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MODAL COUPLING IN THE FREE NONPLANAR FINITE MOTION OF AN ELASTIC CABLE

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SOMMARIO. Nelle oscillazioni di grande ampiezza di un cavo sospeso, il moto nel piano e fuori del piano risulta accoppiato, a differenza di quanto predetto dalla teoria delle piccole oscillazioni. Questo problema viene studiato facendo riferimento ad un modello del cavo, semplice ma significativo, a due soli gradi di libertà, dei quali uno tiene conto del moto pendolare e l'altro del moto nel piano. La soluzione delle equazioni di moto è ottenuta con una tecnica perturbativa fino al terzo ordine, adatta al problema con nonlinearità quadratiche e cubiche. Si studia la modificazione della legge del moto dovuta al trasferimento di energia tra i due modi per differenti condizioni iniziali in assenza di risonanza interna e si valutano gli effetti dell'accoppiamento modale nel problema nonlineare.

SUMMARY. In the finite motions of a suspended elastic cable the in-plane and out-of-plane oscillations are coupled, which is in contrast with what is predicted by the theory of small oscillations. To study the phenomenon of nonlinear coupling, a simple but meaningful two degree-of-freedom model is referred here, one parameter being used to describe the in-plane motion and the other the out-of-plane motion. The solution of the dynamic equilibrium equations is accomplished by an order-three perturbational expansion, which furnishes the time solution of the two displacement parameters. The modification of the free oscillations due to the exchange of energy between the two modes in absence of internal resonance is studied for different initial conditions and the effect of modal coupling is evidenced.

1. INTRODUCTION

The analysis of the dynamics of continuum systems usually leads to the study of nonlinear partial differential equations. The complexity of the problem often suggests to find the solution of the linearized equations of motion, though linear models can not disclose some aspects of the actual behaviour [1, 2].

To tackle a more suitable description of the problem, retaining the most important effects of the nonlinearities, it is convenient to introduce a certain number of simplifying assumptions, mainly to reduce the equations of motion to a system of ordinary differential equations by representing the spatial configuration with a finite number of prescribed shapes [3 - 5] or with a finite element approach [6 - 9]. Sometimes, for a particular phenomenon as the relationship between the frequency and the amplitude of the oscillation, it is possible to adopt only one shape to represent the mode under consideration; in this way the problem is reduced to the study of a simple nonlinear oscillator [10 - 13]. This circumstance can be viewed as a particular case of the general discrete approach where, for a system uncoupled in the linear part, the solution of the single equation is obtained by neglecting the contribution of the other modes in the nonlinear terms.

A better description of the frequency-amplitude relationship of a multidegree-of-freedom system is given by the analysis of monofrequent oscillations; in these cases the motion is still periodic and has the correct nonlinear frequency of the prevailing component which makes the other components to arise [14, 15].

If the attention is focused on the nonlinear coupling of multidegree-of-freedom structures, even simple mechanical model can be conveniently utilized to study the problem, as it was made for the special case of internal resonance [16, 17]. The complete study of the coupling requires the general solution of the equations of motion; taking into account that the numerical importance of this problem is strongly influenced by the sequence and the ratio of the natural frequencies, it is often possible to obtain well approximated results by considering only the two modes mainly involved in the coupling.

Within this frame, in the present work the solution of the coupled motion of a suspended cable is found by referring to a simple but meaningful two degrees-of-freedom model; the model, utilized also in [14], is implemented here to consider the contribution of longitudinal displacements, which play an important role in the nonlinear behaviour [18]. The solution is obtained by an order-three perturbational procedure suitable for studying problems with quadratic and cubic nonlinearities; the time law of the two quantities which define the deformed configuration of the cable is obtained. The modification of the motion due to the exchange of energy between the two modes considered is studied for different initial conditions; for particular values of the latter, the nonlinear motion is characterized by a unique frequency, i.e. monofrequent oscillations occur. Finally, the solution obtained is used as a case example to show what is lost when the coupling terms are omitted.

2. NONLINEAR MODEL OF CABLE. EQUATIONS OF MOTION

Consider a heavy elastic cable suspended between two fixed supports at the same level. The initial static equilibrium

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configuration of the cable, which is assumed as reference configuration of length ℓ_c, lies in the xy plane and is represented by the function $y(s)$. The dynamic configuration is described through the displacement coordinates $\tilde{q}_1(s, t)$, $\tilde{q}_2(s, t)$, $\tilde{q}_3(s, t)$ of a point $P(s)$ (Fig. 1), which are connected with the components u, v, w in an orthogonal system $0xyz$ by the relationships:

$$u = \tilde{q}_3$$
$$v = (y + \tilde{q}_1)\cos\phi - y \qquad (1a-c)$$
$$w = (y + \tilde{q}_1)\sin\phi$$

It is assumed [14] that during the motion the cable always remains in a plane, whose position with respect to the xy plane is defined by the angle:

$$\phi(t) = \tilde{q}_2(0, t)/y(0) = q_2(t)/d \qquad (2)$$

while the deformed in-plane configuration is described through $\tilde{q}_1$ and $\tilde{q}_3$. Such simplified kinematics are considered to be adequate for studying nonlinear coupling between the in-plane modes and the out-of-plane first symmetric mode. With these assumptions, and using the Lagrangian strain as the strain measure, the extensional strain of the cable axis is:

$$\epsilon(s, t) = (\partial\tilde{q}_1/\partial s)(\mathrm{d}y/\mathrm{d}s) + (\partial\tilde{q}_3/\partial s)(\mathrm{d}x/\mathrm{d}s) + + \frac{1}{2}[(\partial\tilde{q}_1/\partial s)^2 + (\partial\tilde{q}_3/\partial s)^2] \qquad (3)$$

In order to obtain an analytical solution to the problem of free vibrations of the cable the following assumptions are made [18]: i) the static equilibrium configuration is represented through the parabola:

$$y = 4d[x/\ell - (x/\ell)^2] \qquad (4)$$

which entails $\mathrm{d}s \simeq \mathrm{d}x$ and permits approximation of the cable initial tension T^I with its horizontal component H; ii) the initial strain is negligible with respect to unity; iii) the gradient of the horizontal component of the dynamic displacement is negligible with respect to unity, i.e. moderately large rotations are considered in the cable motion. An integral relationship between the two in-plane displacement coordinates $\tilde{q}_1$ and $\tilde{q}_3$ is obtained by neglecting the longitudinal inertia forces of the cable, from which the extensional strain results to be function of time only:

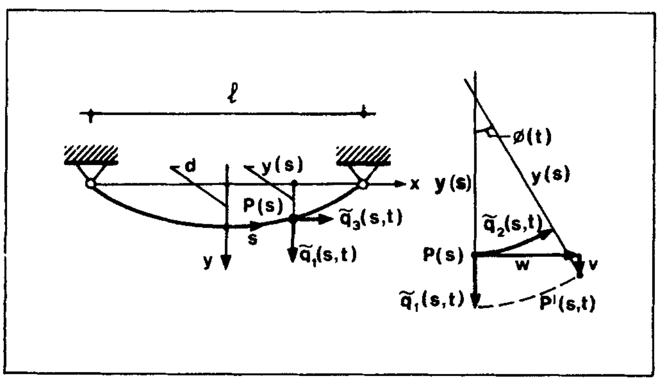

Fig. 1. Cable configuration.

