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On the distribution of flood volume in partial duration series analysis of flood phenomena

V. Choulakian

Dep. of Mathematics

N. El-Jabi

School of Engineering

J. Moussi

Dep. of Mathematics, Université de Moncton, Moncton, N.B. E1A 3E9, Canada

Abstract: A methodology based on the theory of stochastic processes is applied to the analysis of floods. The approach will be based on some results of the theory of extreme values over a threshold. In this paper, we focus on the estimation of the distribution of the flood volume in partial duration series analysis of flood phenomena, by using a bivariate exponential distribution of discharge exceedances and durations over a base level.

Key words: Exponential distribution, bivariate exponential distribution, distribution of flood volume, partial duration series.

1 Introduction

An important facet in the large-scale planning of water resources projects and the development of water resources is the prediction of future water supply, mainly from rainfall and runoff. Of particular importance for the improvement of flood plain management and the design of various hydraulic structures are excessive flows, commonly described as floods.

Understanding the stochastic nature of the flood phenomenon, which is a complex geophysical time process, is the basis of a general methodology of flood analysis.

Since in hydrology most processes involve several random variables, the best approach to the analysis of such processes is through the use of multivariate distributions; and in many aspects of water management and environmental planning, the knowledge of the size and time distribution of streamflows is essential. Peak flood discharge, flood volume and flood duration are three random variables whose probability distributions are of interest to hydrologists.

In many practical problems of flood control engineering, it is important to consider not only the sequences of flood peak exceedances but also the volume of water associated with these exceedances.

Mainly there are two probabilistic methods to analyse flood phenomena: the first method is based on the annual flood series, which corresponds to fitting a distribution function - the most frequently used are log-normal, log Pearson type 3, two parameter gamma and Gumbel or extreme value type I and generalized extreme value - distributions to sampled values of maximum annual floods; the second one is based on the partial duration series (Langbein, 1949; Dalrymple, 1960; Borgman, 1963; Shane and Lynn, 1964; Bernier, 1967; Todorovic and Zelenhasic, 1969; Zelenhasic, 1970; Todorovic and

Woolhiser, 1972; Todorovic, 1978). In this article we shall focus on the latter method. A brief review of both methods is presented by Smith (1990).

In section 2, we introduce some notations and review briefly some known results; in section 3, we present a bivariate exponential distribution to fit the discharge exceedances and the discharge durations over a threshold; and in section 4, we model flood volume.

2 Preliminaries and literature review

Referring to Figure 1, denote by $\xi(t)$, $t \geq 0$ a hydrograph which represents the discharge rate of river flow at a given site. We select a certain base level x_0 and consider the water flows (exceedances) which exceed the base level x_0. Denote by T_1, T_2, T_3 the duration of these exceedances, that is the time between an upcrossing of the level x_0 and the subsequent downcrossing, and τ_1, τ_2, τ_3, ... the times of local maxima of water levels during these exceedances. Define also by

$$X_0 = 0 \text{ and } X_k = \xi(\tau_k) - x_0 \qquad \text{for } k = 1, 2, \cdots \tag{1}$$

the series of maximum value of exceedances; the series (X_k) is called the partial duration series. The exceedances X_k can be assumed to be independent and identically distributed or dependent and identically distributed within a homogeneous time interval such as the year or the "season" (see below). To examine the dependence of successive exceedances multivariate distribution functions might be used, see in particular Rojsberg (1987), who applied Marshall-Olkin bivariate exponential distribution to model two successive exceedances under Markovian assumption. For a brief review of multivariate extremes, refer to Smith (1990). In this paper, we suppose that the exceedances X_k are independent and identically distributed. We define also flood counts or number of exceedances within a fixed time interval $[0, t_1]$ by

$$\eta(t_1) = sup\{k \mid \tau_k \leq t_1\} \tag{2}$$

In this article the time interval will be considered exclusively one year, however, if there are significant seasonal variations in the river flow process, the time interval can with no loss of generality be taken to be one season. Hence in the sequel, we shall omit the index t_1 and write the random variable (r.v.) $\eta(t_1)$ in (2) as η. The r.v. η which is the number of annual exceedances has often been shown to follow a Poisson distribution of parameter λ, see for instance Borgman (1963); Shane and Lynn (1964); Bernier (1967)

$$Pr(\eta = k) = e^{-\lambda}\lambda^k/k! \qquad \text{for } k = 0, 1, 2, \cdots \quad \text{or } \eta \sim P(\lambda) \tag{3}$$

