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Excluded volume effect for the nuclear matter equation of state

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Abstract. We present the thermodynamically consistent procedure to introduce the excluded volume effect into the equation of state of nuclear matter. Implications are discussed in the framework of a mean-field model for hadrons with eigenvolume.

1 Introduction

The determination of the nuclear matter equation of state (EOS) remains nowadays one of the foremost goals in (both theoretical and experimental) investigations on heavy-ion physics (see, for example, [1]). The EOS is a necessary input for hydrodynamical models of nuclear collisions. Any reasonable model for the nuclear matter EOS must account at least for the following two features:

(A) The reproduction of the ground state properties of nuclear matter. At temperature $T=0$ and baryonic density $n \equiv n_0 \cong 0.16\ \mathrm{fm}^{-3}$ nuclear matter saturates, i.e., the energy per particle $W(n) \equiv (\varepsilon/n)_{T=0} - M$ (ε is the energy density, M is the free rest mass of the nucleon) has a minimum (i.e., the pressure $p_0 = 0$) and assumes the value $W(n_0) \cong -16\,\mathrm{MeV}$. Furthermore, experimental information suggests for the effective nucleon mass at the ground state the value $M_0^* = (0.7 \pm 0.1)\,M$ and for the incompressibility $K_0 = 9(\partial p/\partial n)_{T=0, n=n_0}$ values between 200 and 300 MeV (see [2]).

(B) The transition to a phase of deconfined quarks and gluons (quark–gluon plasma) at very high temperatures and/or baryonic densities. Lattice simulations of quantum chromodynamics (QCD) predict this transition to occur at $T_c \cong 200\,\mathrm{MeV}$ for vanishing net baryon number (see, for example, [3]).

Since quarks and gluons are asymptotically free at very large T and n, one can apply QCD perturbation theory in this (T, n)-region to derive an EOS for the quark–gluon plasma [4]. One has to add some phenomenological terms to take into account the difference between the perturbative and the true vacuum (like, e.g., in the bag model [5]). To derive an EOS for confined matter directly from QCD is impossible to date. Thus, one has to invent a more or less phenomenological model EOS for hadron matter. The hadron and the quark–gluon plasma EOS are amalgamated via Gibbs' conditions of phase coexistence.

However, to our knowledge, there is yet *no* hadron matter EOS which is able to account for both above requirements *simultaneously*! Let us clarify this statement.

Hadron matter as an ideal gas of point-like particles does certainly not reproduce the ground state properties (A). Furthermore, there is no reasonable phase transition to the quark–gluon plasma (property (B)) [5].

The meson mean-field model [6] and its phenomenological generalizations [7–10] can account for property (A). However, the repulsive interaction in these models is proportional to n and vanishes for $n \mapsto 0$. Thus, at large T (and $n \cong 0$) one can in principle excite thermally a large number of point-like hadronic resonances which do not interact repulsively. It is clear that due to the large number of degrees of freedom their pressure becomes larger than that of the quark–gluon plasma for sufficiently high T. According to Gibbs' conditions, the hadronic phase would become stable at $T \mapsto \infty$ [11, 12], which is in contradiction with requirement (B).

To overcome this difficulty a hard-core repulsion for hadron gas models has been widely discussed in recent publications [12–16]. By construction, such an approach violates causality at high densities (there is no relativistic rigid body!), but it can help to remedy the above mentioned shortcomings of the point-like particle models. However, there are two faults in the approaches proposed in [12–16]:

1) The excluded volume effect was introduced in the ideal gas model only (so that the property (A) could not be satisfied).

2) The excluded volume procedures in [12–16] are thermodynamically inconsistent.

As a first step in the present work we reconsider the implementation of the excluded volume in ideal gas models to obtain the thermodynamically consistent

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formulation and to correct the results of [12–16]. Then we generalize our approach to the nuclear matter EOS in mean-field theoretical models. It enables us to combine the desired features (A) and (B) in a self-consistent model. The implications will be considered in the framework of the meson mean-field model [6] extended to include a hard-core repulsion.

