood and fiber reinforced plastics are some examples. So far very few papers have been devoted to the investigation of the effect of nonhomogeneity on the frequencies of elastic bodies. Bose [1] has analysed the vibrations produced in a thin nonhomogeneous circular plate whose Young's modulus and density both vary linearly with radius vector with constant Poisson's ratio. Roy [2] has also assumed linear variation in the modulus of elasticity of a square plate with x-coordinate by keeping both Poisson's ratio and density constant.

The present work is a continuation and an enlargement of the previous work [3] where the statistical homogeneity and statistical isotropy were assumed with the exponential variation in Young's modulus and density with x-coordinate. These assumptions would include «glass-spheres in epoxy-resin» and certain mixture of metals for example lead-aluminium. In this investigation the first assumption remains the same and the modulus of elasticity and density of the plate material are supposed to vary linearly along x-axis with constant Poisson's ratio which is apparent in many nonhomogeneous structures, particularly when they are made of concrete or when they are raw planks from branch side toward the root side used extensively on temporary bridges in war time [2].

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2. FORMULATION OF THE PROBLEM

Here, the plate is assumed to be of infinite extent in one of the directions (the y-direction) and the thickness of the plate h, Young's modulus E, density ρ and flexural rigidity D are assumed to vary only in x-direction. Adopting these assumptions, the equation of motion of the isotropic infinite elastic plate of variable thickness, in accordance with Winkler's assumption [4], is derived as

$$Eh^2 \frac{\partial^4 w}{\partial x^4} + 2\left\{h^3 \frac{\partial E}{\partial x} + 3h^2\left(\frac{\partial h}{\partial x}\right)E\right\}\frac{\partial^3 w}{\partial x^3} + \left\{h^3 \frac{\partial^2 E}{\partial x^2} + 6h^2\left(\frac{\partial h}{\partial x}\right)\frac{\partial E}{\partial x} + 6Eh\left(\frac{\partial h}{\partial x}\right)^2 + 3h^2E\frac{\partial^2 h}{\partial x^2}\right\}\frac{\partial^2 w}{\partial x^2} + 12(1-\nu^2)\,\rho h \frac{\partial^2 w}{\partial t^2} + 12(1-\nu^2)\,k_f w = 0, \tag{1}$$

where w is the transverse displacement, ν the Poisson's ratio which has been assumed to be constant and k_f is the foundation modulus.

If the non-dimensional variables, $H = h/a$, $X = x/a$ and $W = w/a$ are introduced, equation (1) takes the form

$$\bar{E}H^3 \frac{\partial^4 W}{\partial X^4} + 2\left\{H^3 \frac{\partial \bar{E}}{\partial X} + 3H^2 \frac{\partial H}{\partial X}\bar{E}\right\}\frac{\partial^3 W}{\partial X^3} + \left\{H^3 \frac{\partial^2 \bar{E}}{\partial X^2} + 6H^2\left(\frac{\partial H}{\partial X}\right)\frac{\partial \bar{E}}{\partial X} + 6\bar{E}H\left(\frac{\partial H}{\partial X}\right)^2 + 3\bar{E}H^2\frac{\partial^2 H}{\partial X^2}\right\}\frac{\partial^2 W}{\partial X^2} \tag{2}$$

$$+\bar{\rho}H \cdot a^2 12(1-\nu^2)\frac{\partial^2 W}{\partial t^2} + 12(1-\nu^2)k_f W = 0, \quad (2)$$

where $\bar{E} = E/a$, $\bar{\rho} = \rho/a$ and a is the width of the plate.

In this investigation, the thickness of the plate H, modulus of elasticity $\bar{E}$ and mass density $\bar{\rho}$ are assumed to vary as

$$H = H_0(1-\alpha X^2),\ \bar{E} = \bar{E}_0(1+\beta X) \text{ and } \bar{\rho} = \bar{\rho}_0(1+\beta X)$$

where

$$H_0 = H|_{X=0},\ \bar{E}_0 = \bar{E}|_{X=0},\ \bar{\rho}_0 = \bar{\rho}|_{X=0},$$

α is a taper constant and β represents the parameter of non-homogeneity.

