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Magnetofluiddynamic flow with a pressure gradient and fluid injection

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SUMMARY
The nonlinear partial differential equation of motion for an incompressible fluid flowing over a flat plate under the influence of a magnetic field and a pressure gradient, and with or without fluid injection (or ejection) through the plate is transformed to a nonlinear, third order ordinary differential equation by using a stream function and a similarity transformation.

The necessary boundary conditions are developed for flow with and without fluid injection (or ejection), and an example is presented to illustrate the solution to the flow problem.

The controlling equation reduces to the well known Falkner–Skan equation when the magnetic field is zero, and if additionally the pressure gradient is zero, the equation reduces to the Blasius equation.

1. Introduction

The current world energy shortage has spurred interest in unconventional ways to generate electrical power. Indeed magnetofluiddynamic power generation has already commenced in Russia [1]. Interest in magnetofluiddynamic flow began in 1918, when Hartmann invented the electromagnetic pump [2].

The first published papers treating the flow of electrically conducting fluid were by Hartmann [3], and Hartmann and Lazarus [4], in 1937. Since then a large body of literature has developed, in which references [5] through [8], for flow through a channel are typical. References [9] and [10] treat the subject from the textbook point of view.

With the exception of linear problems, there are very few exact problems solved in this literature. One difficulty is, to quote from Rossow [11], "It is then to be expected that in flow over a semi-infinite plate in the presence of a magnetic field, a similar solution will not exist." Lykoudis [12] dealing with a compressible fluid, including disassociation effects, showed that similarity solutions do exist for power law flow from a stagnation point.

This paper develops a unique similarity differential equation for incompressible flow over a semi-infinite flat plate in the presence of a magnetic field and a pressure gradient with or without fluid injection or ejection.

2. Theory

The motion equation for an incompressible fluid flowing over a semi-infinite flat plate, see Figure 1, under the influence of a magnetic field, and a pressure gradient is

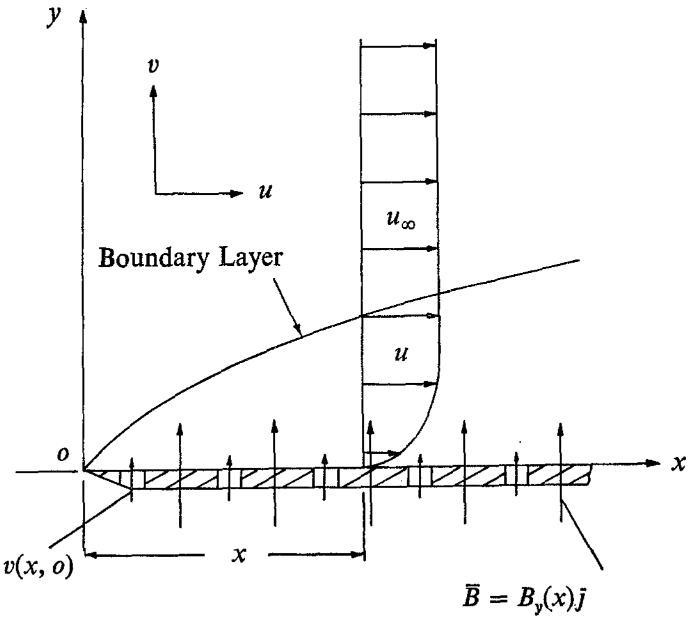

Figure 1. Magnetofluiddynamic boundary layer with fluid injection.

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{dy}=\frac{-g}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}-\frac{g\sigma B_y^2(x)u}{\rho}, \tag{1}$$

and the continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

where

u = fluid velocity in the x-direction,
v = fluid velocity in the y-direction,
g = acceleration of gravity,
ρ = fluid density,
P = pressure,
ν = kinematic viscosity,
σ = electrical conductivity,
B_y = magnetic field strength.

If we define a similarity variable, η, as

$$\eta = ay/x^\alpha = \eta(x, y) \tag{3}$$

where

$$a=\sqrt{\frac{\beta U_0}{\nu}}, \tag{4}$$

$$\alpha = (1 - m)/2, \quad (\alpha > 0, \text{ for uniqueness of } \eta) \tag{5}$$

$$\beta = (1 + m)/2, \quad (\beta \neq 0) \tag{6}$$

and

$$\lim_{y \to \infty} u(l, y) = U_\infty = U_0 l^m, \qquad (l > 0). \tag{7}$$

Also, defining a stream function, ψ, as

$$\psi = bx^\beta f(n) = G(x) f(n) \tag{8}$$

Where

$$b = \sqrt{\frac{\nu U_0}{\beta}} \tag{9}$$

and where

$$\psi_x = \left(\frac{\partial \psi}{\partial x}\right)_y = -v \tag{10}$$

and

$$\psi_y = \left(\frac{\partial \psi}{\partial y}\right)_x = u \tag{11}$$

then, continuity is automatically satisfied, and the motion equation becomes

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = H(x) + \nu \psi_{yyy} - S(x) \psi_y \tag{12}$$