2 Excluded volume in ideal gas models

For simplicity, we first consider one particle species with eigenvolume v_0. The pressure p is related to the grand partition function $\mathscr{Z}$ according to

$$p(T,\mu)=\lim_{V\mapsto\infty} T\frac{\ln\mathscr{Z}(T,\mu,V)}{V}, \tag{1}$$

where μ is the chemical potential, V is the volume of the system. $\mathscr{Z}$ is defined as

$$\mathscr{Z}(T,\mu,V)=\sum_{N=0}^{\infty} e^{\mu N/T} Z(T,N,V) \tag{2}$$

To introduce the excluded volume (á la Van-der-Waals) it is necessary to substitute the canonical partition function Z in (2) by

$$Z^{\text{excl}}(T,N,V)=Z(T,N,V-v_0N)\theta(V-v_0N). \tag{3}$$

This ansatz is motivated by considering N particles with eigenvolume v_0 in a volume V as N point-like particles in the "available volume" $V-v_0N$. Then the following grand partition function results

$$\mathscr{Z}^{\text{excl}}(T,\mu,V)=\sum_{N=0}^{\infty} e^{\mu N/T} Z(T,N,V-v_0N)\theta(V-v_0N). \tag{4}$$

The main problem in the calculation of (4) is the dependence of the available volume on the varying number of particles N. To overcome this difficulty we perform a Laplace transformation of (4). This method of the "isobaric partition function" [17] was successfully used [18–20] to investigate the excluded volume effect in a gas of quark–gluon bags. We obtain

$$\hat{\mathscr{Z}}^{\text{excl}}(T,\mu,\xi)\equiv\int_0^{\infty} dV e^{-\xi V}\mathscr{Z}^{\text{excl}}(T,\mu,V) = \int_0^{\infty} dx e^{-\xi x}\mathscr{Z}(T,\hat{\mu},x), \tag{5}$$

where $\hat{\mu}\equiv\mu-v_0T\xi$. The second equality in (5) results by changing the integration variable V to $x=V-v_0N$.

To proceed further we remind the reader of the following facts (see [18]). It can be proven from the properties of the Laplace transforms that the pressure is equal to

$$p^{\text{excl}}(T,\mu)\equiv\lim_{V\mapsto\infty} T\frac{\ln\mathscr{Z}^{\text{excl}}(T,\mu,V)}{V}=T\xi^*(T,\mu), \tag{6}$$

where $\xi^*(T,\mu)$ is the extreme right singularity of the function $\hat{\mathscr{Z}}^{\text{excl}}$ in the variable ξ. In our consideration $\hat{\mathscr{Z}}^{\text{excl}}$ has only one singular point (for positive ξ), namely when the integral over x in (5) diverges at the upper limit. Thus

$$\xi^*=\lim_{x\mapsto\infty}\frac{\ln\mathscr{Z}(T,\tilde{\mu},x)}{x};\quad \tilde{\mu}=\mu-v_0T\xi^*.$$

Applying (1) for $\mathscr{Z}(T,\tilde{\mu},x)$ and using (6) to eliminate ξ^* we find

$$p^{\text{excl}}(T,\mu)=p(T,\tilde{\mu});\quad \tilde{\mu}=\mu-v_0p^{\text{excl}}(T,\mu). \tag{7}$$

Therefore, we obtain an implicit equation for $p^{\text{excl}}(T,\mu)$, if p is a known function of its arguments.

Let us consider the ideal gas case for sake of illustration. Then we have in (5)

$$\mathscr{Z}_{\text{id}}(T,\tilde{\mu},x)=\exp[xF(T,\tilde{\mu})], \tag{8}$$

where

$$F(T,\mu)=\frac{1}{a}\frac{g}{(2\pi)^3}\cdot\int d^3\mathbf{k}\ln\left\{1+a\exp\left[\frac{-(\mathbf{k}^2+m^2)^{1/2}+\mu}{T}\right]\right\}. \tag{9}$$

Here g is the degeneracy factor, m is the particle mass, $a=\pm1$ for fermions/bosons and $a\mapsto0$ in the Boltzmann limit. One can directly perform the integral (5) and obtains

$$\hat{\mathscr{Z}}^{\text{excl}}_{\text{id}}(T,\mu,\xi)=\frac{1}{\xi-F(T,\hat{\mu})},$$

which gives

$$p^{\text{excl}}_{\text{id}}(T,\mu)=TF(T,\tilde{\mu})=p_{\text{id}}(T,\mu-v_0p^{\text{excl}}_{\text{id}}(T,\mu)). \tag{10}$$

This is just the special case of (7) for the ideal gas (cf. [21, 22]). We note, however, that the expression (7) is valid also for more general cases.