The r.v. exceedance X in (1) and the r.v. duration T often follow exponential distributions of parameter α and β respectively, see for instance Todorovic (1978); Cunnane (1979); North (1980); Ashkar and Rousselle (1981), that is

$$f(x) = \frac{1}{\alpha} e^{-x/\alpha} \qquad \text{for } x \geq 0 \quad or\ X \sim Exp(\alpha) \tag{4}$$

and

$$g(t) = \frac{1}{\beta} e^{-t/\beta} \qquad \text{for } t \geq 0 \quad or\ T \sim Exp(\beta) \tag{5}$$

In general, the use of the exponential distribution function for both exceedances and the durations is a crude approximation. Davison and Smith (1990), following Pickards (1975), use the generalized Pareto distribution function, whose cumulative distribution function is given by

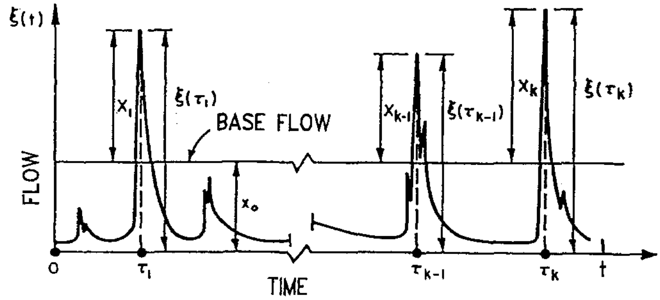

Figure 1. Hydrograph of instantaneous flow of a river at a given station

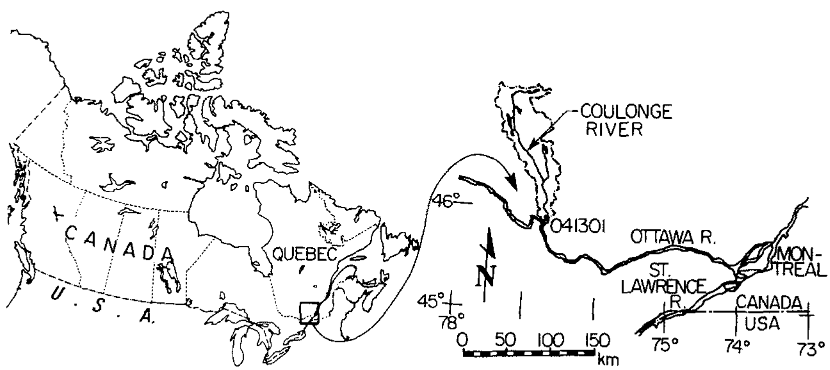

Figure 2. The geographic location of the Coulonge River. *Coulonge River Watershed*: Hydrometric station – 041301; Latitude – 45°52′26″; Longitude – 76°41′09″; Drainage basin – 5150 km^2; Data period – 1927-1980; Population density – 0.4 capita/km^2

$$G(x;\, \alpha, k) = 1 - (1 - kx/\alpha)^{k-1} \tag{6}$$

where $\alpha > 0$ and k is arbitrary; the range of x is $0 < x < \infty$ if $k \leq 0$, $0 < x < \alpha/k$ if $k > 0$. The case $k = 0$ is interpreted as the limit $k \to 0$, i.e. the exponential distribution with mean α. In this paper, crude approximations will be used to simplify the modeling of the complexe phenomena of floods.

