Z. Phys. C - Particles and Fields 56, 285-287 (1992)

Zeitschrift für Physik C Particles and Fields

A Perturbative solution to Gribov–Lipatov equation*

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Received 14 April 1992

Abstract. The so called ad hoc exponentiations of the perturbative solution to Gribov–Lipatov evolution equation for the non-singlet electron structure function are discussed. It is shown that the series resulting from the exponentiation prescription proposed by Jadach and Ward can be understood as a new type of perturbative solution possessing better convergence properties. A recurrence formula for the elements of the Jadach–Ward series is derived. A new perturbative series possessing even better convergence properties and simpler structure is proposed.

In the leading logarithmic approximation the non-singlet electron structure function can be obtained by solving the Gribov–Lipatov evolution equation [1]:

$$\frac{\partial D(x,\beta)}{\partial \beta}=\frac{1}{4}\lim_{\varepsilon\to 0^+}\int_0^1\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\delta(x-x_1x_2)\mathscr{P}_\varepsilon(x_1)D(x_2,\beta), \tag{1}$$

with the boundary condition

$$D(x,0)=\delta(1-x). \tag{2}$$

In (1)

$$\beta(s)=\frac{2}{\pi}\int_{m_e^2}^{s}\frac{\alpha(s')\mathrm{d}s'}{s'}, \tag{3}$$

where α is the (running) coupling constant, and

$$\mathscr{P}_\varepsilon(z)=\delta(1-z)(3/2+2\ln\varepsilon)+\theta(1-\varepsilon-z)\frac{1+z^2}{1-z}. \tag{4}$$

The perturbative solution to (1), (2)

$$D(x,\beta)=\sum_{n=0}^{N}\beta^n d_n(x)+\mathscr{O}(\beta^{N+1}), \tag{5}$$

up to the third order in β has been calculated.* In an important paper [5] Kuraev and Fadin proposed an ad hoc exponentiation procedure in order to improve accuracy of the finite order perturbative solution. According to the prescription of Kuraev and Fadin

$$D(x,\beta)=D_G(x,\beta)+\sum_{n=1}^{N}\beta^n\xi_n(x)+\mathscr{O}(\beta^{N+1}), \tag{6}$$

where the Gribov function

$$D_G(x,\beta)=\frac{\exp[\beta/2(3/4-\gamma)]}{\Gamma(1+\beta/2)}\frac{\beta}{2}(1-x)^{\beta/2-1} \tag{7}$$

solves (1) for $x\to 1^-$, and the functions $\xi_n(x)$ are derived from the requirement that (5) and (6) are identical up to terms of order β^N. In (7) $\gamma=0.57721\ldots$ is Euler's constant.

Inspired by the classic work of Yenni, Frautschi and Suura [6] Jadach and Ward proposed an alternative prescription [7]:

$$D(x,\beta)=D_G(x,\beta)\sum_{n=0}^{N-1}\beta^n\phi_n(x)+\mathscr{O}(\beta^{N+1}). \tag{8}$$

At first glance it seems surprising that one is able to express $N+1$ coefficient functions $d_n(x)$ in terms of N coefficient functions $\phi_n(x)$. However, $d_0(x)=0$ for $x\neq 1$, and the number of independent coefficient functions is N. It has been shown by studying the numerical solutions to Gribov–Lipatov equation that the Jadach and Ward exponentiation, as defined by (8), provides a particularly good approximation [4, 8].

In this paper I show that the Jadach–Ward series can be considered as a systematic perturbative expansion, and derive a recurrence formula for its coefficient functions. As a first step we eliminate the infrared regulator ε and derive

* Work partly supported by KBN Grant No. 20-38-09101
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* For a review and the list of important contributions to this subject see [2–4]

the following form of the evolution equation:

$$\frac{\partial D(x,\beta)}{\partial\beta}=[\tfrac{3}{8}+\tfrac{1}{2}\ln(1-x)-\tfrac{1}{2}\ln x]\,D(x,\beta)+\frac{1}{4}\int_x^1\frac{dy}{y-x}\left[\left(1+\frac{x^2}{y^2}\right)D(y,\beta)-2D(x,\beta)\right]. \quad (9)$$