The particle number density, the entropy density and the energy density are found from (10)

$$n^{\text{excl}}_{\text{id}}(T,\mu)\equiv\left(\frac{\partial p^{\text{excl}}_{\text{id}}}{\partial\mu}\right)_T=\frac{n_{\text{id}}(T,\tilde{\mu})}{1+v_0n_{\text{id}}(T,\tilde{\mu})}, \tag{11}$$

$$s^{\text{excl}}_{\text{id}}(T,\mu)\equiv\left(\frac{\partial p^{\text{excl}}_{\text{id}}}{\partial T}\right)_\mu=\frac{s_{\text{id}}(T,\tilde{\mu})}{1+v_0n_{\text{id}}(T,\tilde{\mu})}, \tag{12}$$

$$\varepsilon^{\text{excl}}_{\text{id}}(T,\mu)\equiv Ts^{\text{excl}}_{\text{id}}-p^{\text{excl}}_{\text{id}}+\mu n^{\text{excl}}_{\text{id}}=\frac{\varepsilon_{\text{id}}(T,\tilde{\mu})}{1+v_0n_{\text{id}}(T,\tilde{\mu})}, \tag{13}$$

where n_{id}, s_{id} and ε_{id} are the well-known expressions for an ideal gas of point-like particles. We stress that our relations (10–13) are thermodynamically consistent (fundamental thermodynamical relations are fulfilled). This is in contrast with the formulations in [12–16], where the typical errors are $n\neq\left(\frac{\partial p}{\partial\mu}\right)_T$, $s\neq\left(\frac{\partial p}{\partial T}\right)_\mu$. The reason is that these formulations do not account for the necessary modification of the chemical potential $\mu\mapsto\tilde{\mu}$ in the ideal gas functions $n_{\text{id}}, s_{\text{id}}, \varepsilon_{\text{id}}, p_{\text{id}}$.

We mention that, instead of the ansatz (3), one could introduce the excluded volume directly in the grand partition function,

$$\mathscr{Z}^{\text{excl}}(T,\mu,V)=\mathscr{Z}(T,\mu,V-v_0\bar{N}^{\text{excl}}), \tag{14}$$

where $\bar{N}^{\text{excl}}$ is the mean particle number. The resulting pressure reads

$$p^{\text{excl}}(T,\mu)=p(T,\mu)(1-v_0 n^{\text{excl}}(T,\mu)). \tag{15}$$

If $v_0 p(T,\mu)\ll 1$, this approach is similar to the above. To see this one uses the identity $n^{\text{excl}}(T,\mu)=(\partial p^{\text{excl}}/\partial\mu)_T$, rearranges the terms and uses Taylor's theorem. After a shift of μ by $-v_0 p(T,\mu)$ we obtain

$$p^{\text{excl}}(T,\mu)\cong p(T,\mu-v_0 p(T,\mu)). \tag{16}$$

If in addition $v_0 n^{\text{excl}}(T,\mu)\ll 1$, $p^{\text{excl}}(T,\mu)\cong p(T,\mu)$ according to (15) and (16) even coincides with (7). The disadvantage of the ansatz (14) is that only an *average* excluded volume enters the calculation and that the expressions for other thermodynamical quantities become more complicated. However, these expressions do not lead to contradictions with thermodynamics, if we refrain from the use of "ad hoc prescriptions" to construct them.