To provide an example, we shall consider a 54 year record of the Coulange River (Quebec - Canada) at station number 41301 (Figure 2). Flood data cover the period 1927-1980.

This watershed drains from north to south and it's a part of the Ottawa river watershed. The total drainage area is 5232 km^2 and 98.4% of this area forms the drained basin of the hydrometric station. The winter is a low-flow season because winter precipitation is largely in the form of snow; and spring is a high-flow season due to the contribution of snow melt to river runoff. Data for the hydrometric station is in the form of mean daily flows. The accuracy of the data is good (less than 10 percent error), and in the period of ice effect it is fair (less than 15 percent error). The mean annual temperature in the watershed is 4.5^oC with 81 cm mean annual precipitation.

Different truncation levels were chosen corresponding to mean number of peaks per year between 0.8 and 5.3. In practice, it is often recommended (for example in the Flood Studies Report, N.E.R.C., 1975) that the threshold be chosen to fix a value between one and five peaks per year. For the truncation level x_0 = 197.60 m^3/s (which corresponds to 2.2 peaks per year), we noted that the assumption of the Poisson distribution of the random variable number of annual exceedances η is valid, that is $\eta \sim P(2.2)$. We assume

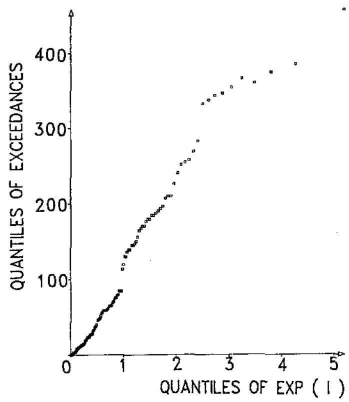

Figure 3

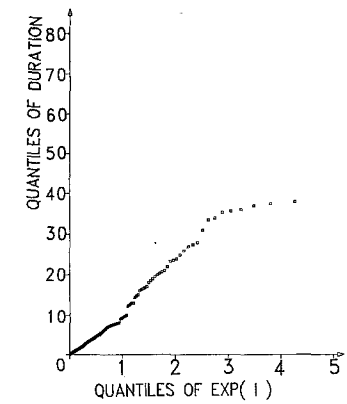

Figure 4

Figure 3. Q - Q plot of $Exp(1)$ versus exceedances when the truncation level $x_0 = 197.60$

Figure 4. Q - Q plot of $Exp(1)$ versus durations

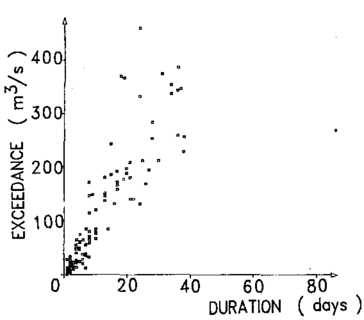

Figure 5

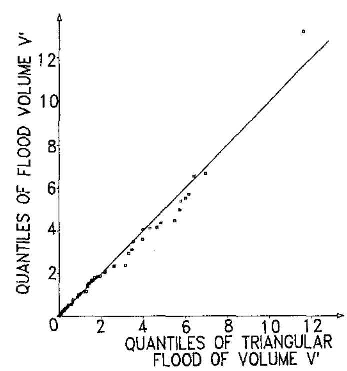

Figure 6

Figure 5. Plot of duration versus exceedance

Figure 6. Q - Q plot of the triangular flood volume V' versus observed flood volume V

also that the exceedance X in m^3/s and duration T in days follow

$$X \sim Exp(115) \quad ; \quad T \sim Exp(11.3)$$

when the truncation level $x_0 = 197.60$ m^3/s. To check whether X and T follow an exponential distribution, we applied quantile probability plots ($Q - Q$), presented in Figures 3 and 4, which were proposed by Wilk and Gnanadesikan (1968) and are commonly used in hydrology. In Figure 3 (and similarly in Figure 4) if $E \sim Exp(1)$ and $X \sim Exp(\alpha)$, then the variables E and X are related by