As we are interested in the non-singlet electron structure function, which is strongly peaked near $x=1$, it is natural to eliminate the term $\sim\ln(1-x)$ in (9). Thus, we write

$$D(x,\beta)=\frac{\beta}{2}(1-x)^{\beta/2-1}\,\tilde{\Phi}(x,\beta), \quad (10)$$

and the boundary condition (3) implies

$$\tilde{\Phi}(1,0)=1, \quad (11)$$

because

$$\lim_{a\to 0^+}\int_0^1 dx\,a(1-x)^{a-1}f(x)=f(1). \quad (12)$$

The evolution equation for $\tilde{\Phi}$ can be written in the following form:

$$\frac{\partial\tilde{\Phi}(x,\beta)}{\partial\beta}=\mathscr{C}(\beta)\tilde{\Phi}(x,\beta)+\frac{1}{\beta}[\tfrac{1}{2}(1+x^2)\tilde{\Phi}(1,\beta)-\tilde{\Phi}(x,\beta)]-\tfrac{1}{2}\ln x\,\tilde{\Phi}(x,\beta)+\tfrac{1}{4}\int_x^1 dy\left(\frac{1-y}{1-x}\right)^{\beta/2}\cdot\left\{2\,\frac{\tilde{\Phi}(y,\beta)-\tilde{\Phi}(x,\beta)}{y-x}-\frac{x+y}{y^2}\tilde{\Phi}(y,\beta)+\frac{1}{1-y}[(1+x^2/y^2)\tilde{\Phi}(y,\beta)-(1+x^2)\tilde{\Phi}(1,\beta)]\right\}, \quad (13)$$

where

$$\mathscr{C}(\beta)=\frac{3}{8}-\frac{1}{2}\gamma-\frac{1}{2}\psi(1+\beta/2), \quad (14)$$

and $\psi(x)=\dfrac{d\ln\Gamma(x)}{dx}$ is Euler's digamma function. For $x\to 1^-$ only the first term in rhs of (13) does not vanish. After an obvious substitutuion:

$$\tilde{\Phi}(x,\beta)=\exp\left[\int_0^\beta d\beta'\,\mathscr{C}(\beta')\right]\Phi(x,\beta), \quad (15)$$

which implies, c.f. (7) and (10),

$$D(x,\beta)=D_G(x,\beta)\Phi(x,\beta) \quad (16)$$

we derive the following condition

$$\Phi(1,\beta)=1. \quad (17)$$

In fact we have shown that Gribov function $D_G(x,\beta)$ solves Gribov–Lipatov equation for $x\to 1^-$. The evolution equation for $\Phi(x,\beta)$ reads:

$$\frac{\partial\Phi(x,\beta)}{\partial\beta}=\frac{1}{\beta}[\tfrac{1}{2}(1+x^2)-\Phi(x,\beta)]-\tfrac{1}{2}\ln x\,\Phi(x,\beta)+\frac{1}{4}\int_x^1 dy\left(\frac{1-y}{1-x}\right)^{\beta/2}\left\{\frac{1}{y-x}\left[\frac{1-x}{1-y}(1+x^2/y^2)\cdot\Phi(y,\beta)-2\Phi(x,\beta)\right]-\frac{1+x^2}{1-y}\right\}. \quad (18)$$

Let us remark that in contrast to (1), (2) the above equation admits a smooth solution for $1\geq x\geq x_0$ and $0\geq\beta\geq\beta_0$, where x_0 and β_0 are positive. Since the kernel of the integral operator is non-singular in this region one can solve (18) numerically, or write the solution in terms of power series in β, or $1-x$, or a double power series*. In particular, by inserting the expansion

$$\Phi(x,\beta)=\sum_{n=0}^{\infty}(\beta/2)^n\,\phi_n(x), \quad (19)$$

we derive a recurrence formula for the coefficient functions of Jadach–Ward series:

$$\phi_0(x)=\tfrac{1}{2}(1+x^2), \quad (20)$$

$$\phi_1(x)=-\tfrac{1}{8}[2(1-x)^2+(1+3x^2)\ln x], \quad (21)$$

$$\phi_{n+1}(x)=\frac{1}{n+2}\left\{\frac{1-x}{4}\lambda_n(x)-\phi_n(x)\ln x+\frac{1-x}{2}\sum_{k=1}^{n}\frac{1}{(n-k)!}\cdot\int_x^1\frac{dy}{y-x}\ln^{n-k}\left(\frac{1-y}{1-x}\right)\cdot\left[(1+x^2/y^2)\frac{\phi_k(y)}{1-y}-2\frac{\phi_k(x)}{1-x}\right]\right\}, \quad (22)$$

where

$$\lambda_n(x)=-(1/n!)\int_x^1 dy(1+y)(x+y)/y^2\ln^n[(1-y)/(1-x)]. \quad (23)$$

The above integrals can be expressed in terms of Nielsen's polylogarithms

$$S_{n,m}(x)=(-1)^{n+m-1}/[(n-1)!m!]\int_0^1 dt\,t^{-1}\ln^{n-1}t\ln^m(1-xt). \quad (24)$$