The consideration of above hypothesis and substitution of $W(x, t) = \bar{W}(X)\, e^{i\omega t}$ for harmonic vibrations in equation (2) yield

$$(1+\beta X - 3\alpha X^2 - 3\alpha\beta X^3 + 3\alpha^2 X^4 + 3\alpha^2\beta X^5 - \alpha^3 X^6 - \alpha^3\beta X^7)\frac{\partial^4 \bar{W}}{\partial X^4}$$

$$-(2\beta - 12\alpha X - 18\alpha\beta X^2 + 24\alpha^2 X^3 + 30\alpha^2\beta X^4 - 12\alpha^3 X^5 - 14\alpha^3\beta X^6)\frac{\partial^3 \bar{W}}{\partial X^3}$$

$$+(-6\alpha - 18\alpha\beta X + 36\alpha^2 X^2 + 60\alpha^2\beta X^3 - 30\alpha^3 X^4 - 42\alpha^3\beta X^5)\frac{\partial^2 \bar{W}}{\partial X^2}$$

$$-\left\{(1+\beta X - \alpha X^2 - \alpha\beta X^3)\frac{\Omega^2}{I^x} - \frac{F_p}{C^x}\right\}\bar{W} = 0, \quad (3)$$

where

$$I^x = \frac{H_0^2}{12},\ F_p = \frac{k_f}{\bar{E}_0},\ C^x = \frac{H_0^3}{12(1-\nu^2)},$$

$$\Omega^2 = \frac{\bar{\rho}_0 a^2 \omega^2 (1-\nu^2)}{\bar{E}_0},$$

ω the radian frequency and Ω is frequency parameter.

3. SOLUTION AND ITS CONVERGENCE

Adopting the method of Frobenius for the solution, the series expression

$$\bar{W}(X) = \sum_{\lambda=0}^{\infty} a_\lambda X^{c+\lambda},\ a_0 \neq 0, \quad (4)$$

for $\bar{W}$ is substituted in equation (3) which reduces to

$$\sum_{\lambda=0}^{\infty} a_\lambda F_1(\lambda) X^{c+\lambda-4} + \sum_{\lambda=0}^{\infty} a_\lambda F_2(\lambda) X^{c+\lambda-3} + \sum_{\lambda=0}^{\infty} a_\lambda F_3(\lambda) X^{c+\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda F_4(\lambda) X^{c+\lambda-1}$$

$$+ \sum_{\lambda=0}^{\infty} a_\lambda F_5(\lambda) X^{c+\lambda} + \sum_{\lambda=0}^{\infty} a_\lambda F_6(\lambda) X^{c+\lambda+1} + \sum_{\lambda=0}^{\infty} a_\lambda F_7(\lambda) X^{c+\lambda+2} + \sum_{\lambda=0}^{\infty} a_\lambda F_8(\lambda) X^{c+\lambda+3} = 0, \quad (5)$$

where

$$b_\lambda^{(0)} = (c+\lambda),$$

$$b_\lambda^{(1)} = (c+\lambda)(c+\lambda-1),$$

$$b_\lambda^{(2)} = (c+\lambda)(c+\lambda-1)(c+\lambda-2),$$

$$b_\lambda^{(3)} = (c+\lambda)(c+\lambda-1)(c+\lambda-2)(c+\lambda-3),$$

$$F_1(\lambda) = b_\lambda^{(3)},$$

$$F_2(\lambda) = \beta(b_\lambda^{(3)} + 2b_\lambda^{(2)}),$$

$$F_3(\lambda) = -3\alpha(b_\lambda^{(3)} + 4b_\lambda^{(2)} + 2b_\lambda^{(1)}),$$

$$F_4(\lambda) = -3\alpha\beta(b_\lambda^{(3)} + 6b_\lambda^{(2)} + 6b_\lambda^{(1)}),$$