$$X = \alpha E \tag{7}$$

In figure 3, we note that at the value $E = 1$, there is a jump in the graph (with a change in slope). For values of $X > 100$ and except for some large values the assumption of the exponential distribution with a parameter value of 115 seems valid, while for values of $X < 100$ the assumption of the exponential distribution with a parameter value of 90 seems good; but given that we are mainly interested in large values of the exceedances, so we accept with some doubt the assumption ($X \sim Exp(115)$). Figure 4 shows that, except for some large values of the duration, the assumption of the exponential distribution ($T \sim Exp(11.3)$) seems valid. There is no doubt, that the distributions of the exceedances and the durations depend on the choice of the threshold value, but if the distributions are chosen carefully, the choice of the threshold level should not have a significant effect on the estimation of large exceedance values. The choice of the threshold value $x_0 = 197.60\ m^3/s$ in the present study was done empirically.

Very few of the numerous papers written on flood hydrology deal with volume prediction. Various models of unit hydrographs or instantaneous unit hydrographs, as well as curve fitting models, assume that the runoff volume of each individual storm is known or predictable. For example, the objective of the study done by Chiang (1975), was to develop storm runoff volume prediction equations through a process by which watershed wetness can be calibrated. Another approach requiring a direct correlation between runoff volume and its causal factors was developed also by Chiang (1975). Todorovic (1978) and Ashkar and Rousselle (1982) provide also, some remarks about the flood volumes. Todorovic models the flood volume over a threshold with the assumption of the stochastic independence of the exceedances X and durations T. Our modeling of the flood volume over a threshold supposes no such restrictive assumption. In the sequel, flood volume will mean flood volume over a threshold.

3 Bivariate exponential distribution

In this section, we study the simultaneous behaviour of the exceedance and duration (X, T) to model their joint distribution $h(x, t)$.

Figure 5 represents the plot of bivariate data (exceedance, duration). The data exhibit a linear trend with the following properties

$$S^2(X/t) \text{ and } \bar{X}/t \text{ are increasing functions of } t$$

$$S^2(T/x) \text{ and } \bar{T}/x \text{ are increasing functions of } x \quad (8)$$

where $S^2(X/t)$ and $\bar{X}/t$ are respectively the variance and the mean of the exceedances conditional upon duration.

The value of the linear coefficient of correlation between exceedance and duration is $\rho = corr(X, T) = 0.887$ calculated from the whole sample; this value is too high because it is affected by the point-swarm near the origin. We recalculate $\rho = 0.667$ after eliminating points with exceedances < 100; after all quantities are calculated to serve some purpose and in this case, it is to emphasize the large values of the exceedances and of the durations. Later on, we shall see that an optimal value of $\rho \approx 0.70$ shall be obtained by minimizing the Kolmogorov distance.

The bivariate exponential distribution of the exceedance and duration that we shall use is

$$h(x, t) = exp[-\frac{1}{1-\rho}(\frac{x}{\alpha} + \frac{t}{\beta})]\frac{1}{\alpha\beta(1-\rho)}\sum_{k\geq 0}[\frac{\rho x t}{\alpha\beta(1-\rho)^2}]^k \frac{1}{(k!)^2} \quad (9)$$

for $(x, t) \in R^+ \times R^+$ and $\alpha, \beta \in R^+$ and $corr(X, T) = \rho \in [0, 1]$.

An extensive discussion of (9) is found in Nagao and Kadoya (1971). Johnson and Kotz (1972, Vol. 4, Ch. 41, pp 260-263) review bivariate distributions for which both marginal distributions are exponentials; there are half a dozen of them. Equation (9) seems to satisfy (8). Note that the Marshall-Olkin bivariate exponential distribution satisfies also (7). It is worthwhile in future studies to compare the two bivariate exponential distributions of Marshall-Olkin and (9) for the modeling of the flood volume.

