Mh. Math. 96, 157—164 (1983)

Monatshefte für
Mathematik

Preassigning the Shape for Bodies of Constant Width

By

Egon Schulte, Dortmund, and **Siniša Vrećica,** Belgrade

With 1 Figure

(Received 7 March 1983; in revised form 25 April 1983)

Abstract. The paper deals with the problem of preassigning the shape for bodies of constant width. In particular, the free choice of boundary points for sets of constant width is discussed.

1. Introduction

Many problems concerning bodies of constant width are inspired by the famous Borsuk-conjecture stating, that every compact subset of $\mathbb{E}^d$ is the union of at most $d+1$ sets of smaller diameter (cf. [2]). It is well-known that every compact subset C of diameter c is contained in a convex body K of constant width c (cf. [1]), and so it is sufficient to prove the conjecture for sets of constant width.

Indeed, much more information on the properties of K is available, namely, there exists a body K with no additional singularities to those of C. This has independently been proved in [3] and [5], thereby answering a former question of Danzer and Grünbaum (cf. [4]). In this paper we shall refer to the results of [5], since they also include statements on certain symmetry properties of K.

Before posing the problems of this paper and stating the main results we introduce some notations. Throughout the whole paper the underlying space will be the Euclidean space $\mathbb{E}^d$. For two different points x and y in $\mathbb{E}^d$ let $\overline{xy}$, $\overrightarrow{xy}$ and $[xy]$ denote the line through x and y, the ray issuing from x and passing through y and the segment joining x and y, respectively. For x in $\mathbb{E}^d$ and $r>0$ let $B(x,r)$ be the (closed) Euclidean ball with center x and radius r. For each boundary point x of a compact set C of diameter $c\,(>0)$ we define $S(x,C):=$ $:= C \cap \operatorname{bd} B(x,c)$, and in case $S(x,C) \neq \emptyset$ also

$$\operatorname{Sing}(x,C) := \operatorname{conv}\Big(\bigcup_{y\in S(x,C)} \overrightarrow{xy}\Big) \cap \operatorname{bd} B(x,c)\,.$$

Note that each body K of constant width c containing C also contains the sets $\text{Sing}(x, C)$. If x and y are in C and $|x - y| = c$, we call the segment joining x and y a *c-chord of C with ends x and y*. As usual we call a boundary point x of a convex body *regular*, if and only if the body is supported at x by exactly one hyperplane. Otherwise, x is a *singular* boundary point. Note that the word "singular" has been used in a slightly different sense in [5].

The following theorem was proved in [5].

Theorem 1: *Let C be a compact, convex subset of $\mathbb{E}^d$ of diameter $c\,(> 0)$. Then C can be embedded into a convex body K of constant width c with the following properties.*

(a) *Each point in $C \cap \text{bd}\, K$ is an end of a c-chord of C.*

(b) *The symmetry group of C is contained in that of K.*

(c) *Each singular boundary point x of K is a singular boundary point of C (relative to the affine hull* $\text{aff}(C)$ of *C), for which the set of antipodes in K is the spherical convex hull of the antipodes in C, that is* $S(x, K) = \text{Sing}(x, C)$.

The present paper deals with a problem raised by D. G. LARMAN at the Oberwolfach symposium on convexity in July 1982. It concerns the free choice of boundary points for sets of constant width. Therefore, it fits in the more general problem of preassigning the shape for bodies of constant width. The exact formulation of the general problem as well as its solution are contained in the next section.

2. Preassigning the Boundary

Given a convex body L of diameter 2 and a compact subset C of L with diameter c, $c < 2$, we may ask whether there exists a convex body K of constant width c such that $C \subset K \subset L$ and $C \cap \text{bd}\, L = K \cap \text{bd}\, L$. The latter property would of course imply that we can arbitrarily preassign any compact subset of $\text{bd}\, L$ with diameter c for the set of contact points of $\text{bd}\, L$ with a suitable body K of constant width c contained in L. Since we wish to prove a result for all c sufficiently close to 2, the body L has to be of constant width 2, which we assume from now on. An affirmative answer to the problem for the particular case L is the circumscribed ball of C has been given in [8] (compare also [6] and [7]). We shall generalize the result to a larger class of sets L and C, also requiring additional properties for K such as symmetry and regularity properties. In particular we shall prove the following theorem and its corollary.