$$F_5(\lambda) = 3\alpha^2(b_\lambda^{(3)} + 8b_\lambda^{(2)} + 12b_\lambda^{(1)}) - \frac{\Omega^2}{I^x} + \frac{F_p}{C^x},$$

$$F_6(\lambda) = 3\alpha^2\beta(b_\lambda^{(3)} + 10b_\lambda^{(2)} + 20b_\lambda^{(1)}) - \frac{\Omega^2\beta}{I^x},$$

$$F_7(\lambda) = -\alpha^3(b_\lambda^{(3)} + 12b_\lambda^{(2)} + 30b_\lambda^{(1)}) + \frac{\alpha\Omega^2}{I^x} \text{ and}$$

$$F_8(\lambda) = -\alpha^3\beta(b_\lambda^{(3)} + 14b_\lambda^{(2)} + 42b_\lambda^{(1)}) + \frac{\alpha\beta\Omega^2}{I^x}.$$

The series expression (4) to be the solution for equation (5), the condition of vanishing coefficients of all the powers of X is enforced and the indicial roots $c = 0, 1, 2, 3$ are obtained by equating to zero the coefficient of the lowest power of X. Also, when the coefficients of the next subsequent powers of X are equated to zero, a_1, a_2 and a_3 are found to be indeterminate for $c = 0$ and rest of the a_λ's ($\lambda = 4, 5, \ldots$) are obtainable from the following recurrence relation

$$a_\lambda = a_0 A_\lambda^{(0)} + a_1 A_\lambda^{(1)} + a_2 A_\lambda^{(2)} + a_3 A_\lambda^{(3)}, \quad (6)$$

where $A_\lambda^{(i)}$'s have been defined in appendix.

Now, after substituting the values of unknown constants a_λ's the following solution, corresponding to $c = 0$ for equation (5), is obtained

$$\bar{W} = a_0\left[1 + \sum_{\lambda=4}^{\infty} A_\lambda^{(0)} X^\lambda\right] + a_1\left[X + \sum_{\lambda=4}^{\infty} A_\lambda^{(1)} X^\lambda\right] + a_2\left[X^2 + \sum_{\lambda=4}^{\infty} A_\lambda^{(2)} X^\lambda\right] + a_3\left[X^3 + \sum_{\lambda=4}^{\infty} A_\lambda^{(3)} X^\lambda\right]. \quad (7)$$

Apparently no new solution arises for other values of c since these are already contained in the solution (7).

The convergence of the solution (7) has been tested by adopting the technique used by Lamb [5] and found to be convergent in the interval $0 \leqslant X \leqslant 1$ for $|\mu| < 1$ (where $\mu = \lim_{\lambda \to \infty} a_{\lambda+1}/a_{\lambda}$) and hence uniformly convergent for all $|\sqrt{\alpha}| < 1$ and $|\beta| < 1$.

4. BOUNDARY CONDITIONS AND FREQUENCY E-QUATIONS

The frequency equations for clamped-clamped and clamped-simply supported infinite plates are obtained by using the following boundary conditions.

(C-C)-PLATE

For an infinite plate clamped at both the edges $X = 0$ and $X = 1$, the deflection and slope of the plate element at the edges should be zero

$$\text{i.e.}\ \bar{W}\Big|_{X=0,1} = \frac{\partial \bar{W}}{\partial X}\Big|_{X=0,1} = 0. \tag{8}$$

Enforcing the boundary conditions (8) on (7) the frequency equation is obtained as

$$\begin{vmatrix} G_1(\Omega) & G_2(\Omega) \\ G_3(\Omega) & G_4(\Omega) \end{vmatrix} = 0, \tag{9}$$

where

$$G_1(\Omega) = 1 + \sum_{\lambda=4}^{\infty} A_{\lambda}^{(2)}, \qquad G_2(\Omega) = 1 + \sum_{\lambda=4}^{\infty} A_{\lambda}^{(3)},$$

$$G_3(\Omega) = 2 + \sum_{\lambda=4}^{\infty} \lambda A_{\lambda}^{(2)} \text{ and } G_4(\Omega) = 3 + \sum_{\lambda=4}^{\infty} \lambda A_{\lambda}^{(3)}.$$