Bivariate exponential density (9) has the following characteristics:

1) $$h_1(x) = \int_0^\infty h(x, t)dt = \frac{1}{\alpha}e^{-x/\alpha} \quad \text{for } x \in R^+$$

$$h_2 = \int_0^\infty h(x, t)dx = \frac{1}{\beta}e^{-t/\beta} \quad \text{for } t \in R^+ \tag{10}$$

That is the marginal distributions of X and T are exponential; the estimated values of the parameter α and β are 115 and 11.3 respectively. We saw in Figures 3 and 4 that the exponential distribution fitted quite well the marginal series of the exceedances and the durations so Equations (10) seem to be adequate for the data.

2) $$E(T/x) = \beta(1 - \rho + \frac{\rho x}{\alpha}) \quad \text{is an increasing function of } x$$

$$E(X/t) = \alpha(1 - \rho + \frac{\rho t}{\beta}) \quad \text{is an increasing function of } t \tag{11}$$

The $E(T/x)$ and $E(X/t)$ are estimated from the data by $\bar{T}/x$ and $\bar{X}/t$ which have been shown in (8) to be increasing functions of t and x respectively. So we can assume that the data satisfy (11).

3) $$Var(T/x) = \beta^2[(1-\rho)^2 + \frac{2}{\alpha}\rho(1-\rho)x] \text{ is an increasing function of } x$$

$$Var(X/t) = \alpha^2[(1-\rho)^2 + \frac{2}{\beta}\rho(1-\rho)t] \text{ is an increasing function of } t \tag{12}$$

The $Var(T/x)$ and $Var(X/t)$ are estimated from the data by $s^2(T/x)$ and $s^2(X/t)$ which were shown in (8) to be increasing functions of t and x respectively. So we can also assume that the data satisfy (12).

4) $$h(t/x) = \frac{1}{\beta(1-\rho)}exp[-\frac{1}{(1-\rho)}(\frac{\rho x}{\alpha} + \frac{t}{\beta})]\sum_{k\geq 0}[\frac{\rho xt}{\alpha\beta(1-\rho)^2}]^k\frac{1}{(k!)^2}$$

$$h(x/t) = \frac{1}{\alpha(1-\rho)}\exp[-\frac{1}{(1-\rho)}(\frac{\rho t}{\beta} + \frac{x}{\beta})]\sum_{k\geq 0}[\frac{\rho xt}{\alpha\beta(1-\rho)^2}]^k\frac{1}{(k!)^2} \tag{13}$$

It is difficult to see if the bivariate data satisfy (13).

So on an overall basis, it seems valid to assume that the bivariate exponential distribution (9) fits the data.

4 Distribution of the flood volume

By denoting α_k the time of occurrence of the k-th upcrossing of the base level x_0 by process $\xi(t)$, then the flood volume will be equal to

$$V_k = \int_{\alpha_k}^{\alpha_k+T_k} \xi(s)ds - x_0 \cdot T_k \tag{14}$$

where as usual T_k represents the duration of the k-th exceedance. The flood volume V_k will be estimated by the triangular flood volume

$$V_k' = X_k \cdot T_k/2 \tag{15}$$

where as usual x_k represents the k-th exceedance level. Let v_k and v_k' represent the sample values of the flood volumes V_k and V_k'.

Table 1. Kolmogorov-Smirnov criterion sup e_i, expected and standard deviation of the triangular flood volume for some values of ρ, α = 115 and β = 11.3

ρ	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80
$E(v')$	907	942	975	1007	1040	1072	1105	1137	1170
$Var(v')$	1958	2045	2130	2214	2295	2376	2455	2532	2609
$Sup\ e_i$	0.1121	0.1063	0.1001	0.9416	0.8952	0.8682	0.8612	0.9025	0.1091

Figure 6 displays the Q - Q sample plot of the flood volume V and the triangular flood volume V'; it is evident that the flood volume V and the triangular volume V' almost have the same distribution; that is $Pr[V<v] \approx Pr[V'<v]$. To calculate the distribution of the triangular flood volume V', we use the bivariate distribution $h(x, t)$ of the exceedance and duration given by (9).

