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Peristaltic motion of a non-Newtonian fluid

Part II. Visco-elastic fluid

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With 2 figures and 1 table

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Introduction

The study of peristaltic transport of fluids is found to be of great importance in physiology. Particularly, the theoretical investigations on peristaltic motion by *Burns* and *Parkes* (1), *Fung* and his group (2, 3), *Shapiro* et al. (4), etc.[1]) have excited some interest in recent years. However, these studies are confined to only *Newton*ian fluids. Since many of the biological fluids including blood exhibit non-*Newton*ian behaviour [see *Whitmore* (5)], we (6)[2]) made an attempt to study the peristaltic motion of a power law fluid in a tube with a sinusoidal wave of small amplitude travelling down its wall. The solution for the stream function is obtained as a power series in terms of *Reynolds* numbers and the effect of non *Newton*ian parameter on the streamline pattern is discussed in detail.

The aim of the present note is to extend our previous analysis (6) to include the discussions about the peristaltic motion for a visco-elastic fluid taking the constitutive equation of a simple fluid with fading memory.

2. The fluid model

We choose as a model of visco-elastic fluid a simple fluid with fading memory. Employing the sequence of co-rotational kinematic tensors as described by *Giesekus* (7, 8), the constitutive equation has the form

$$T = -pI + 2\eta_0(f^{(1)} + \chi^{(2)} f^{(2)} + \chi^{(11)} f^{(1)2}), \quad [2.1]$$

where

$$f^{(1)} = \tfrac{1}{2}(\Delta v + v\Delta) \quad [2.2]$$

$$\omega = \tfrac{1}{2}(\Delta v - v\Delta) \quad [2.3]$$

$$f^{(2)} = \frac{\partial f^{(1)}}{\partial t} + v \cdot \Delta f^{(1)} + \omega \cdot f^{(1)} - f^{(1)} \cdot \omega, \quad [2.4]$$

mean respectively the usual rate of deformation tensor, the vorticity tensor and the co-rotational kinematics tensor; η_0 representing the viscosity parameter, $\chi^{(2)}$ and $\chi^{(1)}$ having dimension of time characterising the elasticity of the fluid.

3. Formulation of the problem

Consider the axisymmetric flow of a visco-elastic fluid characterised by [2.1] to [2.4] in a circular cylindrical tube with a sinusoidal wave of small amplitude travelling down its wall. The wall of the tube is taken as

$$R = a\left\{1 + \varepsilon \cos \frac{2\pi}{\lambda}(Z - c\tau)\right\}, \quad [3.1]$$

where a is the radius of the undisturbed tube, $a\varepsilon$ is the amplitude of the wave, λ is the wave length, c is the wave speed, R and Z are the cylindrical polar coordinates with Z-axis measured along the axis of the tube, R is the radius in the radial direction, τ is the time.

The equations of motion are,

$$\frac{\partial U}{\partial R} + \frac{U}{R} + \frac{\partial W}{\partial Z} = 0, \quad [3.2]$$

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial R} + W\frac{\partial U}{\partial Z} = \frac{\partial T_{RR}}{\partial R} + \frac{\partial}{\partial Z} T_{ZZ} + \frac{T_{RR}}{R}, \quad [3.3]$$

[1]) A detailed literature about the peristaltic motion is presented in the first part of this paper.

[2]) (6) is referred as Part I in sequel.

$$\frac{\partial W}{\partial \tau} + U \frac{\partial W}{\partial R} + W \frac{\partial W}{\partial Z} = \frac{\partial T_{RZ}}{\partial R} + \frac{\partial}{\partial Z} T_{ZZ} + \frac{T_{RZ}}{R}, \qquad [3.4]$$

where $T_{RR} \ldots$ are the stress components given by [2.1] to [2.4] (U, W) are the velocity components in the direction of R and Z respectively.

The above equations are rendered dimensionless by introducing the dimensionless variables r, z, t, u, w with the help of characteristic length a, characteristic velocity c.