(C-SS)-PLATE

For an infinite plate clamped at $X = 0$ and simply supported at $X = 1$, the boundary conditions are

$$\bar{W}\Big|_{X=0,1} = \frac{\partial \bar{W}}{\partial X}\Big|_{X=0} = \frac{\partial^2 \bar{W}}{\partial X^2}\Big|_{X=1} = 0. \tag{10}$$

Applying the boundary conditions (10) on (7), the characteristic equation for the determination of frequencies has been derived as

$$\begin{vmatrix} G_1(\Omega) & G_2(\Omega) \\ G_5(\Omega) & G_6(\Omega) \end{vmatrix} = 0 \tag{11}$$

where

$$G_5(\Omega) = 2 + \sum_{\lambda=4}^{\infty} \lambda(\lambda-1)A_{\lambda}^{(2)}$$

and

$$G_6(\Omega) = 6 + \sum_{\lambda=4}^{\infty} \lambda(\lambda-1)A_{\lambda}^{(3)}.$$

5. RESULTS AND DISCUSSION

The numerical results have been computed by using DEC 20 digital computer and terms of the series upto an accuracy of 10^{-8} in their absolute values have been retained. The results for the variation of frequency parameter $\Omega = a\omega\sqrt{(1-\nu^2)/(\bar{E}_0/\rho_0)}$ with taper constant α have been displayed in Fig. 1. The frequency parameter Ω for homogeneous plates without foundation decreases with the increase in α for both the combinations of boundary conditions in

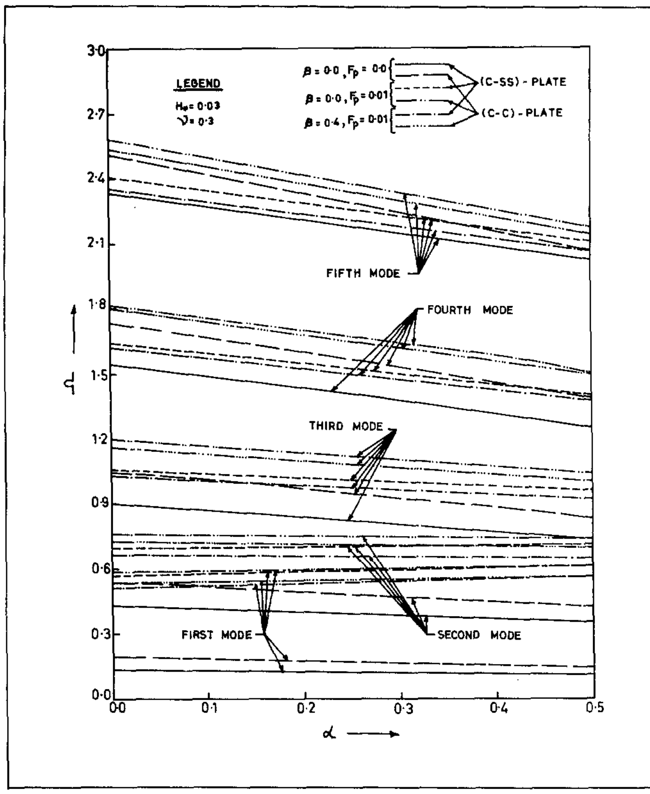

Fig. 1. Effect of taper constant «α» on frequency parameter «Ω».