Theorem 2: *Let L be a body of constant width 2. Assume that a ball of radius $r > 0$ slides freely in L, that is, for each x in* bd L *there exists a ball B of radius r such that $x \in B \subset L$. Let $(1 \leqslant) 2 - r < c < 2$ and C be a compact, convex subset of L with diameter c. Then there exists a body K of constant width c such that*

(a) *$C \subset K \subset L$ and $K \cap \operatorname{bd} L = C \cap \operatorname{bd} L$.*

(b) *Each point in $(C \cap \operatorname{bd} K) \setminus \operatorname{bd} L$ is at the end of a c-chord of C.*

(c) *Each common symmetry of C and L is a symmetry of K.*

(d) *Each singular boundary point x of K is a singular boundary point of C (relative to* $\operatorname{aff}(C)$*) and, if $x \in K \setminus \operatorname{bd} L$, then $S(x, K) = \operatorname{Sing}(x, C)$.*

Corollary: *Let L be as in theorem 2, $2 - r < c < 2$, and C be a compact subset of* bd L *with diameter c. Then there exists a body K of constant width c such that*

(a) *$C \subset K \subset L$ and $K \cap \operatorname{bd} L = C$.*

(b) *Each common symmetry of C and L is a symmetry of K.*

(c) *Each singular boundary point x of K lies in C and is a singular boundary point of the convex hull of C (relative to* $\operatorname{aff}(C)$*).*

A careful inspection of the proofs will show that both results also hold for $2 - r = c$, if each ball of radius r contained in L touches bd L in at most one point. But nevertheless, the results will not hold with weaker assumptions on L and c, that is, theorem 2 is in a sense best possible. In fact, we shall prove the following reverse statement to theorem 2.

Theorem 3: *Let L be a set of constant width 2 and $1 \leqslant c < 2$. Each subset C of diameter c is embeddable into a body K of constant width c contained in L (and with $K \cap \operatorname{bd} L = C \cap \operatorname{bd} L$), if and only if a ball of radius $r = 2 - c$ slides freely in L (with exactly one point of contact with* bd L*).*

The following characterization of the ball is an immediate consequence of theorem 3.

Corollary: *The unit ball is the only set L of diameter 2, for which each* *subset of diameter c, $c \geqslant 1$, is embeddable into a body K of constant width c contained in L.*

We turn to the proofs of the above-mentioned results. The assumptions on L will imply that L is a Minkowski-sum of two sets.

Lemma: *Let a ball of radius $r\,(>0)$ slide freely in the body L of constant width 2. Define $L_r := \bigcap_{x \in \operatorname{bd} L} B(x, 2-r)$. For x in $\operatorname{bd} L$ and its respective antipode x' in L let $p(x)$ denote the unique point of $[x\,x']$ at distance $2-r$ from x. Then we have*

(a) *L_r is a body of constant width $2-2r$ with boundary $\{p(x) \mid x \in \operatorname{bd} L\}$.*

(b) *$L = L_r + B(0, r)$, the Minkowski-sum of the two sets.*

(c) *If r is maximal with respect to L, then no ball of positive radius slides freely in L_r.*

(d) *If K is a body of constant width $2-r$ with $L_r \subset K$, then $K \subset L$. Furthermore, $x \in K \cap \operatorname{bd} L$ implies $p(x) \in \operatorname{bd} K$.*

Proof: If x and x' are antipodes in L, then

$$L_r \subset B(x, 2-r) \cap B(x', 2-r),$$

and so the width of L_r in the direction determined by $\overline{x\,x'}$ is at most $2-2r$. On the other hand $p(x)$ and $p(x')$ are in L_r. Indeed, by the assumptions on L, $x' \in B(p(x), r) \subset L$ and $x \in B(p(x'), r) \subset L$. Since $\operatorname{diam}(L) = 2$, this implies $|p(x) - z|$, $|p(x') - z| \leqslant 2 - r$ for each z in $\operatorname{bd} L$, hence $p(x), p(x') \in L_r$. Consequently, L_r is of constant width $2-2r$. Furthermore, if $z \in \operatorname{bd} L_r$, there is a point x in $\operatorname{bd} L$ with $|z - x| = 2 - r$, for which its respective supporting plane to L is orthogonal to $\overline{xz}$. Hence, $z = p(x)$ as required. That proves part (a).