The boundary conditions in terms of the dimensionless variables are

$$\left.\begin{aligned} u &= \varepsilon \alpha \sin \alpha (z - t) \\ w &= 0 \end{aligned}\right\} \text{ on } \quad r = 1 + \varepsilon \cos \alpha (z - t), \qquad [3.5]$$

where $\alpha = \dfrac{2\pi a}{\lambda}$.

Introducing the stream function Ψ in the form

$$\left.\begin{aligned} u &= -\frac{1}{r} \frac{\partial \Psi}{\partial z}, \\ w &= \frac{1}{r} \frac{\partial \Psi}{\partial r} \end{aligned}\right\} \qquad [3.6]$$

the equation of continuity can be identically satisfied.

4. Solution for the stream function

Assuming the amplitude of the wave ε to be small, the solution for Ψ is taken as a power series in terms of ε in the form

$$\Psi = \Psi_0 + \varepsilon \Psi_1 + \varepsilon^2 \Psi_2 + \ldots$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots \qquad [4.1]$$

Substituting [4.1] along with [3.6] in [3.3] and [3.4], collecting the coefficients of various powers of ε, we obtain the differential equations for $\Psi_0, \Psi_1, \ldots$

In the absence of peristaltic wave along the boundary of the tube, we obtain the usual *Poiseuille* flow of a viscoelastic fluid in a cylindrical tube of circular cross section. As the axial velocity in this case is only a function of r, we find that Ψ_0 is a function of r only and $\dfrac{\partial \Psi_0}{\partial z} = \dfrac{\partial \Psi_0}{\partial t} = 0$.

The solution for Ψ_0 is given by

$$\Psi_0(r) = k_0 \left(\frac{r^4}{2} - r^2\right), \qquad [4.2]$$

where $k_0 = \dfrac{Re}{8} \dfrac{\partial p_0}{\partial z}$, $\dfrac{\partial p_0}{\partial z}$ being the constant axial pressure gradient.

It is observed that the stream function for the viscoelastic fluid in the absence of peristaltic wave coincides with that of the *Newton*ian fluid. Such situations, namely the primary flow of the non-*Newton*ian fluid coinciding with that of *Newton*ian fluid, do occur in many of the secondary flows of non-*Newton*ian fluids [see *Bhatnagar* (9)]. On the other hand, the primary flow in the case of power law fluids does not coincide with that of *Newton*ian fluid [see (7)].

Similarly, the differential equation for Ψ_1 is obtained by substituting [4.1] along with [3.6] in [3.3] and [3.4], equating the coefficients of ε and eliminating p_1. Thus the equation for Ψ_1 is

$$\left[\left(\frac{1}{Re} + K L\right) D^2 - L\right] D^2 \Psi_1 + K$$
$$\times \left[\frac{1}{r^2} L \left(3 D^2 \Psi_1 - 5 \frac{\partial^2 \Psi_1}{\partial z^2} - \frac{\partial^3 \Psi_1}{\partial r^3}\right)\right.$$
$$+ 4 k_0 \frac{\partial}{\partial z} \left\{4 \frac{\partial^2 \Psi_1}{\partial z^2} + \frac{2}{r} \left(\frac{\partial \Psi_1}{\partial r} - 2 \frac{\Psi_1}{r}\right) - 3 D^2 \Psi_1\right\}$$
$$\left. + \frac{1}{r} w_0 \frac{\partial^4 \Psi_1}{\partial r \partial z^3}\right] - 4 k_0 S r \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \Psi_1}{\partial z}\right) = 0, \qquad [4.3]$$

where

$$D^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{r} \frac{\partial}{\partial r}\right), \qquad [4.4]$$

$$L = \left(\frac{\partial}{\partial t} + w_0 \frac{\partial}{\partial z}\right), \qquad [4.5]$$

$Re = \dfrac{a c \varrho}{\eta_0}$ – *Reynolds* number,

$$K = \frac{\eta_0 \chi^{(2)}}{\varrho a^2}, \qquad S = \frac{\eta_0 \chi^{(11)}}{\varrho a^2}, \qquad [4.6]$$

and w_0 being the axial velocity in the absence of peristaltic wave.