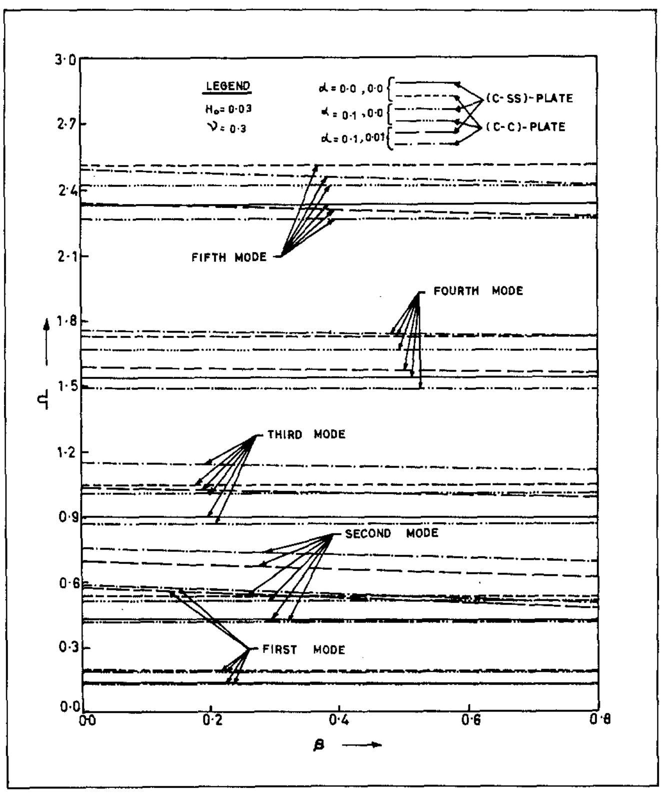

Fig. 2. Effect of non-homogeneity parameter «β» on frequency parameter «Ω».

all the five modes. Also, the frequencies, in the case of homogeneous plates resting on elastic foundation, are larger than those in the case of nonhomogeneous plates on elastic foundation.

Fig. 2 shows the effect of nonhomogeneity parameter β on the frequency parameter Ω. For $\alpha = 0.0$ and $F_p = 0.0$, Ω increases slowly with the increase in β while for $\alpha = 0.1$ and $F_p = 0.01$, it decreases for the edge conditions and modes discussed here. It is also noted that the frequencies of uniform plates are larger than those of plates of variable thickness. Figs. 3 and 4 depict that the frequencies of (C-SS) and (C-C) plates increase with the increase in the value of foundation modulus for all the sets of values of α and β. The non-dimensional transverse displacement $\bar{W}$ and moment parameters $\bar{M}$ have been shown in Figs. 5 and 6 respectively.

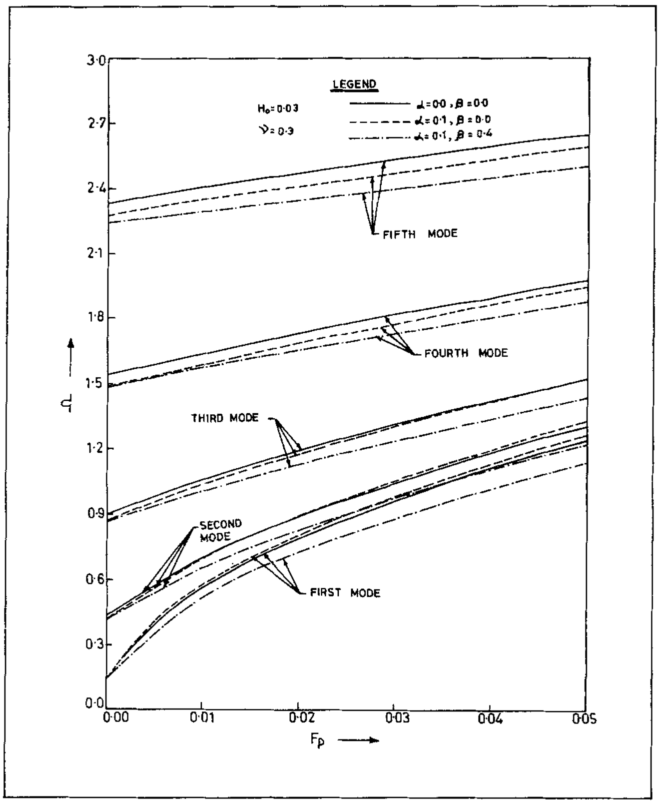

Fig. 3. Effect of foundation modulus «F_p» on frequency parameter «Ω» of a (C-SS)-PLATE.