$$\partial\tilde{q}_3/\partial x + (\partial\tilde{q}_1/\partial x)(\mathrm{d}y/\mathrm{d}x) + \frac{1}{2}(\partial\tilde{q}_1/\partial x)^2 = e(t) \qquad (5)$$

By integrating Eq. (5) and accounting for the boundary condition $\tilde{q}_3(\ell, t) = 0$ to determine the constant $e(t)$, it follows:

$$\tilde{q}_3 = \frac{x}{\ell}\int_0^\ell \left[\frac{\partial\tilde{q}_1}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\partial\tilde{q}_1}{\partial x}\right)^2\right]\mathrm{d}x + - \int_0^x \left[\frac{\partial\tilde{q}_1}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\partial\tilde{q}_1}{\partial x}\right)^2\right]\mathrm{d}x \qquad (6)$$

The equations of motion of a simple two-parameter model of the suspended cable can be deduced via the Lagrange equations. The strain energy, the kinetic energy and the gravitational energy are as follows:

$$U = U^I + \int_0^\ell \left(He + \frac{1}{2}EAe^2\right)\mathrm{d}x,$$
$$K = \int_0^\ell \frac{1}{2}m(\dot{v}^2 + \dot{w}^2)\,\mathrm{d}x, \qquad W = W^I - \int_0^\ell mgv\,\mathrm{d}x \qquad (7a-c)$$

where U^I, W^I are the values in the initial configuration, E, A and m are the elastic modulus, cross-sectional area and mass per unit length of cable, respectively. By substituting for $e(t)$ the expression obtainable from Eqs. (5), (6), for v and w the relations ($1b$, c) – the former being expanded up to order-four terms in the two components $\tilde{q}_1$ and ϕ assumed of the same order when substituted in Eq. (7c) –, the Lagrange equations are written in terms of just $\tilde{q}_1$ and ϕ. The cable transverse coordinate $\tilde{q}_1$ is represented through separate variables as follows:

$$\tilde{q}_1(x, t) = q_1(t)f(x) \qquad (8)$$

the in-plane shape function $f(x)$ being assumed as the eigenfunction of the linearized dynamic problem [18]. To obtain a dimensionless form of the equations of motion, the following positions are made:

$$u_1 = q_1/d,\ u_2 = \phi = q_2/d,\ \tilde{x} = x/\ell,\ \tilde{y} = y/d,\ \tau = \omega_2 t \qquad (9)$$

and the parameter $\lambda = \omega_1/\omega_2$ is introduced, ω_1 and ω_2 being the frequencies of the in-plane and out-of-plane motions in the associated linear problem. The equations of motion then read:

$$\ddot{u}_1 + \lambda^2 u_1 = (c_1 u_1^2 + c_2 u_2^2 + c_3\dot{u}_2^2) + (c_7 u_1\dot{u}_2^2 + c_8 u_1^3)$$
$$\ddot{u}_2 + u_2 = (c_4 u_1\ddot{u}_2 + c_5\dot{u}_1\dot{u}_2 + c_6 u_1 u_2) + + (c_9 u_2^3 + c_{10}u_1\dot{u}_1\dot{u}_2 + c_{11}u_1^2\ddot{u}_2) \qquad (10a, b)$$

where the dot indicates $d/d\tau$. The dimensionless coefficients c_i, which are given in Appendix 1, depend on the initial configuration $\tilde{y}$ and on the assumed transverse eigenfuction f; c_1 and c_8 also contain the parameter $\Lambda^2 = (EA/mg\ell)(8d/\ell)^3$ which, according to the nondimensionalization with respect to sag, accounts by itself for the mechanical and geometrical properties of the cable and governs its planar nonlinear vibrations [13, 18].

Nonlinear quadratic and cubic terms exist in Eqs. (10). Analysis of them gives some indications as regards the phenomena which can occur in the finite free dynamics of the cable. The occurrence of geometric and inertial terms of pure nature (u_1^2, u_2^2, $\dot{u}_2^2$, u_1^3, u_2^3) in the two equations assures that both monofrequent oscillations can exist under particular initial conditions for the remaining variable, with frequency dependent on the amplitude of oscillation and time solution different from the linear one. Neglecting the contribution of the horizontal in-plane displacement, they were studied in [14] where it was shown that, due to the presence in the equation for u_1 of pure forcing terms associated with u_2, the out-of-plane monofrequent oscillation is characterized by some nonlinear coupling between the two coordinates. Eqs. (10) can be used as well for studying the general case u_1, u_2 of the same order, in which phenomena of either internal resonance or general modal coupling occur; the latter will be analyzed in the following.

To obtain the solution to system (10) the multiple scale method [2] is adopted. A perturbation parameter ϵ of the order of the amplitude is introduced and the variables u_i are considered functions of a sequence of three independent time scales T_0, T_1, T_2, which are related to τ by the expressions $T_n = \epsilon^n \tau$, and are expanded in powers of ϵ up to the ϵ^3-order, to attain a solution with the same accuracy as is involved in the differential equations:

$$u_i = \epsilon u_{i1}(T_0, T_1, T_2) + \epsilon^2 u_{i2}(T_0, T_1, T_2) + \\ + \epsilon^3 u_{i3}(T_0, T_1, T_2) + 0(\epsilon^4 \tau) \tag{11}$$

By expressing the time derivatives in terms of the T_n variables and substituting Eqs. (11) in Eqs. (10), a system of two partial differential equations with the unknowns u_{ij} is obtained. By equating coefficients of like powers of ϵ, a sequence of three linear systems follows:

$$\text{order } \epsilon : D_{00}u_{11} + \lambda^2 u_{11} = 0, \quad D_{00}u_{21} + u_{21} = 0 \tag{12}$$

$$\begin{aligned} \text{order } \epsilon^2 &: D_{00}u_{12} + \lambda^2 u_{12} = -2D_{01}u_{11} + c_1 u_{11}^2 + \\ &\quad + c_2 u_{21}^2 + c_3(D_0 u_{21})^2 \\ &\quad D_{00}u_{22} + u_{22} = -2D_{01}u_{21} + c_4 u_{11} D_{00}u_{21} + \\ &\quad + c_5 D_0 u_{11} D_0 u_{21} + c_6 u_{11} u_{21} \end{aligned} \tag{13}$$

$$\begin{aligned} \text{order } \epsilon^3 &: D_{00}u_{13} + \lambda^2 u_{13} = -2D_{01}u_{12} - 2D_{02}u_{11} - D_{11}u_{11} + \\ &\quad + 2c_1 u_{11}u_{12} + 2c_2 u_{21}u_{22} + \\ &\quad + 2c_3(D_0u_{21}D_0u_{22} + D_0u_{21}D_1u_{21}) + \\ &\quad + c_7 u_{11}(D_0u_{21})^2 + c_8 u_{11}^3 \\ &\quad D_{00}u_{23} + u_{23} = -2D_{01}u_{22} - 2D_{02}u_{21} - D_{11}u_{21} + \\ &\quad + c_4(u_{11}D_{00}u_{22} + 2u_{11}D_{01}u_{21} + u_{12}D_{00}u_{21}) + \\ &\quad + c_5(D_0u_{11}D_0u_{22} + D_0u_{11}D_1u_{21} + \\ &\quad + D_0u_{12}D_0u_{21} + D_1u_{11}D_0u_{21}) + \\ &\quad + c_6(u_{11}u_{22} + u_{12}u_{21}) + c_9 u_{21}^3 + \\ &\quad + c_{10}u_{11}D_0u_{11}D_0u_{21} + c_{11}u_{11}^2 D_{00}u_{21} \end{aligned} \tag{14}$$

where the notations $D_i = \partial/\partial T_i$ and $D_{ij} = \partial^2/\partial T_i \partial T_j$ have been used for the sake of simplicity. The problem is completed with the initial conditions:

$$u_i(0) = \epsilon \bar{u}_i \qquad \dot{u}_i(0) = \epsilon \bar{\dot{u}}_i \tag{15}$$