In the remainder of this section, we present the distribution function of the triangular flood volume V' and its moments $E(V'^n)$ for $n \in N$ (set of natural numbers). The proofs of the propositions and other related mathematical results are given in the Appendix.
Proposition 1: The distribution function of the flood volume V' in (15) using (9) is

$$G(v) = Pr[\frac{XT}{2} < v]$$

$$= 1 - (1-\rho)\sum_{k\geq 0}\frac{\rho^k}{k!}\sum_{j=0}^{k}\frac{1}{j!}[\frac{2v}{\alpha\beta(1-\rho)^2}]^j\int_0^\infty e^{-y}exp[-\frac{2v}{\alpha\beta(1-\rho)^2}\frac{1}{y}]y^{k-j}dy \tag{16}$$

which might be reexpressed as modified Bessel function.
Corollary 1: In (16), when $\rho \to 0$, that is when the r.v exceedance X and duration T are independent, then

$$G(v) = 1 - \int_0^\infty e^{-y}exp(-\frac{2v}{\alpha\beta y})dy \tag{17}$$

Proposition 2: For $n \in N$

$$E(V'^n) = E(\frac{X^nT^n}{2^n}) = \frac{\alpha^n\beta^n}{2^n}(1-\rho)^{2n+1}\sum_{k\geq 0}\left(\frac{(k+n)!}{k!}\right)^2\rho^k \tag{18}$$

$$\textit{Corollary 2:}\quad E(V') = (1+\rho)\frac{\alpha\beta}{2} \quad ; \quad Var(V') = (3\rho^2 + 14\rho + 3)\frac{\alpha^2\beta^2}{4} \tag{19}$$

We apply the above results to the Coulonge River data. Let us designate by

$$G_N(v) = \frac{i - 1/3}{N + 1/3} \qquad \text{for } v_{(i)} \leq v < v_{(i+1)},\ i = 1, ..., N \tag{20}$$

the sample or the empirical cumulative distribution function (ecdf) of the flood volume; $v_{(i)}$ are the ordered sample values of the observed flood volume data for $i = 1, ..., N$. $G(v)$ is the theoretical cumulative distribution function. Define $e_i = |G_N(v) - G(v)|$ for $v_{(i)} \leq v < v_{(i+1)}$; then the Kolmogorov-Smirnov test criterion is taken to be sup e_i (Lindgren, 1976), which are displayed in Table 1 for some values of ρ and α = 115, β = 11.3. It is to be noted that for almost all cases sup e_i is attained at the flood volume = 1476 $m^3 days/second$. The Kolmogorov-Smirnov critical values for 5%, 10%, 15% and 20% are respectively 0.1245, 0.1118, 0.1045 and 0.0981. For α = 115 m^3/s, β = 11.3 *days*, $0.50 \leq \rho \leq 0.75$, the $G(v)$ of (18) is widely acceptable. But our aim is to find an optimal