The inclusion $L_r + B(0, r) \subset L$ is obvious. Conversely, if x and x' are antipodes in L, then $x \in B(p(x'), r)$, hence $x \in L_r + B(0, r)$ as required. Part (c) is an immediate consequence of (b), and so we are left with the proof of (d).

Now, let K be given. Since a body of constant width $2-r$ is the intersection of all balls of radius $2-r$ containing it (cf. [1]), we obviously have

$$K \subset \bigcap_{y \in \operatorname{bd} L_r} B(y, 2-r),$$

but the right hand side is just L. If $x \in \operatorname{bd} L \cap K$, the plane at x orthogonal to $\overline{xp(x)}$ supports K. Its parallel supporting plane of K at distance $2 - r$ contains $p(x)$, and so $p(x) \in \operatorname{bd} K$.

Proof of theorem 2: We shall apply theorem 1 to a suitable set C_2 of diameter c. Note that the assumptions on L imply in particular, that a ball of radius $2 - c$ slides freely in L.

Define $C_1 := \operatorname{conv}(C \cup L_{2-c})$ with L_{2-c} as in the lemma. Since $|x - y| \leqslant c$ for each x in L and y in L_{2-c}, C_1 is a convex body of diameter c. Each c-chord of C_1 has either both ends in C or one in C and the other in L_{2-c}. Also $C_1 \cap \operatorname{bd} L = C \cap \operatorname{bd} L$.

Define the non-negative functions $\sigma: L \mapsto \mathbb{R}$ and $\varrho: C_1 \mapsto \mathbb{R}$ by

$$\sigma(x) := \tfrac{1}{2} \sup \{t \mid B(x, t) \subset L\}$$

and

$$\varrho(x) := \tfrac{1}{2}(c - \sup\{|y - x| \mid y \in C_1\}) \cdot \sigma(x).$$

Then, ϱ vanishes at x, if and only if x is in $C \cap \operatorname{bd} L$ or is at the end of a c-chord in C_1. An easy calculation shows

$$\varrho(x_1) + |x_1 - x_2| + \varrho(x_2) \leqslant c \quad (x_1, x_2 \in C_1) \qquad (*)$$

with equality, if and only if $|x_1 - x_2| = c$. Let

$$C_2 := \operatorname{conv}\Big(\bigcup_{x \in C_1} B(x, \varrho(x))\Big).$$

By (*), C_2 is a convex body of diameter c, for which each c-chord has its ends in C_1, hence either both in C or one in C and the other in L_{2-c}. Let x be at the end of a c-chord of C_2 and consider $S(x, C_2)$.

Assume $x \in L_{2-c}$, hence $x = p(z)$ for some z in $\operatorname{bd} L$. By the assumptions on L, there is a ball B of radius $r > 2 - c$ such that $B(x, 2 - c) \subset B \subset L$. With respect to $\operatorname{diam}(L) = 2$, this implies that z is the only point in L with $|z - x| = c$. Consequently, as z is at the end of a c-chord of C_2, z is in $C \cap \operatorname{bd} L$. In particular, $S(x, C_2) = \{z\} = \operatorname{Sing}(x, C_2)$.

If $x \in C \cap \operatorname{int}(L)$, the above considerations show that no antipode of x in C_2 can lie in L_{2-c}. Hence $S(x, C_2) = S(x, C)$.