3. PERTURBATION SOLUTION

In order to examine the modification of the law of motion of the i-th coordinate induced by the j-th, the periodic solution:

$$u_{11} = A_1(T_1, T_2)e^{i\lambda T_0} + c.c., \quad u_{21} = A_2(T_1, T_2)e^{iT_0} + c.c. \tag{16}$$

is introduced in system (12), $A_j(T_1, T_2)$ being unknown complex amplitudes. In Eqs. (16) and the following, c.c. and the overbar ($\bar{A}_j$) indicate the complex conjugate. Substituting in system (13) gives:

$$\begin{aligned} &D_{00}u_{12} + \lambda^2 u_{12} = -2i\lambda D_1 A_1 e^{i\lambda T_0} + c_1 A_1^2 e^{2i\lambda T_0} + c_1 A_1 \bar{A}_1 + \\ &+ (c_2 - c_3)A_2^2 e^{2iT_0} + (c_2 + c_3)A_2\bar{A}_2 + \text{c.c.} \\ &D_{00}u_{22} + u_{22} = -2iD_1A_2e^{iT_0} - (c_4 + c_5\lambda - c_6)A_1A_2e^{i(\lambda+1)T_0} + \\ &- (c_4 - c_5\lambda - c_6)A_1\bar{A}_2e^{i(\lambda-1)T_0} + \text{c.c.} \end{aligned} \tag{17}$$

Analysis of system (17) shows that internal resonance occurs at the order ϵ^2 for $\lambda = 2$. If this circumstance is not verified, zeroing of the secular terms gives $A_j = A_j(T_2)$, showing that no frequency correction occurs at this order for either one of the coordinates. Solving Eqs. (17) gives:

$$\begin{aligned} u_{12} &= k_1 A_1^2 e^{2i\lambda T_0} + k_2 A_2^2 e^{2iT_0} - 3k_1 A_1 \bar{A}_1 + k_3 A_2 \bar{A}_2 + \text{c.c.} \\ u_{22} &= k_4 A_1 A_2 e^{i(\lambda+1)T_0} + k_5 A_1 \bar{A}_2 e^{i(\lambda-1)T_0} + \text{c.c.} \end{aligned} \tag{18}$$

the coefficients $k_1, \ldots, k_5$ depending on λ and c_i; they are reported in Appendix 2 together with coefficients $k_6, \ldots, k_{15}$ introduced subsequently. By substituting Eqs. (16) and (18) into (14), the following system is obtained:

$$\begin{aligned} &D_{00}u_{13} + \lambda^2 u_{13} = [-2i\lambda D_2A_1 + k_6A_1^2\bar{A}_1 + k_7A_1A_2\bar{A}_2]e^{i\lambda T_0} + \\ &- 8\lambda^2 k_{10}A_1^3 e^{3i\lambda T_0} - 4(\lambda+1)k_{11}A_1A_2^2e^{i(\lambda+2)T_0} + \\ &+ 4(\lambda-1)k_{12}A_1\bar{A}_2^2e^{i(\lambda-2)T_0} + \text{c.c.} \\ &D_{00}u_{23} + u_{23} = [-2iD_2A_2 + k_8A_1\bar{A}_1A_2 + k_9A_2^2\bar{A}_2]e^{iT_0} + \\ &- 8k_{13}A_2^3e^{3iT_0} - 4\lambda(\lambda+1)k_{14}A_1^2A_2e^{i(2\lambda+1)T_0} + \\ &- 4\lambda(\lambda-1)k_{15}A_1^2\bar{A}_2e^{i(2\lambda-1)T_0} + \text{c.c.} \end{aligned} \tag{19a, b}$$

from which it is observed that internal resonance occurs at the order ϵ^3 for $\lambda = 1$ in Eq. (19b). However this condition has no significance in the actual mechanical problem since it corresponds to the taut string, whose behaviour can not be studied through the model used herein, in which the two parameters considered describe the in-plane and the pendulum motions. If this resonance condition is excluded too, zeroing secular terms again, when the polar forms

$$A_h(T_2) = a_h(T_2)e^{i\varphi_h(T_2)} \qquad (h = 1, 2) \tag{20}$$

are introduced and the real and imaginary parts are separated, provides a differential system with respect to time scale T_2 having the unknowns a_h, φ_h, whose solution reads:

$$a_h = \text{const.} \qquad \varphi_h = \hat{\varphi}_h \tau + \varphi_h^0 \tag{21a, b}$$

wherein

$$\hat{\varphi}_1 = -\frac{1}{2\lambda}(k_6 a_1^2 + k_7 a_2^2)\epsilon^2, \quad \hat{\varphi}_2 = -\frac{1}{2}(k_8 a_1^2 + k_9 a_2^2)\epsilon^2 \tag{22}$$

The values of the real amplitudes a_h and phases $\varphi_h^0 = \varphi_h(0)$ will be determined by means of conditions (15). The solution to system (19) reads:

$$\begin{aligned} u_{13} &= k_{10} A_1^3 e^{3i\lambda T_0} + k_{11} A_1 A_2^2 e^{i(\lambda+2)T_0} + \\ &+ k_{12} A_1 \bar{A}_2^2 e^{i(\lambda-2)T_0} + \text{c.c.} \\ u_{23} &= k_{13} A_2^3 e^{3iT_0} + k_{14} A_1^2 A_2 e^{i(2\lambda+1)T_0} + \\ &+ k_{15} A_1^2 \bar{A}_2 e^{i(2\lambda-1)T_0} + \text{c.c.} \end{aligned} \tag{23}$$

Accounting for Eqs. (21b), the amplitudes (20) are rewritten as $A_h(\tau) = A_h^* e^{i\hat{\varphi}_h \tau}$, with $A_h^* = a_h e^{i\varphi_h^0}$. Substituting them in the relations (16), (18), (23) and these in Eq. (11) the complete solution at the order ϵ^3 follows:

$$\begin{aligned} u_1 &= \epsilon A_1^* e^{i\Omega_1 \tau} + \epsilon^2 \{k_1 A_1^{*2} e^{2i\Omega_1 \tau} + k_2 A_2^{*2} e^{2i\Omega_2 \tau} + \\ &- 3k_1 A_1^* \bar{A}_1^* + k_3 A_2^* \bar{A}_2^*\} + \epsilon^3 \{k_{10} A_1^{*3} e^{3i\Omega_1 \tau} + \\ &+ k_{11} A_1^* A_2^{*2} e^{i(\Omega_1 + 2\Omega_2)\tau} + k_{12} A_1^* \bar{A}_2^{*2} e^{i(\Omega_1 - 2\Omega_2)\tau}\} + \text{c.c.} \\ u_2 &= \epsilon A_2^* e^{i\Omega_2 \tau} + \epsilon^2 \{k_4 A_1^* A_2^* e^{i(\Omega_1 + \Omega_2)\tau} + \\ &+ k_5 A_1^* \bar{A}_2^* e^{i(\Omega_1 - \Omega_2)\tau}\} + \epsilon^3 \{k_{13} A_2^{*3} e^{3i\Omega_2 \tau} + \\ &+ k_{14} A_1^{*2} A_2^* e^{i(2\Omega_1 + \Omega_2)\tau} + k_{15} A_1^{*2} \bar{A}_2^* e^{i(2\Omega_1 - \Omega_2)\tau}\} + \text{c.c.} \end{aligned} \tag{24}$$

where

$$\Omega_1 = \lambda + \hat{\varphi}_1, \qquad \Omega_2 = 1 + \hat{\varphi}_2 \tag{25}$$

denote the nonlinear frequencies of the two coordinates. Taking into account Eqs. (22), it is observed that each frequency depends on the square of the oscillation amplitude both of the corresponding coordinate and of the other one.