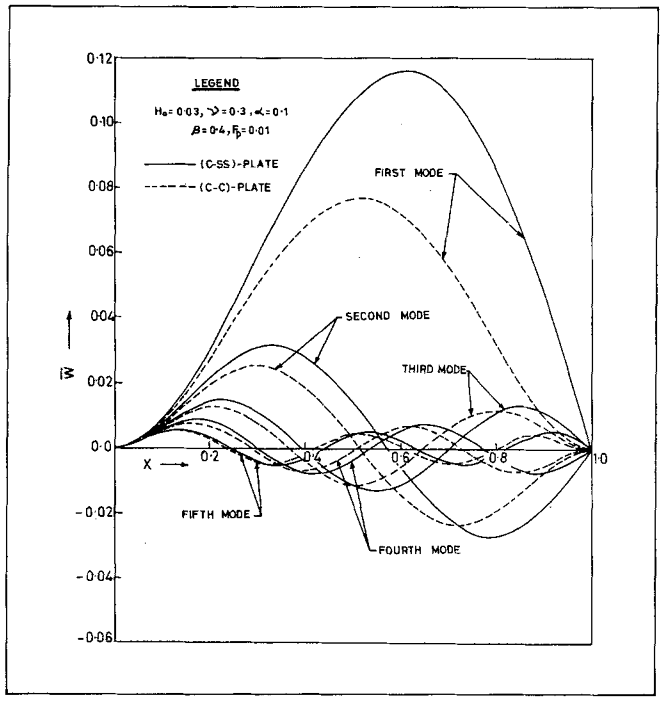

Fig. 5. Transverse displacement «W» corresponding to the first five modes of vibration.

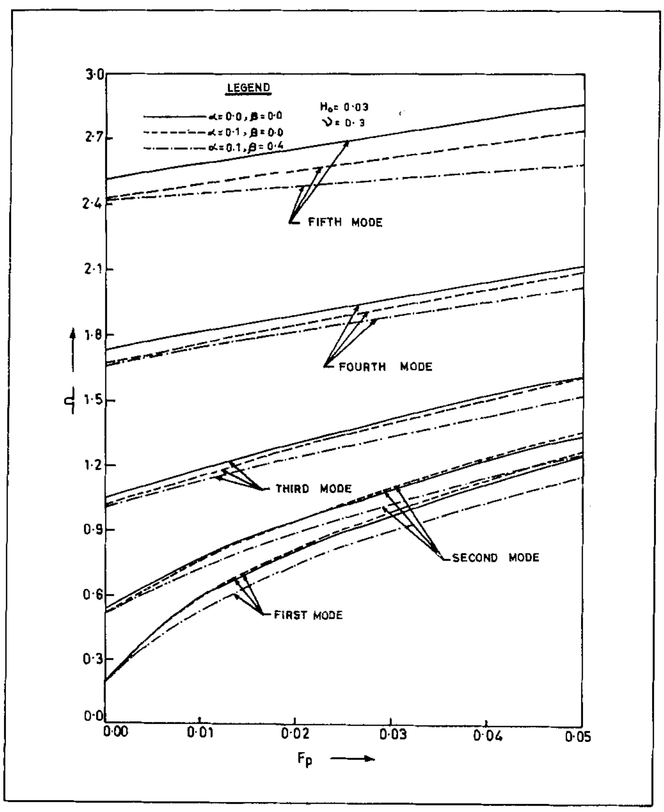

Fig. 4. Effect of foundation modulus «F_p» on frequency parameter «Ω» on a (C-C)-PLATE.