For x in $C \cap \operatorname{bd} L$ the set of antipodes may increase, even in such a way that $\operatorname{Sing}(x, C) \subsetneqq \operatorname{Sing}(x, C_2)$, since $p(x) \in S(x, C_2)$ but possibly $p(x) \notin \operatorname{Sing}(x, C)$. But this would of course imply that x is a singular boundary point of C.

Next we apply theorem 1 to C_2 yielding a body K of constant width c with $C_2 \subset K$ and the properties of theorem 1. Since $L_{2-c} \subset C_2$, our lemma shows $K \subset L$. Also $x \in K \cap \operatorname{bd} L$ implies $p(x) \in \operatorname{bd} K$ and, by the above considerations, $p(x)$ is at the end of a c-chord of C_2 and x is in $C \cap \operatorname{bd} L$. This proves part (a) of theorem 2.

By theorem 1 (a), each point x in $(C \cap \operatorname{bd} K) \setminus \operatorname{bd} L$ is at the end of a c-chord of C_2. As x is an interior point of L, no antipode of x in L can lie in L_{2-c}. That proves part (b) of theorem 2. Part (c) is clear from the construction.

By theorem 1 (c) each singular boundary point x of K is a singular boundary point of C_2 such that $S(x, K) = \operatorname{Sing}(x, C_2)$. Since $\operatorname{Sing}(x, C_2)$ is not a single point, this implies $x \notin L_{2-c}$, hence $x \in C$. Since $C \subset K$, x must be a singular boundary point of C. If $x \notin \operatorname{bd} L$, then $S(x, C) = S(x, C_2)$ and therefore $\operatorname{Sing}(x, C) = \operatorname{Sing}(x, C_2) = S(x, K)$ as required. But that completes the proof.

Proof of the corollary: We apply theorem 2 to $C' := \operatorname{conv}(C)$. Note that $C' \cap \operatorname{bd} L = C \cap \operatorname{bd} L$ and that each c-chord of C' has its ends in C.

Let K have the properties of theoree 2. Obviously, (a) and (b) of the corollary hold. By (b) and (d) of theorem 2, each singular boundary point x of K is one of C' and lies in $\operatorname{bd} L$, hence in C as required.

Proof of theorem 3: From theorem 2 and the above remarks we know that the regularity assumptions on L are sufficient for each subset C of diameter c to be embeddable into a suitable body K of constant width c. Now, let us assume conversely, that each subset C is contained in a body K with the required properties.

First we deduce that all boundary points of L are regular. In fact, each singular x in $\operatorname{bd} L$ has more than one antipode in L. Hence, there are two antipodes y_1 and y_2 in $\operatorname{Sing}(x, L)$ with $|y_1 - y_2| \leqslant c$ and a point z in $\operatorname{conv}(x, y_1, y_2)$ with $|y_1 - z| = |y_2 - z| = c$. By the assumptions on c, the set $C := \operatorname{conv}(z, y_1, y_2)$ is embeddable into a set K of constant width c contained in L. But this is in fact impossible, since L does not contain all arcs with radius c and ends y_1 and y_2, which are nevertheless contained in K.

Let r denote the greatest real number, for which a ball or radius r slides freely in L. The case $r = 0$ is a priori not excluded. However we shall prove that $r \geqslant 2 - c > 0$. Let us assume to the contrary that $r < 2 - c$, that is $c < 2 - r$.

By the definition of r, there is an x in $\operatorname{bd} L$, for which no ball B of radius t with $t > r$ and $x \in B \subset L$ exists. However, $x \in B(y, r) \subset L$ for a suitable y in L (if $r = 0$, then $y = x$). Denote the unique antipode of x

in L by x' (compare figure 1). Since $c < 2 - r$, there is an interior point z of $[y\,x']$ with $|z - x'| > c$. By the choice of x, the maximum r_z of the radii t with $B(z, t) \subset L$ satisfies $r_z < |x - z| < 2 - c$. By the definition of r_z, the ball $B(z, r_z)$ touches bd L in at least one point u. Let u' be its antipode in L, hence $u' \in \overrightarrow{uz}$. Then we have

$$|u' - z| = 2 - r_z > 2 - |x - z| = |z - x'|.$$

Now, two cases are possible.