In satisfying the initial conditions, the complex amplitudes A_h^* are expanded in powers of ϵ as:

$$\epsilon A_h^* = \sum_{1}^{3}{}_k\, \epsilon^k A_{hk}^* + 0(\epsilon^4) \tag{26}$$

and substituted in Eqs. (15) via Eqs. (24). Accounting for Eqs. (22), (25) as well, three systems of conditions follow at the orders ϵ, ϵ^2, ϵ^3, from which the complex amplitudes A_{hk}^* ($h = 1, 2$; $k = 1, 2, 3$) are obtained sequentially (see Appendix 3); then, if the polar forms

$$A_{hk}^* = a_{hk} e^{i\varphi_{hk}^0} \tag{27}$$

are introduced and the real and imaginary parts are separated, the real amplitudes a_{hk} and the phases φ_{hk}^0 follow.

Finally, by substituting relations (26), (27) in Eqs. (24), the temporal laws of the motion in circular form are obtained:

$$\begin{aligned} u_1 &= \epsilon 2a_{11}\cos\phi_{11} + \epsilon^2 2\{a_{12}\cos\phi_{12} + a_{11}^2(k_1\cos 2\phi_{11} - 3k_1) + \\ &+ a_{21}^2(k_2\cos 2\phi_{21} + k_3)\} + \epsilon^3 2\{a_{13}\cos\phi_{13} + \\ &+ 2a_{11}a_{12}[k_1\cos(\phi_{11}+\phi_{12}) - 3k_1\cos(\phi_{11}-\phi_{12})] + 2a_{21}a_{22}\,\cdot \\ &\cdot [k_2\cos(\phi_{21}+\phi_{22}) + k_3\cos(\phi_{21}-\phi_{22})] + a_{11}^3 k_{10}\cos 3\phi_{11} + \\ &+ a_{11}a_{21}^2[k_{11}\cos(\phi_{11}+2\phi_{21}) + k_{12}\cos(\phi_{11}-2\phi_{21})]\} \\ u_2 &= \epsilon 2a_{21}\cos\phi_{21} + \epsilon^2 2\{a_{22}\cos\phi_{22} + a_{11}a_{21}[k_4\cos(\phi_{11}+\phi_{21}) + \\ &+ k_5\cos(\phi_{11}-\phi_{21})]\} + \epsilon^3 2\{a_{23}\cos\phi_{23} + \\ &+ a_{11}a_{22}[k_4\cos(\phi_{11}+\phi_{22}) + k_5\cos(\phi_{11}-\phi_{22})] + \\ &+ a_{12}a_{21}[k_4\cos(\phi_{12}+\phi_{21}) + k_5\cos(\phi_{12}-\phi_{21})] + \\ &+ a_{21}^3 k_{13}\cos 3\phi_{21} + a_{11}^2 a_{21}[k_{14}\cos(2\phi_{11}+\phi_{21}) + k_{15}\cos(2\phi_{11}-\phi_{21})]\} \end{aligned} \tag{28}$$

In Eqs. (28) the phases ϕ_{hk} read:

$$\phi_{hk} = \Omega_h \tau + \varphi_{hk}^0 \tag{29}$$

the nonlinear frequencies Ω_h being expressed explicitly in terms of a_{hk}, φ_{hk}^0 as

$$\Omega_h = \bar{\omega}_h - \frac{1}{2\bar{\omega}_h} \sum_{1}^{2}{}_j\, [\epsilon^2 K_{hj} a_{j1}^2 + 2\epsilon^3 K_{hj} a_{j1} a_{j2} \cos(\varphi_{j1}^0 - \varphi_{j2}^0)] \tag{30}$$

wherein: $\bar{\omega}_1 = \lambda$, $\bar{\omega}_2 = 1$, $K_{11} = k_6$; $K_{12} = k_7$, $K_{21} = k_8$, $K_{22} = k_9$.

Equations (28) show considerable modification of the temporal laws of the two coordinates with respect to the linear ones, which is due essentially to the existing nonlinear coupling. Indeed the motion of the h-th component is described by superposing several harmonics – having frequencies combinations of the two nonlinear fundamental frequencies (30) – to the generating solution. This latter splits into three terms – of order ϵ, ϵ^2, ϵ^3 respectively – having different phases φ_{hk}^0; such difference however reduces to zero if the particular case of zero initial velocity for the h-th component is considered.

The following combination frequencies appear explicitly:

$$\begin{aligned} &2\Omega_1, \quad 2\Omega_2, \quad \Omega_1 \pm \Omega_2 \\ &3\Omega_1, \quad 3\Omega_2, \quad \Omega_1 \pm 2\Omega_2, \quad 2\Omega_1 \pm \Omega_2 \end{aligned} \tag{31a, b}$$

Harmonics with frequencies $2\Omega_h$, $3\Omega_h$ are associated with the pure nonlinear terms of equations of motion (10), while the remaining ones are associated with the mixed terms; each of them affects just the component exhibiting the corresponding nonlinear term in the relevant equation. Quadratic and cubic terms of (10) give rise to harmonics with frequencies (31a) and (31b) respectively. Pure quadratic terms give rise to constant contributions (drifts) as well in the laws of motion, which therefore occur for u_1 only: one of them depends on the in-plane oscillation amplitude, the other one is forced by the out-of-plane oscillation. They have different sign: indeed the former is negative due to softening behaviour of the cable for $u_1 < 0$, the latter is always positive; the ensuing in-plane motion occurs about a position different from the initial configuration.

4. PARAMETRIC INVESTIGATION OF NONLINEAR COUPLING

The laws of motion of the two coordinates are now examined to put into evidence the features of the nonlinear response of the cable to a given set of initial conditions.

Two cables are considered, characterized by the values of the elastogeometric parameter Λ^2 corresponding, for a technical value of the mechanical properties ($EA/H = 500$), to two different sag-to-span ratios d/ℓ (Table 1); they are both prestressed cables [13] and have linear frequencies with ratio λ either more close to the resonant condition $\lambda = 1$ (*a*) or equally far from both the resonant conditions $\lambda = 1$ and $\lambda = 2$ (*b*). The laws of motion obtained refer to a dimensionless time interval $0 \leqslant \tau \leqslant 60$ and to different initial oscillation amplitudes, the initial velocities of both coordinates being assumed zero at first.