We finally note that the extension of the procedure (3–7) for several particle species is straightforward. If $v_0^{(1)},\ldots,v_0^{(k)}$ denote the particle eigenvolumes and $\mu_1,\ldots,\mu_k$ are the chemical potentials for k different kinds of particles, then we obtain

$$p^{\text{excl}}(T,\mu_1,\ldots,\mu_k)=p(T,\tilde{\mu}_1,\ldots,\tilde{\mu}_k), \tag{17}$$

where

$$\tilde{\mu}_i=\mu_i-v_0^{(i)}p^{\text{excl}}(T,\mu_1,\ldots,\mu_k),\quad i=1,\ldots,k.$$

3 Excluded volume in mean-field theoretical models

In this section we introduce the excluded volume in the nuclear matter EOS. We restrict our consideration to a nucleon–antinucleon system (poins and resonances are neglected for sake of simplicity), where the pressure can be written as [9]

$$p(T,\mu)=T[F(T,v_N)+F(T,v_{\bar{N}})] + nU(n)-\int_0^n dn'\,U(n')+P(M^*). \tag{18}$$

The function F is given by (9) with $a=1$, $m=M^*$, $g=4$ and

$$v_N\equiv\mu-U(n),\quad v_{\bar{N}}\equiv-\mu+U(n),$$

μ is the baryonic chemical potential ($\mu_N=\mu$, $\mu_{\bar{N}}=-\mu$). Formula (18) defines a special class of thermodynamically self-consistent equations of state for nuclear matter which are phenomenological extensions of the meson mean-field model [6]. Models of this class are fixed by specifying the two functions $U(n)$ and $P(M^*)$. Particular choices of $U(n)$ and $P(M^*)$ reproduce a great variety of nuclear EOS models known from the literature (see [9, 10] for details).

It is seen from (18) that the pressure is a sum of two contributions:

1) The thermal pressure of the nucleons and antinucleons with effective mass M^* in the field $U(n)$.
2) Contributions of the fields themselves.

Therefore, the grand partition function is a product

$$\mathscr{Z}(T,\mu,V)=\mathscr{Z}_{\text{therm}}(T,\mu,V)\times\mathscr{Z}_{\text{field}}(T,\mu,V), \tag{19}$$

where

$$\mathscr{Z}_{\text{therm}}(T,\mu,V)=\exp\{V[F(T,v_N)+F(T,v_{\bar{N}})]\}. \tag{20}$$

The motivation for the introduction of an excluded volume is to restrict the free particle motion. Hence we apply the above Laplace transformation (5) to $\mathscr{Z}_{\text{therm}}$ only. Therefore, the excluded volume EOS for mean-field models reads

$$p^{\text{excl}}(T,\mu)=T[F(T,\tilde{v}_N)+F(T,\tilde{v}_{\bar{N}})] + n^{\text{excl}}U(n^{\text{excl}})-\int_0^{n^{\text{excl}}} dn'\,U(n')+P(M^*), \tag{21}$$

where

$$\tilde{v}_N\equiv v_N-v_0 T[F(T,\tilde{v}_N)+F(T,\tilde{v}_{\bar{N}})],$$

$$\tilde{v}_{\bar{N}}\equiv v_{\bar{N}}-v_0 T[F(T,\tilde{v}_N)+F(T,\tilde{v}_{\bar{N}})].$$

Thermodynamic consistency requires the appearance of n^{excl} in (21), instead of n as in (18) (cf. [9]).

The dependence of the effective nucleon mass M^* on T and μ is defined by extremizing the thermodynamical potential (maximum of the pressure):

$$\left(\frac{\delta p^{\text{excl}}}{\delta M^*}\right)_{T,\mu}=0. \tag{22}$$

The specification of $U(n^{\text{excel}})$ and $P(M^*)$ completely determines the EOS. Other thermodynamical quantities can be obtained from general thermodynamic relations.

4 A meson mean-field model EOS for particles with finite eigenvolume

In this section we study an application of the general result (21) of the preceding section. We put

$$P(M^*)=-\frac{1}{2C_s^2}(M-M^*)^2,\quad U(n^{\text{excl}})=C_v^2 n^{\text{excl}}. \tag{23}$$

Then, in the case $v_0=0$, the EOS reduces to the well-known meson mean-field model of [6].