where

$$H(x) = \frac{-g}{\rho} \frac{\partial P}{\partial x} \tag{13}$$

and

$$S(x) = \frac{g \sigma B_y^2(x)}{\rho}. \tag{14}$$

It can be shown that

$$\psi_x = \left(\frac{\partial \psi}{\partial G}\right)_f G'(x) + \left(\frac{\partial \psi}{\partial f}\right)_G f'(\eta) \left(\frac{\partial \eta}{\partial x}\right)_y \tag{15}$$

and

$$\psi_y = \left(\frac{\partial \psi}{\partial f}\right)_G f'(\eta) \left(\frac{\partial \eta}{\partial y}\right)_x \tag{16}$$

where

$$\left(\frac{\partial \psi}{\partial G}\right)_f = f(\eta) \tag{17}$$

$$\left(\frac{\partial\psi}{\partial f}\right)_G = G(x) \tag{18}$$

and

$$\left(\frac{\partial\eta}{\partial x}\right)_y = \frac{-\alpha\eta}{x} \tag{19}$$

$$\left(\frac{\partial\eta}{\partial y}\right)_x = \frac{a}{x^\alpha} \tag{20}$$

Thus

$$\psi_x = bx^{\beta-1}[\beta f(\eta) - \alpha\eta f'(\eta)] = -v \tag{21}$$

and

$$\psi_y = abx^{\beta-\alpha}f'(\eta) = u. \tag{22}$$

Further substitution in Eq. (12), gives

$$a^2b^2x^{2(\beta-\alpha)-1}[(\beta-\alpha)f'(\eta)^2 - \beta f(\eta)f''(n)]$$
$$= H(x) + a^3bvx^{\beta-3\alpha}f'''(\eta) - abx^{\beta-\alpha}S(x)f'(\eta). \tag{23}$$

Now if

$$H(x) = H_0x^\gamma \tag{24}$$

$$S(x) = S_0x^\delta \tag{25}$$

then, to permit a similarity problem, it follows that

$$\gamma = 2m - 1 \tag{26}$$

and

$$\delta = m - 1 \tag{27}$$

and so, Eq. (23) can be written as

$$f'''(\eta) + f(\eta)f''(\eta) - \bar{\beta}f'(\eta)^2 + H_1 - N_m f'(\eta) = 0 \tag{28}$$

where

$$H_1 = H_0/\beta U_0^2 \tag{29}$$

$$\bar{\beta} = 2m/(m+1) \tag{30}$$

and where we define a dimensionless field strength number, N_m,

$$N_m = \frac{S_0}{\beta U_0}. \tag{31}$$

It is necessary to determine three boundary conditions for Eq. (28), further it is desirable to put Eq. (28) into a more convenient form for computing.

Using the condition that

$$\lim_{y\to\infty} u(x, y) = K(x) = U_0 x^m = U_\infty \left(\frac{x}{l}\right)^m = \lim_{\eta\to\infty} U_0 x^m f'(\eta) \tag{32}$$

so that

$$\lim_{y\to\infty} u(l, y) = K(l) = U_0 l^m = U_\infty = \lim_{\eta\to\infty} U_0 l^m f'(\eta) \tag{33}$$

then, it must be that

$$\lim_{\eta\to\infty} f'(\eta) = 1. \tag{34}$$

Utilizing Eq. (34), it can be shown that

$$H_1 = \bar{\beta} + N_m. \tag{35}$$

To find the other two necessary boundary conditions, we assume at the wall, ($y = 0$), there is no slippage for u, so that

$$u(x, 0) = 0 = \lim_{\eta\to 0} U_0 x^m f'(\eta) \tag{36}$$

and from Eq. (21), we see that

$$\begin{aligned} v(x, 0) &= \lim_{\eta\to 0} \{-b x^{\beta-1}[\beta f(\eta) - \alpha\eta f'(\eta)]\} = F(x) \\ &= v_0 x^\zeta. \end{aligned} \tag{37}$$

Thus, it must be that

$$f(0) = -v_0/b\beta \tag{38}$$

and

$$f'(0) = 0 \tag{39}$$

and

$$F(x) = v_0 x^{(m-1)/2} \tag{40}$$

where

v_0 = magnitude of velocity coefficient for injection, ($v_0 > 0$), or ejection, ($v_0 < 0$), of fluid through the wall.

If $v_0 \equiv 0$, then

$$f(0) = 0 \tag{41}$$

and there is no fluid passing through the plate.