When small initial amplitudes of the same value are considered for both coordinates ($u_1(0) = u_2(0) = 0.1$), the nonlinear phenomena are very modest for very shallow cables, like cable *a*. For the slacker cable *b* some nonlinear effects are observed mostly in the pendulum oscillation (Fig. 2), whose amplitude grows up to 1.4 times its initial value; they are due essentially to the ϵ^2-order harmonic of frequency $(\Omega_1 - \Omega_2)$, which has appreciable amplitude with respect to the harmonic of frequency Ω_2. As the initial amplitudes increase, the coupling effects increase too, owing mainly to the contribution of the higher harmonics having frequencies $(\Omega_1 \pm \Omega_2)$. The motion of the shallower cable *a* does not yet differ very much from the linear motion (Fig. 3), thus showing that the existing proximity to the ϵ^3-order resonance condition $\lambda = 1$ is not sufficient to give rise to strong energy exchange for the amplitudes considered: both the in-plane and the out-of-plane coordinates still exhibit prevailing component oscillations of frequency $\Omega_1 = 1.056$ and $\Omega_2 = 1.002$ respectively, though as time runs they are shifted about the static equilibrium positions, mostly the u_2 coordinate due to the slow harmonic of frequency $(\Omega_1 - \Omega_2) = 0.054$.

Stronger nonlinear effects are observed for the slacker cable *b* for the same initial amplitudes (Fig. 4), being directly associated with the higher energy exchange occurring between the two coordinates. The in-plane motion is influenced notably from the pendulum one: indeed the forced harmonic of frequency $2\Omega_2 = 2.018$ existing in the law for u_1 has amplitude equal to 40% of that of the natural oscillation, which is then almost hidden in the motion, and it is in phase with the pendulum oscillation. This latter – though similar qualitatively to that occurring for lower initial amplitudes (see Fig. 2) – is more markedly nonstationary, the ratio $u_{2\max}/u_2(0)$ reaching the value 2.2 in the time interval considered: it results mainly from the combination of the natural oscillation of frequency $\Omega_2 = 1.009$ and of the harmonic of frequency $(\Omega_1 - \Omega_2) = 0.520$, which is faster than the corresponding one for cable *a* (see Fig. 3). It is interesting to observe how the motion changes as an equal but negative in-plane initial amplitude is considered (Fig. 5, plotted in the same scale as Fig. 4), which is opposite

Table 1. Cable properties.

Cable	Λ^2	d/ℓ	$\omega_1\sqrt{d/g}$	$\omega_2\sqrt{d/g}$	λ
a	1.536	1/144	1.179	1.118	1.054
b	19.20	1/41	1.766	1.118	1.580

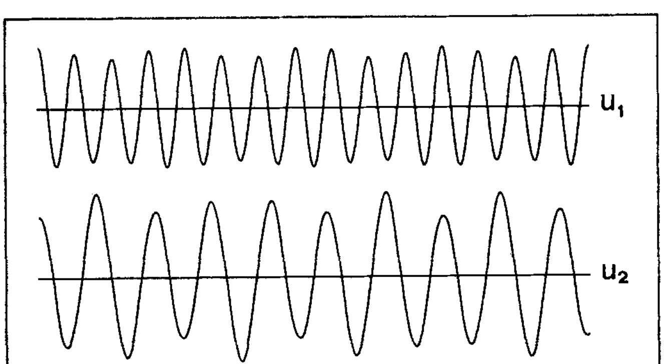

Fig. 2. Motion of cable *b*: $u_1(0) = 0.1, u_2(0) = 0.1$.

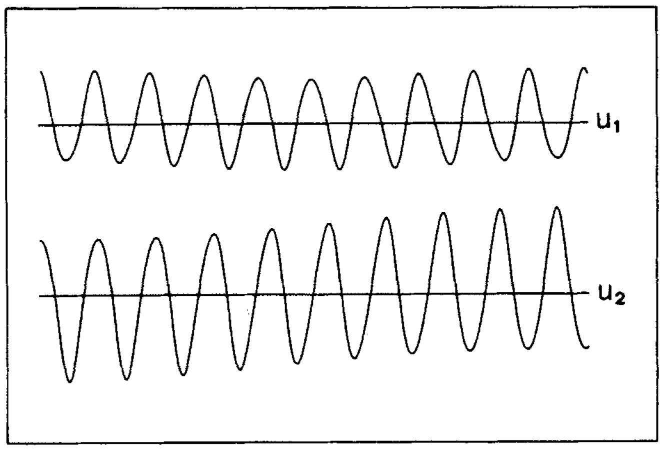

Fig. 3. Motion of cable *a*: $u_1(0) = 0.25, u_2(0) = 0.25$.

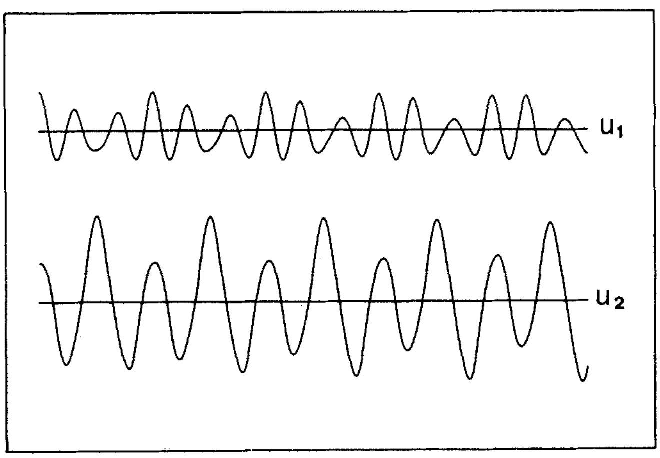

Fig. 4. Motion of cable *b*: $u_1(0) = 0.25, u_2(0) = 0.25$.

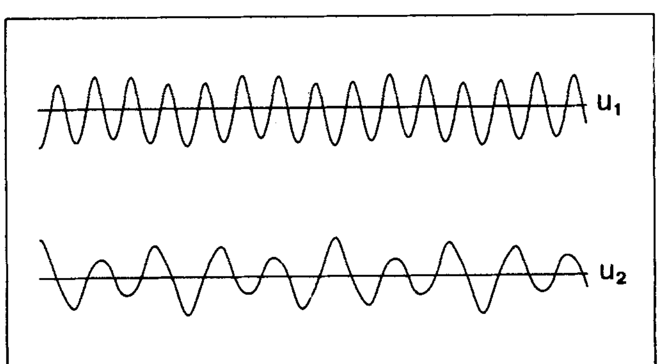

Fig. 5. Motion of cable *b*: $u_1(0) = -0.25, u_2(0) = 0.25$.

in phase with respect to the pendulum amplitude and corresponds to lower potential energy assigned to the system. In the in-plane motion the contributions of the natural oscillation of frequency $\Omega_1 = 1.554$ and of the ϵ^3-order slow harmonic of frequency $(2\Omega_2 - \Omega_1) = 0.394$ are recognizable. The out-of-plane motion remains limited within the values ± 0.25, still being described by the natural oscillation of frequency $\Omega_2 = 0.974$ and by higher order harmonics.