In Fig. 1 we show the ground state properties of the EOS (21–23) as a function of the parameter $R=(3v_0/4\pi)^{1/3}$ (the "radius" of a nucleon*). One observes that the effective mass in the ground state M_0^* increases with R and reaches experimentally measured values around $R\cong 0.7$ fm, which is also a reasonable value for the nucleon radius. The ground state incompressibility coefficient K_0 reaches a minimum of $K_0\cong 523$ MeV around $R\cong 0.7$ fm.

The explanation of these results is obvious: the

* Note that in the present excluded volume approach particles are treated as deformable but incompressible objects

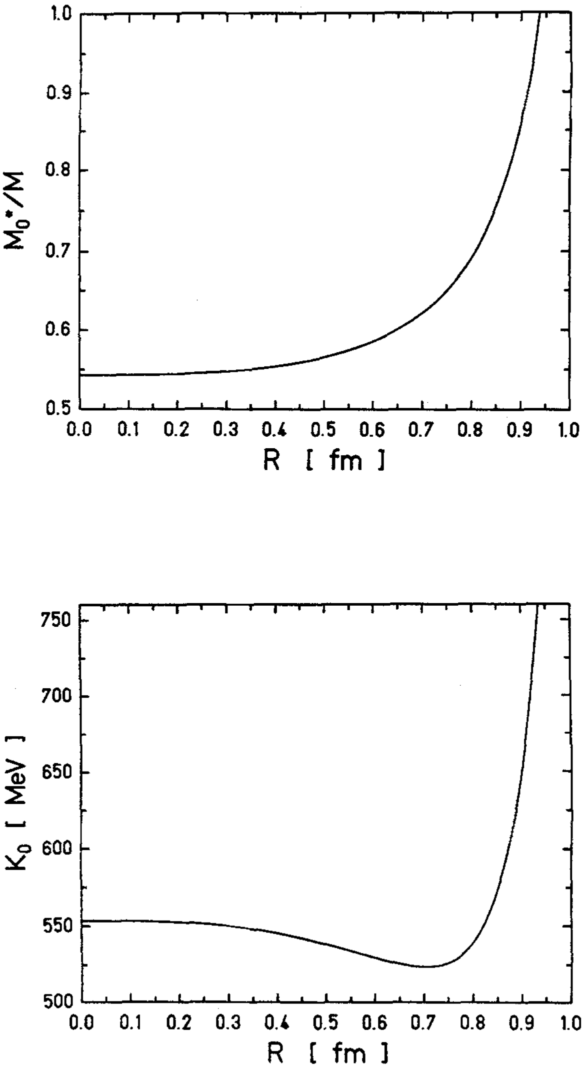

Fig. 1 a, b. The normalized effective mass M_0^*/M **a** and the incompressibility K_0 **b** in the nuclear matter ground state as a function of the "radius" R of a nucleon