The boundary conditions on Ψ_1 are

$$\left.\begin{aligned} \frac{\partial \Psi_1}{\partial r} &= -4 k_0 \cos \alpha (z - t) \\ \frac{\partial \Psi_1}{\partial z} &= -\alpha \sin \alpha (z - t) \end{aligned}\right\} \text{ at } r = 1. \qquad [4.7]$$

The differential eq. [4.3] suggests that the solution for Ψ_1 can be taken in the form

$$\Psi_1(r, z) = F(r) \cos \alpha (z - t) + G(r) \sin \alpha (z - t), \qquad [4.8]$$

where $F(r)$ and $G(r)$ have to be determined from [4.3] and [4.7]. Now, substituting for Ψ_1 from [4.8] in [4.3] and collecting the coefficients of $\cos \alpha (z - t)$ and $\sin \alpha (z - t)$, we get the following equations for $F(r)$ and $G(r)$:

$$(L_1 - \alpha^2)^2 \left(\frac{F}{r}\right) = - Re\, \alpha L_2 \left(\frac{G}{r}\right), \qquad [4.9]$$

$$(L_1 - \alpha^2)^2 \left(\frac{G}{r}\right) = - Re\, \alpha L_2 \left(\frac{F}{r}\right), \qquad [4.10]$$

where

$$L_1 \equiv \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right), \qquad [4.11]$$

$$L_2 \equiv \{2k_0(1-r^2)+1\}(L_1-\alpha^2) - K\Big[2k_0(1-r^2)$$
$$\times \left\{\left(\frac{d^2}{dr^2} - \frac{6}{r^2} - \alpha^2\right)(L_1 - \alpha^2) - 2\alpha^2\left(\frac{1}{r}\frac{d}{dr} + \frac{3}{r^2}\right)\right\}$$
$$+ 4k_0\left\{\frac{3}{r^2}(L_1-\alpha^2) - 2\left(\frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) + 2\frac{\alpha^2}{r^2}\right\}$$
$$- \left\{(L_1-\alpha^2)^2 + \frac{\partial}{\partial r}\left(\frac{1}{r}(L_1-\alpha^2)\right) + 2\alpha^2\left(\frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)\right\}\Big]$$
$$- k_0 S\left(\frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right). \qquad [4.12]$$

The boundary conditions on $F(r)$ and $G(r)$ are given by

$$\left.\begin{aligned} F(1) = 1,\ F'(1) = -4k_0, \\ G(1) = 0,\ G'(1) = 0 \end{aligned}\right\} \qquad [4.13]$$

and

$$\left.\begin{aligned} F(0) = F'(0) = 0, \\ G(0) = G'(0) = 0. \end{aligned}\right\} \qquad [4.14]$$

At it is very clearly seen that the differential equations for $F(r)$ and $G(r)$ are very complicated and it is not possible to get a closed form solution even in *Newton*ian case. Therefore, following *Chow* (10), we introduce further simplification that the *Reynolds* number is very small so that $F(r)$ and $G(r)$ can be represented as power series in terms of *Re*. Thus we take

$$\left.\begin{aligned} F(r) = F_0(r) + Re^2 F_2(r) + \ldots \\ G(r) = Re\, G_1(r) + 0(Re^3). \end{aligned}\right\} \qquad [4.15]$$

Substituting [4.15] in [4.9] and [4.10] and separating the various order terms we get the following differential equations for F_0, $G_1, \ldots$

$$(L_1 - \alpha^2)^2\left(\frac{F_0}{r}\right) = 0, \qquad [4.16]$$

$$(L_1 - \alpha^2)^2 \frac{G_1}{r} = - k_0 \alpha L_2\left(\frac{F_0}{r}\right), \qquad [4.17]$$

$$(L_1 - \alpha^2)^2 \frac{F_2}{r} = - \alpha L_2\left(\frac{G_1}{r}\right). \qquad [4.18]$$

The corresponding boundary conditions on $F_0, F_1, \ldots$ can be deduced using [4.13] and [4.14].