The strong coupling existing between the two coordinates suggests to examine the response of the system to finite initial amplitude assigned to one coordinate and zero (or small) to the other one. With $u_1(0) = 0$, $u_2(0) = 0.5$ (Fig. 6), the in-plane component is completely forced by the pendulum one: however, it vibrates with its frequency $\Omega_1 = 1.508$ and is modulated by the forced harmonic of frequency $2\Omega_2 = 2.122$, the two harmonics having the same amplitude so to give rise, when in-phase, to a maximum value of u_1 reaching 64% of $u_{2max} \equiv u_2(0)$. Of course the pendulum component too is involved in the coupling phenomenon and its natural oscillation is somewhat modified by the harmonics of frequencies $(\Omega_1 - \Omega_2) = 0.447$ and $3\Omega_2 = 3.183$. The motions of the two components are markedly different from those arising in the pendulum monofrequent oscillation [14] which occurs for particular in-plane initial amplitude (Fig. 7). This is a steady-state oscillation characterized by the unique nonlinear frequency $\Omega_2 = 1.061$, in which a forced in-plane component exists of order higher than the out-of-plane one.

When finite initial amplitude is assigned to the in-plane coordinate and zero to the out-of-plane one, no coupling occurs: this is the case of the in-plane monofrequent oscillation [14]. However, a small perturbation assigned to the pendulum is sufficient to produce notable coupling again (Fig. 8, in which a double time interval is considered). The in-plane motion essentially consists of the natural oscillation, drifted towards the soft side of the system; the pendulum motion exhibits beatings due to the presence in the relevant equation of a forcing harmonic of frequency $\Omega_1 - \Omega_2 \cong \Omega_2$. Strong energy exchange between the two coordinates is observed, according to which as the in-plane motion decreases slightly, the out-of-plane one increases up to 8.5 times its initial value; this deep difference in the variation of the two coordinates is obviously a consequence of the much higher flexibility of the cable in the out-of-plane direction.

It is worthwhile to notice that the maximum out-of-plane displacements correspond to the maximum in-plane oscillation towards the soft side, while the cable lies in the vertical plane ($u_2 \simeq 0$) when the in-plane oscillation exhibits a maximum value towards the hard side.

With the same initial conditions, the beating phenomenon just seen does not take place for the shallower cable (Fig. 9).

Finally, a case of initial velocity different from zero for one coordinate only is examined. The set of conditions chosen in Fig. 10 correspond to the same energy furnished to the cable as in the case studied in Fig. 4. Little modification of the shape of the law of motion for u_2 occurs but the maximum value reached ($u_{2max} = 0.38$) is notably lower

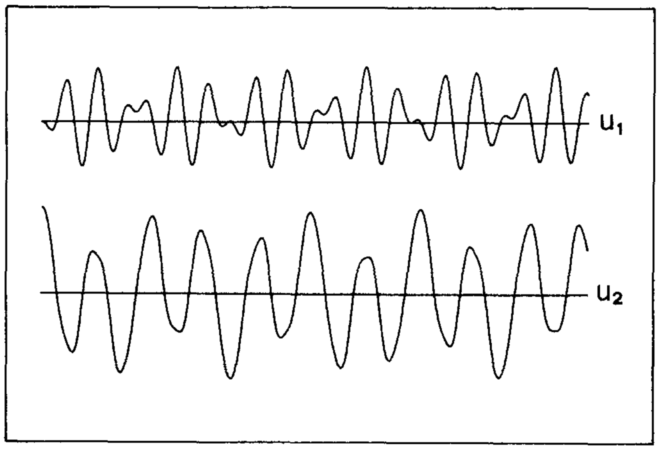

Fig. 6. Motion of cable b: $u_1(0) = 0$, $u_2(0) = 0.5$.

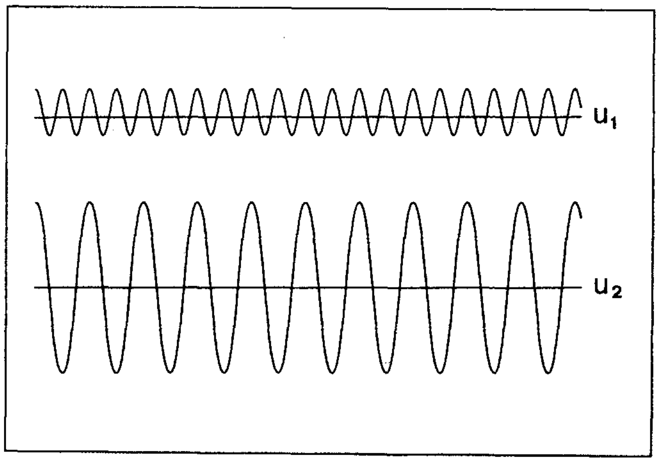

Fig. 7. Motion of cable b: $u_1(0) = 0.163$, $u_2(0) = 0.5$ (pendulum monofrequent oscillation).

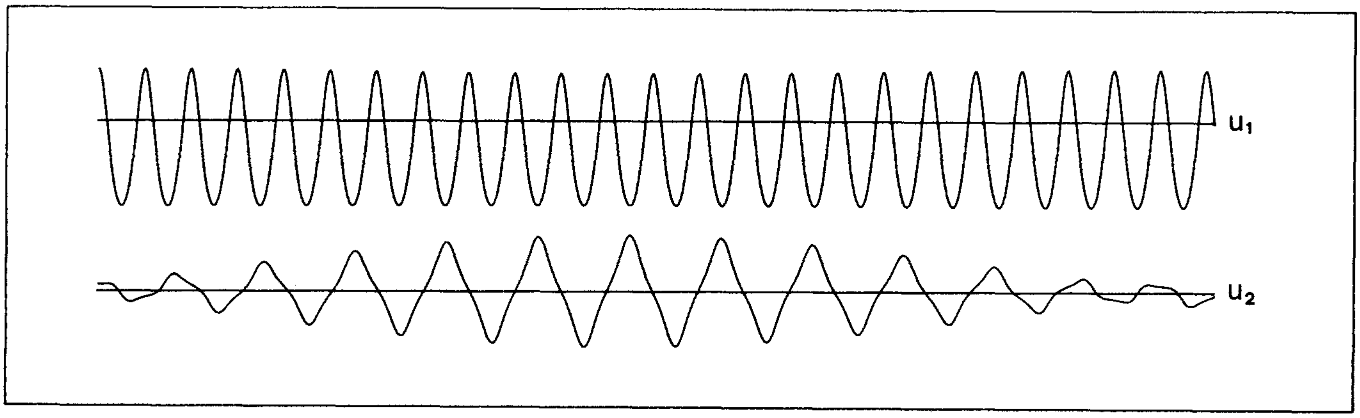

Fig. 8. Motion of cable b: $u_1(0) = 0.8$, $u_2(0) = 0.1$.