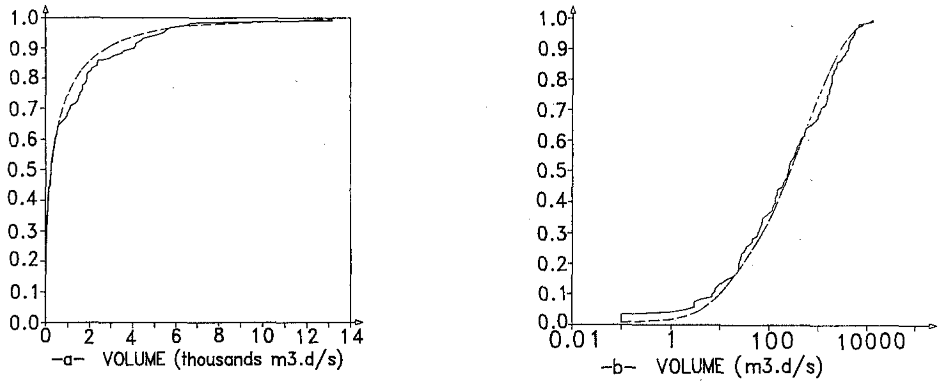

Figure 7. a) Comparison of the empirical $G_N(v)$ and theoretical $G(v)$ cumulative distribution functions of the flood volume for $\rho = 0.70$, $\beta = 11.3$ and $\alpha = 115$. In b), the same plot, but the volumes are plotted in logarithmic scale

value of ρ.

Table 1 displays also the expected value and standard deviation $\sigma(V')$ for some values of ρ calculated by applying (19). While from the sample, we get $V = \sum v_i/N = 1122.4$ m^3d/s and $s(V) = 1919.3$ m^3d/s, $V' = 1151.3$ m^3d/s and $S(V') = 1944.6$ m^3d/s, from Table 1, we deduce an optimal value of $\rho = 0.70$ when $\alpha = 115$ m^3/s and $\beta = 11.3$ days. We note that $\sigma(V')$ overestimates $s(V')$.

The optimal value of $\sigma = 0.70$, by minimizing the Kolmogorov distance given α and β, is far from the moment estimate of $\rho = 0.89$ calculated in section 3. These two values of ρ are different because the criteria of estimating them are different. Of course, other estimating procedures also can be used, such as maximium likelihood, see Nagao and Kadao (1971).

The distribution of the flood volume is highly skewed; the first quartile = 33 m^3d/s, the median = 245 m^3d/s and the third quartile 1553 m^3d/s. Figure 6 compares the empirical $G_N(v)$ and the theoretical $G(v)$ distribution functions of the flood volume for ρ = 0.70, $\beta = 11.3$ days and $\alpha = 115$ m^3/s. The plot $(v_{(i)}, G(v_i))$ in Figure 7a) emphasizes the large values of the flood volume, because of the extreme skewness for the distribution; for this reason, in the lower right hand corner, we have presented the plot $(v_{(i)}, G(v))$ where the volumes are plotted in logarithmic scale (Figure 7b)), to emphasize small values of the volume; the conjunction of the two plots makes the comparison of $G_N(v)$ and $G(v)$ interesting for all values of the flood volume.

5 Conclusion

In this article, we presented a bivariate exponential density to fit the joint distribution of the discharge exceedances and the discharge duration over a base level. We also modeled the distribution of the flood volume over a threshold using the bivariate exponential density. It was shown that many of the observed characteristics of flood exceedance and duration, as well as flood volume were well represented by the proposed model. It is also interesting to note that the results of Todorovic (1978), can be used in conjunction with the results of the present study to calculate the distribution of the maximum of the flood volumes over a threshold.

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Appendix

We need the following results for later on,

Lemma 1: for $k \in N$,

$$\int_0^{2v/t} \exp(-\frac{x}{\alpha(1-p)})x^k dx = k!\{[\alpha(1-p)]^{k+1}[1-\exp(-\frac{2v}{\alpha(1-p)t})\sum_{j=0}^{k}(\frac{2v}{\alpha(1-p)t})^j \frac{1}{j!}]\} \quad (A1)$$

Proof: integrating by parts, it is easily proved. In (A1) when $2v/t \to \infty$, then

$$\int_0^{2v/t} \exp(-\frac{x}{\alpha(1-p)})x^k dx = [\alpha(1-p)]^{k+1}k! \quad (A2)$$