If $|x' - u'| < c$, then there exist u'' in $[u'x']$ and z' in the relative interior of conv(u', x', z) such that $|z' - u''| = |z' - x'| = c$. By our assumptions on c, the set $C := \text{conv}(u'', x', z')$ of diameter c is embeddable into a body K of constant width c contained in L. However, this is impossible, since on the one hand the hyperplane in x' orthogonal to $\overline{x\,x'}$ supports L, but on the other hand its respective halfspace does not contain all arcs with radius c and ends u'' and x', belonging however to K.

Similar arguments apply for the case $|x' - u'| \geqslant c$. If we choose u'' in $[x'u']$ such that $|x' - u''| = c$ and define $C := [u''x']$, then a contradiction arises as in the former case.

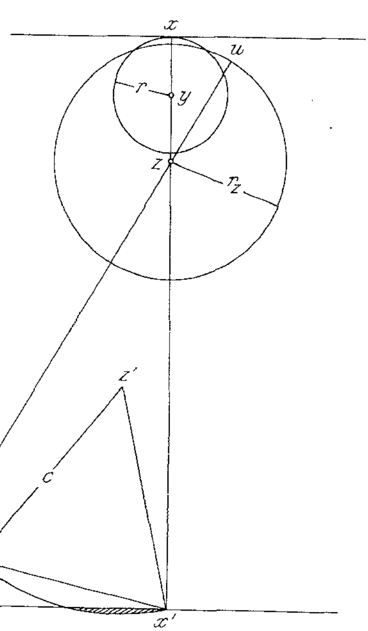

Figure 1

The considerations shows that $r \geqslant 2 - c$. Now, let us assume additionally that each subset C is even embeddable into a body K satisfying $K \cap \operatorname{bd} L = C \cap \operatorname{bd} L$. Suppose to the contrary that there is a ball $B(x, 2 - c)$ with center x and radius $2 - c$ touching $\operatorname{bd} L$ in two points y and z. Let y' and z' respectively denote their antipodes. Since L is a body of constant width 2, its boundary contains the arc s of radius c with ends y' and z' contained in $\operatorname{aff}(x, y, z)$. If we define $C := \operatorname{conv}(x, y', z')$, then each body K containing C also contains s and therefore $K \cap \operatorname{bd} L \neq C \cap \operatorname{bd} L$. So, each ball of radius $2 - c$ contained in L touches $\operatorname{bd} L$ in at most one point. That completes the proof of theorem 3.

The above-mentioned results do not take into consideration the particular shape a given set C may have. So, one might think about more individual assumptions on C and L implying the existence of a suitable body K. However, an answer to this general problem would require more powerful methods.

References

[1] BONNESEN, T., FENCHEL, W.: Theorie der konvexen Körper. Berlin: Springer. 1934.

[2] BORSUK, K.: Drei Sätze über die n-dimensionale euklidische Sphäre. Fund. Math. **20**, 177—190 (1933).

[3] FALCONER, K. J.: Singularities of sets of constant width. Geom. Dedicata **11**, 187—193 (1981).

[4] LARMAN, D. G., ROGERS, C. A.: Durham symposium on the relations between infinite-dimensional and finite-dimensional convexity. Bull. London Math. Soc. **8**, 1—33 (1976).

[5] SCHULTE, E.: Konstruktion regulärer Hüllen konstanter Breite. Mh. Math. **92**, 313—322 (1981).

[6] SCOTT, P. R.: Sets of constant width and inequalities. Quart. J. Math. Oxford (2) **32**, 345—348 (1981).

[7] VREĆICA, S.: A note on sets of constant width. Publ. Inst. Math. Beograd **29** (43), 289—291 (1981).

[8] VREĆICA, S.: Sets of constant width. Preprint.

E. SCHULTE
Mathematisches Institut
Universität Dortmund
D-4600 Dortmund
Federal Republic of Germany

S. VREĆICA
Institute of Mathematics
University of Belgrade
11000 Belgrade
Yugoslavia