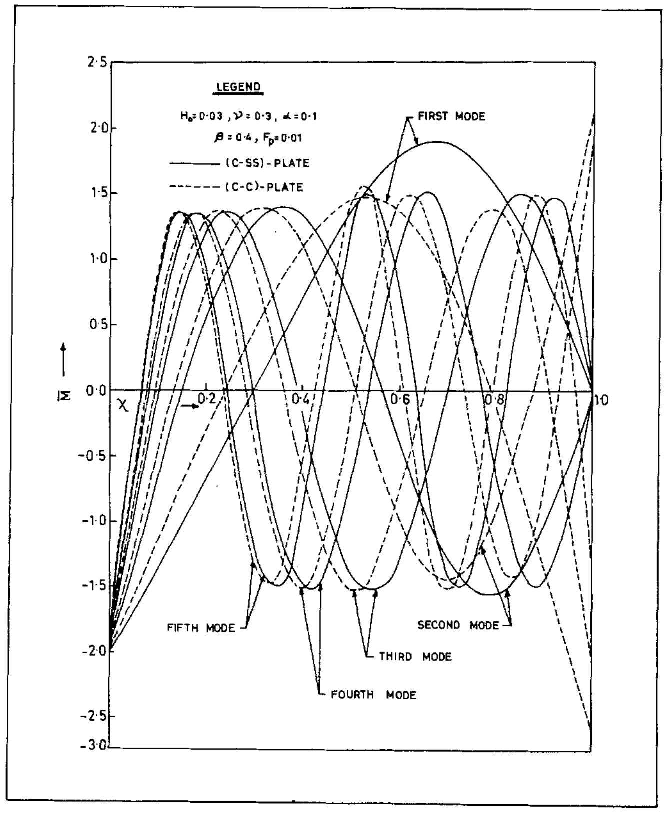

Fig. 6. Moment parameters «M» corresponding to the first five modes of vibration.

The results obtained here compare very well with those given in [3] for $\alpha = 0.0$, $\beta = 0.0$ and $F_p = 0.0$ (Fig. 1). The values of Ω for $\alpha = 0.1$ and $\beta = 0.0$, for the variation of foundation modulus (Figs. 3 and 4) have been found in a good agreement with those obtained by Tomar and Gupta [6] for the available first two modes of vibration while those differ to some extent for $\beta = 0.0$, $F_p = 0.01$ for the variation of α.

APPENDIX

Definition of the parameters involved in equation (6) are as follows for $i = 0, 1, 2, 3$:

$$A_\lambda^{(i)} = 1 \qquad \text{if } \lambda = i$$

$$= 0 \qquad \text{otherwise } (\lambda = 0, 1, 2, 3)$$

$$A_4^{(i)} = -[A_3^{(i)}F_2(3) + A_2^{(i)}F_3(2) + A_1^{(i)}F_4(1) + A_0^{(i)}F_5(0)]/F_4(1),$$

$$A_5^{(i)} = -[A_4^{(i)}F_2(4) + A_3^{(i)}F_3(3) + A_2^{(i)}F_4(2) + A_1^{(i)}F_5(1) + A_0^{(i)}F_6(0)]/F_5(1),$$

$$A_6^{(i)} = -[A_5^{(i)}F_2(5) + A_4^{(i)}F_3(4) + A_3^{(i)}F_4(3) + A_2^{(i)}F_5(2) + A_1^{(i)}F_6(1) + A_0^{(i)}F_7(0)]/F_6(1),$$

$$A_7^{(i)} = -[A_6^{(i)}F_2(6) + A_5^{(i)}F_3(5) + A_4^{(i)}F_4(4) + A_3^{(i)}F_5(3) + A_2^{(i)}F_6(2) + A_1^{(i)}F_7(1) + A_0^{(i)}F_8(0)]/F_7(1)$$

and

$$A_{\lambda+7}^{(i)} = -[A_{\lambda+6}^{(i)}F_2(\lambda+6) + A_{\lambda+5}^{(i)}F_3(\lambda+5) + A_{\lambda+4}^{(i)}F_4(\lambda+4) + A_{\lambda+3}^{(i)}F_5(\lambda+3) + A_{\lambda+2}^{(i)}F_6(\lambda+2) + A_{\lambda+1}^{(i)}F_7(\lambda+1) + A_\lambda^{(i)}F_8(\lambda)]/F_7(\lambda+1) \qquad (\lambda = 1, 2, 3, \ldots).$$

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