For $n\geq 2$

$$\lambda_n=(-1)^{n+1}[1-x+(1+x)S_{n,1}(1-x)+xS_{n-1,1}(1-x)], \quad (25)$$

whereas

$$\lambda_1=1-x+(1+x)\,\mathrm{Li}_2(1-x)-x\ln x, \quad (26)$$

and dilogarithm $\mathrm{Li}_2\equiv S_{1,1}$. The first two coefficient functions ϕ_n have been calculated in [7]. A direct evaluation of Feynman diagrams and resulting leading terms also leads to ϕ_0 as given in (20), c.f. [2]. The next coefficient reads:

$$\phi_2=\frac{1}{8}\left[(1-x)^2+\frac{1}{2}(3x^2-4x+1)\ln x+\frac{1}{12}(1+7x^2)\ln^2 x+(1-x^2)\,\mathrm{Li}_2(1-x)\right], \quad (27)$$

and has been first obtained in [8].

It is evident from the form of the coefficient functions ϕ_n that Jadach–Ward series is poorly convergent for

* Analytical solutions to evolution equations in terms of power series in x and $1-x$ are given in [9], see also references quoted therein

x near zero. It follows from (22) that $\phi_n \sim \ln^n x$ for small x. Thus, it seems reasonable to consider the power series expansion for the function

$$\Psi(x, \beta) = \Phi(x, \beta)/\Phi_0(\beta \ln x), \tag{28}$$

which we require to be finite at $x=0$. By definition $\Phi_0(\beta \ln x)$ contains this part of $\Phi(x, \beta)$ which is divergent for $x \to 0^+$. It is conceivable that the power series for $\Psi(x, \beta)$ has a simpler structure, at least for a few first coefficients, because the requirement of finiteness constrains the number of building blocks for the perturbative expression of a given order*. For the same reason expressions resulting from Jadach and Ward exponentiation are much simpler than the corresponding expressions derived using Kuraev and Fadin prescription. In order to derive $\Phi_0(\beta \ln x)$ one has to study the limit: x and $\beta \to 0^+$ for $\beta \ln x$ finite. In this limit one derives from (18) the following equation for Φ_0:

$$\begin{aligned}\frac{\partial \Phi_0(\beta \ln x)}{\partial \beta} = \frac{1}{4}\int_x^1 dy \{ & 2[\Phi_0(\beta \ln y) - \Phi_0(\beta \ln x)]/(y-x) \\ & - \Phi_0(\beta \ln y)/y\} + \beta^{-1}[1 - \Phi_0(\beta \ln x)] \\ & - \frac{1}{2} \ln x \, \Phi_0(\beta \ln x) + \mathcal{O}(1).\end{aligned} \tag{29}$$

We insert expansion

$$\Phi_0 = \sum_{n=0}^{\infty} c_n (\beta \ln x)^n, \tag{30}$$

and obtain the following expression for the coefficients c_n:

$$c_n = (-1)^n/[4^n (n+1)! \, n!]. \tag{31}$$

Consequently

$$\Phi_0(\beta \ln x) = \frac{2}{\sqrt{-\beta \ln x}} \mathbf{I}_1(\sqrt{-\beta \ln x}), \tag{32}$$

where $\mathbf{I}_1(z)$ is the modified Bessel function.

Taking all the factors together we obtain our final expression for the non-singlet structure function in the leading logarithmic approximation:

$$\begin{aligned}D(x, \beta) = & \frac{\exp(\beta/2(3/4-\gamma))}{\Gamma(1+\beta/2)} \sqrt{-\beta/\ln x} \, \mathbf{I}_1(\sqrt{-\beta \ln x}) \\ & \cdot (1-x)^{\beta/2-1} \Psi(x, \beta),\end{aligned} \tag{33}$$

where the power series for $\Psi(x, \beta)$ can be derived from Jadach–Ward series (20)–(22) and the series (30) for Φ_0. Then from (28) we obtain:

$$\begin{aligned}\Psi(x, \beta) = & \frac{1}{2}(1+x^2) - \frac{\beta}{8}[(1-x)^2 + x^2 \ln x] \\ & + \frac{\beta^2}{32}(1-x)[1 - x - x \ln x \\ & + (1+x) \mathrm{Li}_2(1-x)] + \ldots .\end{aligned} \tag{34}$$

As a final remark let us note that exponentiation in QCD is a highly non-trivial task. Thus, it has been believed that the exponentiation procedures cannot be easily extended to QCD. However, the derivation given in the present article can be extended to the case of Altarelli–Parisi equation [10].

Acknowledgements. I thank S. Jadach, J.H. Kühn and K. Zalewski for discussions and comments on this work.

* A similar strategy for solving differential equations is known in quantum mechanics as the method of Sommerfeld polynomials. I thank Professor Kacper Zalewski for pointing this out to me

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