Proof of Proposition 1

$$G(v) = Pr[\frac{XT}{2} < v] = Pr[XT < 2v] = \int_0^\infty (\int_0^{2v/t} h(x,t)dx)dt$$

$$= \sum_{k\geq 0} \frac{\rho^k}{\alpha\beta(1-\rho^2)]^k} \frac{1}{(k!)^2} \frac{1}{\alpha\beta(1-\rho)} \int_0^\infty \exp[-\frac{t}{\beta(1-\rho)}]t^k \{\int_0^{2v/t} \exp[-\frac{x}{\alpha(1-\rho)}]x^k dx\}dt$$

$$= \sum_{k\geq 0} \frac{\rho^k}{[\alpha\beta(1-\rho)^2]^k} \frac{1}{(k!)^2} \frac{1}{\alpha\beta(1-\rho)} \int_0^\infty \exp[-\frac{t}{\beta(1-\rho)}]t^k$$

$$\{[\alpha(1-\rho)]^{k+1}k![1-exp(-\frac{2v}{\alpha(1-\rho)t})\sum_{j=0}^{k}\frac{1}{j!}(\frac{2v}{\alpha(1-\rho)t})^j]\}dt \qquad \text{(by using (A1))}$$

$$= \sum_{k\geq 0} \frac{\rho^k}{\beta^{k+1}} \frac{1}{(1-\rho)^k} \frac{1}{k!} [\int_0^\infty \exp(-\frac{t}{\beta(1-\rho)})t^k \{1 - exp(-\frac{2v}{\alpha(1-\rho)t})\sum_{j=0}^{k}\frac{1}{j!}(\frac{2v}{\alpha(1-\rho)t})^j\}dt]$$

$$= \sum_{k\geq 0} \frac{\rho^k}{\beta^{k+1}} \frac{1}{(1-\rho)^k} \frac{1}{k!} [[\beta(1-\rho)]^{k+1}k! - \int_0^\infty \exp(-\frac{t}{\beta(1-\rho)})t^k exp(-\frac{2v}{\alpha(1-\rho)t})\sum_{j=0}^{k}\frac{1}{j!}(\frac{2v}{\alpha(1-\rho)t})^j dt]$$

and finally by changing of variable $t/(\beta(1-\rho)) = y$, we get (16).

Proof of corollary 1

Using the fact that, when $k = 0$ $\lim_{\rho\to 0}\rho^k = 1$ and when $k > 0$ $\lim_{\rho\to 0}\rho^k = 0$, (17) follows easily from (16).

Proof of proposition 2

$$E(V^n) = E(\frac{X^nT^n}{2^n}) = \sum_{k\geq 0}(\frac{\rho}{\alpha\beta(1-\rho)^2})^k \frac{1}{(k!)^2} \frac{1}{2^n\alpha\beta(1-\rho)} \int_0^\infty exp(-\frac{x}{\alpha(1-\rho)})x^{k+n}dx \int_0^\infty \exp(-\frac{t}{\beta(1-\rho)})t^{k+n}dt$$

$$= \sum_{k\geq 0}(\frac{\rho}{\alpha\beta(1-\rho)^2})^k \frac{1}{(k!)^2} \frac{1}{2^n\alpha\beta(1-\rho)} [(1-\rho)\alpha]^{k+n+1}[(1-\rho)\beta]^{k+n+1}((k+n)!)^2$$

using (A2) and after some simplification (18) is obtained. Let us denote by, for $n \in N$,

$$I_n(\rho) = \sum_{k\geq 0}(\frac{(k+n)!}{k!})^2\rho^k \quad (A3)$$

It follows that $I_0(\rho) = \frac{1}{1-\rho}$. The next lemma gives a recurrence formula to calculate $I_n(\rho)$ as a function of partial derivatives of $I_{n-1}(\rho)$.

Lemma 2: For $n \in N$ and $n \geq 1$

$$I_n(\rho) = \frac{\partial I_{n-1}(\rho)}{\partial\rho} + \rho\frac{\partial^2 I_{n-1}(\rho)}{\partial\rho^2} \quad (A4)$$

proof: By induction.

Proof of corollary 2: by (A2).

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