Additionally, the following must hold:

$$P(x) = P_0 - \frac{\rho H_0 x^{2m}}{2mg}, \quad (m \neq 0) \tag{42}$$

where

$P_0 = a$ constant

and

$$S(x) = S_0 x^{m-1} = \frac{g\sigma B_{y0}^2 x^{m-1}}{\rho} \tag{43}$$

$$B_y(x) = B_{y0} x^{(m-1)/2}. \tag{44}$$

Lykoudis [13] developed an expression equivalent to Eq. (44). Additionally

$$N_m = \frac{g\sigma B_{y0}^2}{\rho\beta U_0}. \tag{45}$$

Using Eq. (35) in Eq. (28), we obtain

$$f'''(\eta) + f(\eta)f''(\eta) + \bar{\beta}[1 - f'(\eta)^2] + N_m[1 - f'(\eta)] = 0. \tag{46}$$

Boundary conditions, $f(\eta)$:

1. $f(0) = -v_0/b\beta$
2. $f'(0) = 0$
3. $\lim_{\eta \to \infty} f'(\eta) = 1$

Eq. (46) is the general similarity differential equation controlling the effects of magnetic field, pressure gradient and fluid injection, or ejection, through the wall. When the magnetic field is zero, the equation reduces to the well known Falkner–Skan equation. If both the magnetic field and pressure gradient are zero, the equation reduces to the Blasius equation.

For convenience in presenting results, we can write

$$\frac{u}{U_\infty}\left(\frac{l}{x}\right)^m = f'(\eta) \tag{47}$$

and

$$\frac{v}{U_\infty}\left(\frac{Re_x}{\beta}\right)^{\frac{1}{2}}\left(\frac{l}{x}\right)^{m/2} = -f(\eta) + (1 - m)\eta f'(\eta)/(1 + m) \tag{48}$$

where the Reynolds number, Re_x, is

$$Re_x = \frac{U_\infty x}{\nu}. \tag{49}$$

Example

Given:

$$f'''(\eta) + f(\eta)f''(\eta) + \bar{\beta}[1 - f'(\eta)^2] + N_m[1 - f'(\eta)] = 0. \tag{46}$$

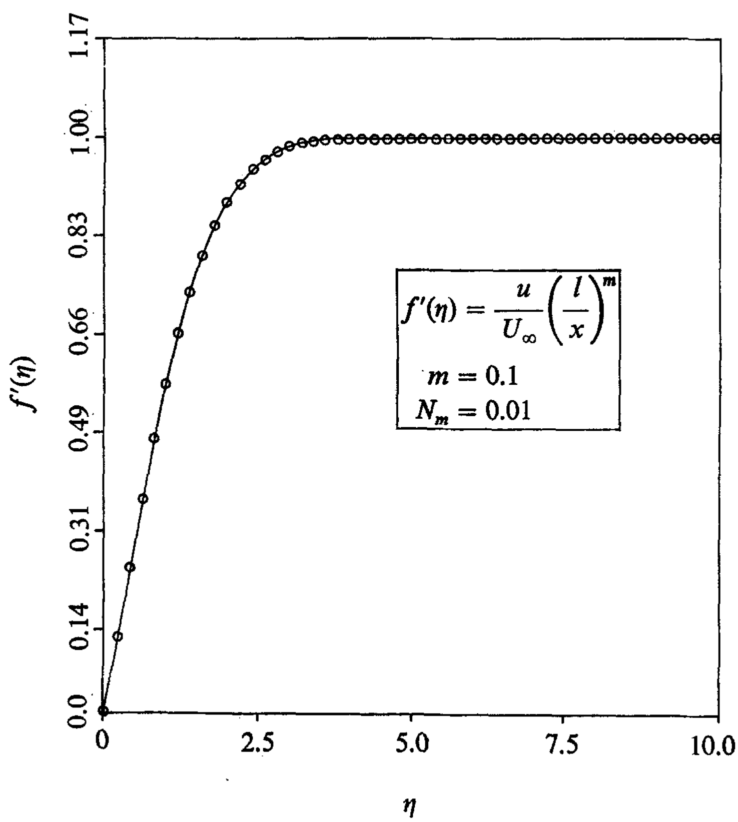

Figure 2. Magnetofluiddynamic flow with a pressure gradient.

Boundary conditions, $f(\eta)$:

1. $f(0) = 0$, (no fluid injection or ejection)
2. $f'(0) = 0$
3. $\lim_{\eta \to \infty} f'(\eta) = 1$

Assume:

$m = 0.1, \; N_m = 0.01.$

Figure 2 shows a plot of $u(l/x)^m/U_\infty = f'(\eta)$ vs $\eta = ay/x^\alpha$ for $0 \leqq \eta \leqq 10$.

Conclusion

Using a stream function ψ, which satisfies continuity, and a similarity variable, η, the nonlinear partial differential equation of motion for an incompressible fluid flowing over a flat plate, under the influence of a magnetic field, and a pressure gradient, with or without fluid injection or ejection, is transformed to a nonlinear, third order, ordinary differential equation. The necessary boundary conditions have been established from a physical basis.

The derived controlling differential equation includes the classical Falkner–Skan equation and the Blasius equation as special cases.

An example is solved numerically and $f'(\eta)$ is shown plotted against the similarity variable η.

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