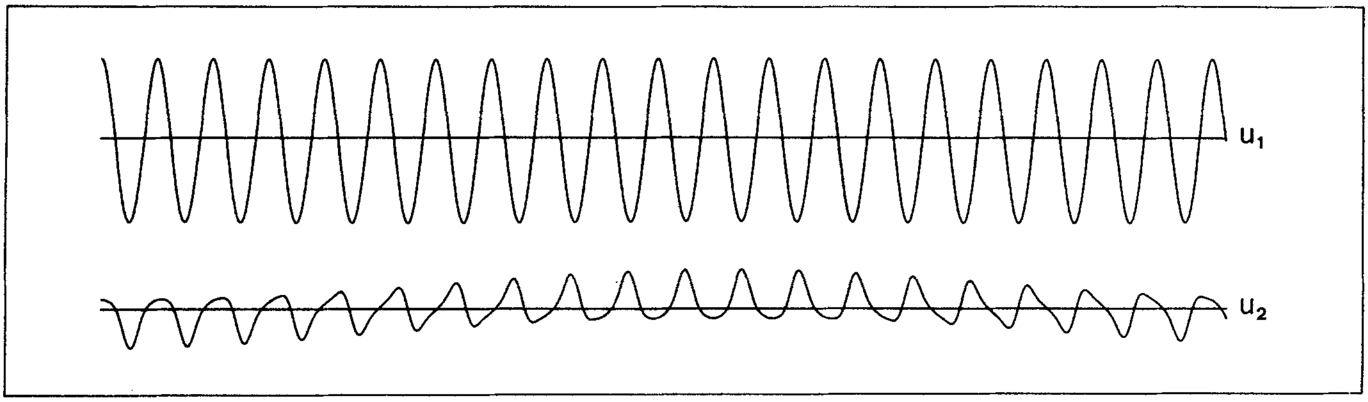

Fig. 9. Motion of cable *a*: $u_1(0) = 0.8$, $u_2(0) = 0.1$

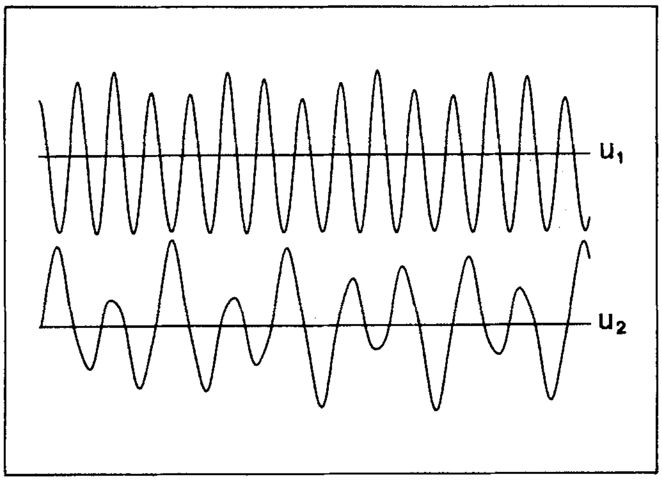

Fig. 10. Motion of cable *b*: $u_1(0) = 0.25$, $\dot{u}_1(0) = 0$; $u_2(0) = 0$, $\dot{u}_2(0) = 0.25$.

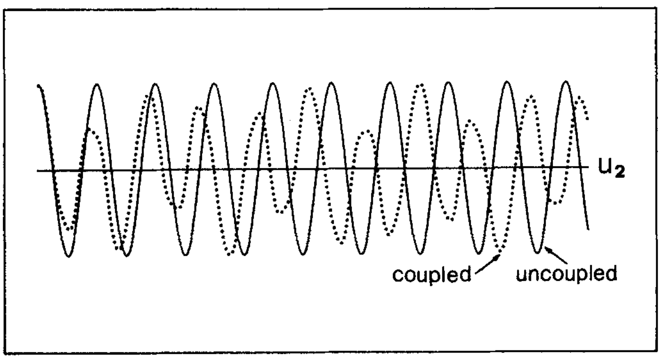

Fig. 11. Effect of disregarding modal coupling (cable *b*): $u_1(0) = 0$, $u_2(0) = 0.5$.

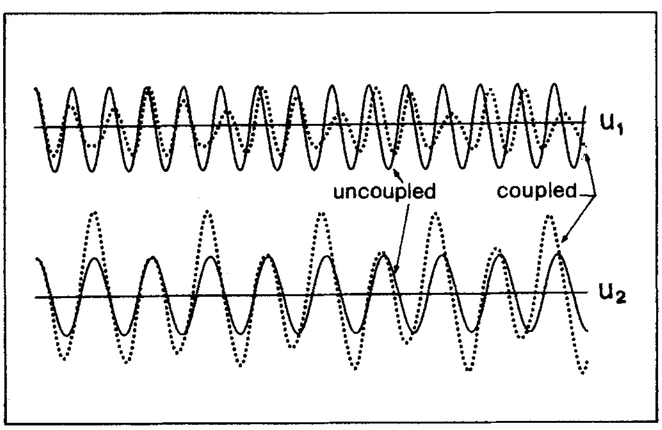

Fig. 12. Effect of disregarding modal coupling (cable *b*): $u_1(0) = u_2(0) = 0.25$.

than the maximum (0.55) in Fig. 4. Correspondingly, the peaks in the law of u_1 are amplified with respect to the initial value, $u_{1\max}$ reaching the same value as $u_{2\max}$. This shows how differently the motion evolves depending on the way, as either amplitude or velocity, a given energy is assigned initially to the system.

5. EFFECT OF DISREGARDING MODAL COUPLING

Some approximations are often introduced in the kinematics of a mechanical problem to reduce the system of equations of motion to one or two equations only. In cable continuum dynamics, this reduction has been accomplished by assuming zero the longitudinal displacement component in [14], or by expressing it in terms of the transverse displacement component through different procedures in [12, 18] and in the present paper as well (see Eq. 6); similar assumptions have been made in the literature for the taut string and the shallow arch. But the main reduction of the number of variables is often effected on the discretized model by retaining few degrees-of-freedom, sometimes only one, associated with the modes of interest of the system [4, 6]. In this way the contribution of the omitted modes on the law of motion of the retained coordinate is lost; however, it must be noticed that this contribution is not always important, since it depends on the values of the dynamical characteristics of the system.

For the cable model considered herein the effects which arise among the in-plane modes are negligible while the coupling between in-plane and out-of-plane modes just described by two parameters is important.

Indeed, if the coupling terms are omitted in Eqs. (10), no in-plane motion is forced by the out-of-plane one when the initial amplitudes $u_1(0) = 0$, $u_2(0) = 0.5$ are assigned to the cable; the pendulum oscillation in turn remains uncorrectly stationary and with nonlinear frequency $\Omega_2 = 0.996$ lower than the actual one (Fig. 11). It is then easily understood how if finite initial amplitudes ($u_1(0) = u_2(0) = 0.25$) are given to both uncoupled components, the motions which arise differ completely from the nonstationary actual ones (Fig. 12).

The results obtained clearly show the entity of the errors which can be made in some mechanical systems when modal coupling is disregarded.

6. CONCLUSIONS

A two degree-of-freedom model of an elastic cable, simple but able to represent the main features of its dynamic behaviour, has been used to study the coupled in-plane and out-of-plane finite free oscillations.

It has been shown that, as the initial amplitudes increase with respect to very low values, the response differs notably from the linear one just due to coupling between the two coordinates, which gives rise to non-periodic motions with continuous energy exchange. The slacker is the cable, the stronger are the coupling phenomena.

The peculiar characteristics of the nonlinear dynamic response of the cable are associated with its initial curvature, which causes quadratic nonlinearities in the equations of motion. They consist of: i) considerably different forced out-of-plane motion arising if the initial finite in-plane amplitude is given towards the hard or the soft side, the resultant out-of-plane motion being increased notably in the former case also if it is simply perturbed initially; ii) strong in-plane motion being forced from the out-of-plane one for zero initial condition assigned to the former as well.