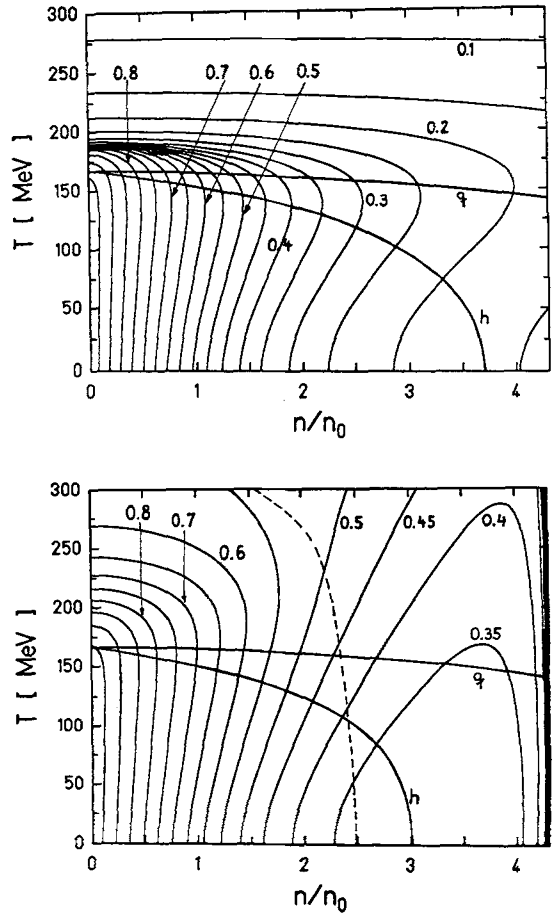

Fig. 2 a, b. Contour plot of the normalized effective mass M^*/M as a function of T and n/n_0 for **a** the original mean-field model with $R=0$ and **b** the mean-field model (21–23) with $R=0.7$ fm. The curves labeled (h) and (q) are the boundaries of the Gibbs phase coexistence region between hadron matter and a quark–gluon plasma (described by the bag model EOS with $B^{1/4}=235$ MeV). To the right of the dashed line in **b** the EOS (21–23) becomes acausal, due to the inherently non-relativistic nature of the eigenvolume approach

nucleonic eigenvolume provides an additional repulsive force. Thus, the vector field repulsion is reduced (C_v^2 is smaller; this lowers K_0). Simultaneously, the scalar field is reduced, since it is no longer necessary to balance a large vector field in order to reproduce the ground state binding energy. This rises the effective mass. We conclude that the eigenvolume-corrected mean-field model (21–23) exhibits more realistic ground state features than the original one (18).

We also investigated the behaviour of (21–23) for $R=0.7$ fm at large T and n. First of all we mention that, since the hadrons are incompressible, we have a limiting value for the total particle density, $(n_N+n_{\bar{N}})_{\rm lim}^{\rm excl}=1/v_0 \cong 4.4\,n_0$. Approaching this value, ε, n, and s assume finite values, while p diverges. In Fig. 2 we show contour plots of the normalized effective mass, M^*/M, as a function of T and n/n_0 for the original mean-field model ($v_0=0$) and the modified model (21–23) with $R=0.7$ fm. One observes that the transition to a massless nucleon–antinucleon plasma ($M^*\mapsto 0$) at $T\cong 200$ MeV and small density n exhibited by the mean-field model [23] vanishes, if nucleons have a finite eigenvolume. Moreover, M^* never becomes smaller than ~ 0.3 M. A point of criticism of the meson mean-field model was that the mean-field approximation is no longer valid for $M^*/M \ll 1$.

The Gibbs phase coexistence region, calculated for quark matter described by the bag model with $B^{1/4}=235$ MeV and hadron matter described by (18) and (21–23), respectively, is also depicted in Fig. 2. One observes that the reduction of the hadronic pressure due to the excluded volume reduces the domain of thermodynamic stability of hadron matter in the ($T-n/n_0$)-plane. However, the exact position of the coexistence region depends on the value of the bag constant and whether we include pions and resonances on the hadronic side [24].

As pointed out in the introduction the excluded volume description is acausal for large n and T. For our model EOS (21–23) this happens to the right of the dashed line in Fig. 2(b), near the phase transition to the quark–gluon plasma. The inclusion of pions and resonances will alter the position of this line, as well as another effect which we have up to now neglected: the reduction of the hadronic eigenvolume due to the pressure in a medium (as, e.g., in a bag model of hadrons, see [22, 25]). Then $v_0 = v_0(T, \mu)$ decreases with increasing T and/or n. In this case we have no limiting value $(n_N + n_{\bar{N}})_{\mathrm{lim}}^{\mathrm{excl}}$ in the hadron phase and the domain of causality is considerably enlarged.

A main advantage of the new EOS presented here is that the inclusion of resonances (with their eigenvolumes) will not favour the hadronic phase at $n \cong 0$ and large T as in the standard mean-field model approach.