We notice that [4.16] coincides with the equation given by *Chow* (10) for the *Newton*ian fluids, the solution of which can be obtained in closed form in terms of the modified *Bessel* functions. The solution for the further functions F_1, $G_1, \ldots$ in our analysis are solved numerically. The important feature of these two point boundary value problems is that they have regular singular points at the origin. The *Runge-Kutta-Gill* integration procedure is employed and the values of these functions are obtained on IBM 360/44 computer, and some of the numerical values are presented in the table. Since the numerical values of higher order terms are relatively small, they have not been recorded here.

Table 1

r	$k_0 = 0$, $K = -0.2$, $S = 0.6$	$k_0 = 0.05$, $K = -0.5$, $S = 0.6$	$k_0 = 1.00$, $K = -0.2$, $S = 0.6$
0	0	0	0
	0	0	0
0.2	0.07615	0.07979	0.14803
	−0.00023	−0.00025	−0.00206
0.4	0.28633	0.29898	0.53855
	−0.00074	−0.00083	−0.00680
0.8	0.86376	0.88616	1.31151
	−0.00078	−0.00087	−0.00703
1.0	1.00	1.00	1.00
	0.0	0.0	0.0

Table giving the values of F_0 and G_1.
(The first and the second values in each column are F_0 and G_1 respectively.)

5. Discussion

The velocity components can be evaluated using the stream function. As in our previous analysis (6), here also we notice many interesting features in the streamline pattern as we change the pressure gradient. The streamlines for the low pressure gradient are depicted in fig. 1, which shows that the streamline form closed loops with $z(= z - t)$ axis as a $\Psi = 0$ (separating) streamline, and the further $\Psi = 0$ lines run approximately perpendicular to the z-axis. This type of behaviour was observed in Part I for power law fluids. From the numerical calculations we find that for a relatively high pressure gradient, the streamline pattern changes completely. In other words, the streamlines

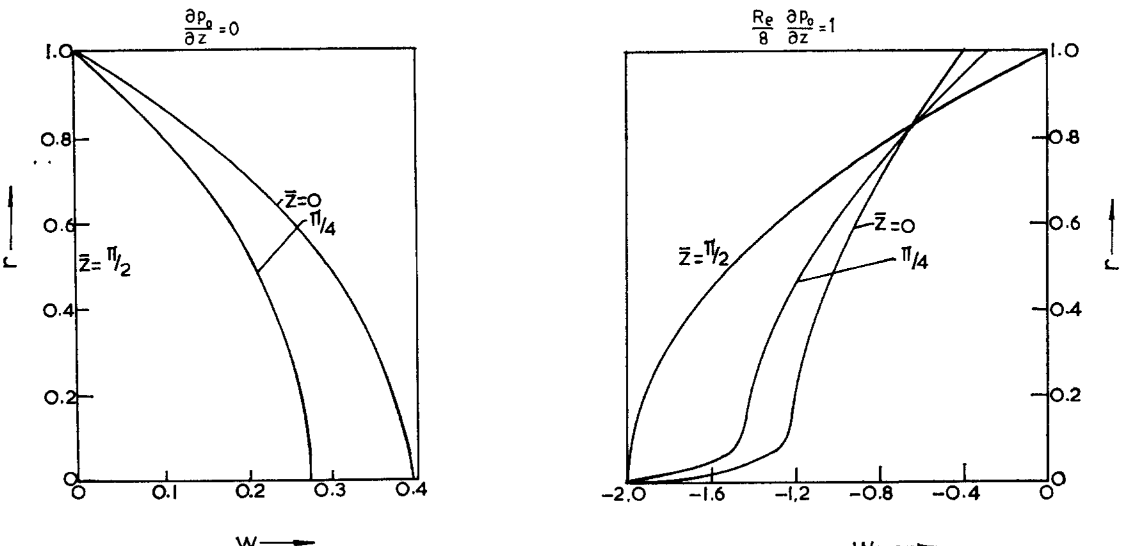

Fig. 1. Streamlines for the visco-elastic fluids

(a) $\left(\frac{Re}{8}\frac{\partial p_0}{\partial z} = 0.05,\ \varepsilon = 0.05,\ K = -0.5,\ S = 0.6\right)$ (b)