An evaluation has also been made of how the laws of motion change if the coupling terms in the nonlinear equations of motion are cancelled. The information obtained can be of interest mostly whenever a given configuration variable is represented through one, rather than several, modes.

APPENDIX 1: Expressions of the dimensionless coefficients c_i

$$c_1 = -\frac{3}{1024}\Lambda^2 I_1 I_2 I_{cc}/I_{ff}I_c \qquad c_2 = -\frac{1}{2} I_f I_{cc}/I_{ff}I_c$$

$$c_3 = I_{cf}/I_{ff} \qquad c_4 = c_5 = -2I_{cf}/I_{cc} \qquad c_6 = -I_f/I_c \qquad c_7 = 1$$

$$c_8 = -\frac{1}{1024}\Lambda^2 I_2^2 I_{cc}/I_{ff}I_c \qquad c_9 = 1/6 \qquad c_{10} = 2c_{11} = -2I_{ff}/I_{cc}$$

where $\Lambda^2 = (EA/mg\ell)(8d/\ell)^3$ and the integrals I read

$$I_1 = \int_0^1 f'y'\,dx, \qquad I_2 = \int_0^1 f'^2\,dx, \qquad I_c = \int_0^1 y\,dx,$$

$$I_{cc} = \int_0^1 y^2\,dx, \quad I_{cf} = \int_0^1 yf\,dx, \quad I_{ff} = \int_0^1 f^2\,dx, \quad I_f = \int_0^1 f\,dx$$

the prime denoting $d/d\tilde{x}$ and the tilde being omitted for the sake of simplicity.

APPENDIX 2: Expressions of the dimensionless coefficients k_i

$$k_1 = -c_1/3\lambda^2 \qquad k_2 = (c_2 - c_3)/(\lambda^2 - 4) \qquad k_3 = (c_2 + c_3)/\lambda^2$$

$$k_4 = (c_4 + \lambda c_5 - c_6)/\lambda(\lambda + 2) \qquad k_5 = (c_4 - \lambda c_5 - c_6)/\lambda(\lambda - 2)$$

$$k_6 = -10c_1k_1 + 3c_8$$

$$k_7 = 2\{2c_1k_3 + c_2(k_4 + k_5) + c_3[(\lambda+1)k_4 - (\lambda-1)k_5] + c_7\}$$

$$k_8 = 6k_1(c_4 - c_6) + k_4[\lambda(\lambda+1)c_5 - (\lambda+1)^2c_4 + c_6] + k_5[\lambda(\lambda-1)c_5 - (\lambda-1)^2c_4 + c_6] - 2c_{11}$$

$$k_9 = (c_6 - c_4)(k_2 + 2k_3) + 2c_5k_2 + 3c_9$$

$$k_{10} = -(2c_1k_1 + c_8)/8\lambda^2$$

$$k_{11} = \{-2c_1k_2 + 2k_4[(\lambda+1)c_3 - c_2] + c_7\}/2(\lambda+2)$$

$$k_{12} = \{2c_1k_2 + 2k_5[(\lambda-1)c_3 + c_2] - c_7\}/2(\lambda-2)$$

$$k_{13} = [(c_4 + 2c_5 - c_6)k_2 - c_9]/8$$

$$k_{14} = \{c_4[k_1 + (\lambda+1)^2k_4] + c_5[2\lambda k_1 + \lambda(\lambda+1)k_4] + -c_6(k_1 + k_4) + \lambda c_{10} + c_{11}\}/4\lambda(\lambda+1)$$

$$k_{15} = \{c_4[-k_1 + (\lambda-1)^2k_5] - c_5[2\lambda k_1 - \lambda(\lambda-1)k_5] + -c_6(k_1 + k_5) - \lambda c_{10} + c_{11}\}/4\lambda(\lambda-1)$$

APPENDIX 3: Expressions of the complex amplitudes A^*_{hk} ('*' being omitted for the sake of simplicity)

$$A_{11} = \frac{1}{2}\left(\bar{u}_1 - i\,\frac{\dot{\bar{u}}_1}{\lambda}\right)$$

$$A_{21} = \frac{1}{2}(\bar{u}_2 - i\,\dot{\bar{u}}_2)$$

$$A_{12} = -\frac{1}{2}\left[k_1(3A_{11}^2 - \bar{A}_{11}^2 - 6A_{11}\bar{A}_{11}) + k_2\left(\frac{\lambda+2}{\lambda}A_{21}^2 + \frac{\lambda-2}{\lambda}\bar{A}_{21}^2\right) + 2k_3A_{21}\bar{A}_{21}\right]$$

$$A_{22} = -\frac{1}{2}\{k_4[(\lambda+2)A_{11}A_{21} - \lambda\bar{A}_{11}\bar{A}_{21}] + k_5[\lambda A_{11}\bar{A}_{21} - (\lambda-2)\bar{A}_{11}A_{21}]\}$$

$$\begin{aligned}A_{13} = -\frac{1}{2}\Big[&2k_1(3A_{11}A_{12} - \bar{A}_{11}\bar{A}_{12} - 3A_{11}\bar{A}_{12} - 3\bar{A}_{11}A_{12}) + \\ &+ 2k_2\left(\frac{\lambda+2}{\lambda}A_{21}A_{22} + \frac{\lambda-2}{\lambda}\bar{A}_{21}\bar{A}_{22}\right) + 2k_3(A_{21}\bar{A}_{22} + \bar{A}_{21}A_{22}) + \\ &- \frac{1}{2\lambda^2}(k_6A_{11}\bar{A}_{11} + k_7A_{21}\bar{A}_{21})(A_{11} - \bar{A}_{11}) + \\ &+ 2k_{10}(2A_{11}^3 - \bar{A}_{11}^3) + 2k_{11}\left(\frac{\lambda+1}{\lambda}A_{11}A_{21}^2 - \frac{1}{\lambda}\bar{A}_{11}\bar{A}_{21}^2\right) + \\ &+ 2k_{12}\left(\frac{\lambda-1}{\lambda}A_{11}\bar{A}_{21}^2 + \frac{1}{\lambda}\bar{A}_{11}A_{21}^2\right)\Big]\end{aligned}$$

$$\begin{aligned}A_{23} = -\frac{1}{2}\{&k_4[(\lambda+2)(A_{11}A_{22} + A_{12}A_{21}) - \lambda(\bar{A}_{11}\bar{A}_{22} + \bar{A}_{12}\bar{A}_{21})] + \\ &+ k_5[\lambda(A_{11}\bar{A}_{22} + A_{12}\bar{A}_{21}) - (\lambda-2)(\bar{A}_{11}A_{22} + \bar{A}_{12}A_{21})] + \\ &- \frac{1}{2}(k_8A_{11}\bar{A}_{11} + k_9A_{21}\bar{A}_{21})(A_{21} - \bar{A}_{21}) + 2k_{13}(2A_{21}^3 - \bar{A}_{21}^3) + \\ &+ 2k_{14}[(\lambda+1)A_{11}^2A_{21} - \lambda\bar{A}_{11}^2\bar{A}_{21}] + 2k_{15}[\lambda A_{11}^2\bar{A}_{21} - (\lambda-1)\bar{A}_{11}^2A_{21}]\}\end{aligned}$$

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