5 Summary

We have presented the thermodynamically consistent EOS for free particles with eigenvolume. We have also shown that the excluded volume can be consistently introduced in mean-field theoretical model equations of state for hadron matter. We have applied the results to the meson mean-field model. The reproduction of the ground state properties of nuclear matter is more realistic than in the case $v_0 = 0$. Furthermore, there is no transition to a massless nucleon–antinucleon plasma at large T, and the effective mass M^* is never smaller than $\sim 0.3\,M$.

The main advantage of the new EOS is that it combines realistic ground state properties of nuclear matter with the physically expected behaviour at large T and n (properties (A) and (B) discussed in the introduction).

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References

1. H. Stöcker, W. Greiner: Phys. Rep. 137 (1986) 277
2. A. van der Woude in: "The Nuclear Equation of State. Part A: Discovery of Nuclear Shock Waves and the EOS", proceedings of the NATO Advanced Study Institute on the Nuclear Equation of State, May 22–June 3, 1989, Peñiscola, Spain (ed. W. Greiner, H. Stöcker), p. 649; M.M. Sharma, ibid., p. 661; J.P. Blaizot, ibid., p. 679; N.K. Glendenning, ibid., p. 751
3. F. Karsch: Z. Phys. C – Particles and Fields 38 (1988) 147
4. B.A. Freedman, L.D. McLerran: Phys. Rev. D16 (1977) 1130, 1147, 1169; J.I. Kapusta: Nucl. Phys. B148 (1979) 461
5. J. Cleymans, R.V. Gavai, E. Suhonen: Phys. Rep. 130 (1986) 217
6. J.D. Walecka: Ann. Phys. 83 (1974) 491; B.D. Serot, J.D. Walecka: Phys. Lett. 27B (1979) 172
7. D.H. Rischke, B.L. Friman, H. Stöcker, W. Greiner: J. Phys. G14 (1988) 191
8. J. Zimanyi, B. Lukacs, P. Levai, J. Bondorf, N.L. Balazs: Nucl. Phys. A484 (1988) 647
9. K.A. Bugaev, M.I. Gorenstein: Z. Phys. C – Particles and Fields 43 (1989) 261
10. M.I. Gorenstein, D.H. Rischke, K.A. Bugaev, H. Stöcker, W. Greiner: Preprint UFTP 239/1989, Frankfurt, 1989
11. D.H. Rischke: Diploma thesis, Frankfurt, 1988 (unpublished)
12. U. Heinz, P.R. Subramanian, H. Stöcker, W. Greiner: J. Phys. G12 (1986) 1237
13. J. Cleymans, K. Redlich, H. Satz, E. Suhonen: Z. Phys. C – Particles and Fields 33 (1986) 151
14. E. Suhonen, S. Sohlo: J. Phys. G13 (1987) 1487
15. H. Kuono, F. Takagi: Z. Phys. C – Particles and Fields 42 (1989) 209
16. J. Cleymans, E. Suhonen: Z. Phys. C – Particles and Fields 37 (1987) 51; J. Cleymans, H. Satz, E. Suhonen, D.W. von Oertzen: Phys. Lett. 242B (1990) 111
17. A. Münster: Statistical Physics, Berlin, Heidelberg, New York: Springer 1969
18. M.I. Gorenstein, V.K. Petrov, G.M. Zinovjev: Phys. Lett. 106B (1981) 327
19. R. Hagedorn: Z. Phys. C – Particles and Fields 17 (1983) 265
20. J.I. Kapusta, K.A. Olive: Nucl. Phys. A408 (1983) 478
21. L.P. Csernai, J.I. Kapusta, G. Fai, D. Hahn, J. Randrup, H. Stöcker: Phys. Rev. C35 (1986) 1297
22. A.G. Schnabel: Preprint UCT-TP 97/88, Cape Town, 1988 (unpublished)
23. J. Theis, G. Graebner, G. Buchwald, J. Maruhn, W. Greiner, H. Stöcker, J. Polonyi: Phys. Rev. D28 (1983) 2286
24. B.M. Waldhauser, D.H. Rischke, J.A. Maruhn, H. Stöcker, W. Greiner: Z. Phys. C – Particles and Fields 43 (1989) 411
25. J.I. Kapusta: Phys. Rev. D23 (1981) 2444