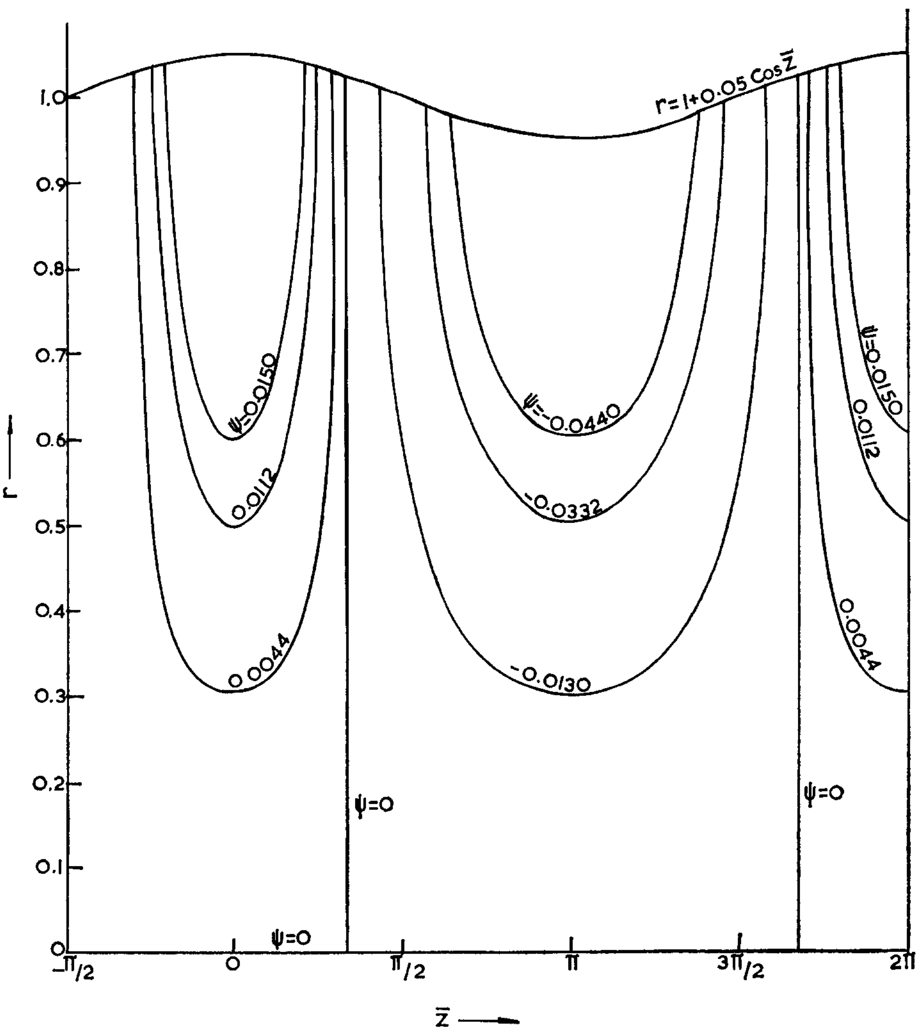

Fig. 2

form continuous lines running parallel to the axis of the tube when considered near the axis, while a considerable deformation is seen when considered near the boundary. A detailed physical explanation for such a behaviour was provided in Part I (6).

Thus the critical value of the parameter k_0 ($= Re\, \partial p_0/\partial$) connected with the pressure gradient for a set of parameters involved have been obtained with much tedious computational work. For example when $E = 0.05$, $K = -0.2$, $S = 0.6$, the critical value of k_0 is 0.25. This explains that for the values of k_0 less than $k_{0c} = 0.25$, we obtain the streamline pattern with closed loops and separating $\Psi = 0$ lines, while for $k_0 > k_{0c}$ we get continuous streamlines. Hence k_{0c} can be determined for given K and S. In a broader sense we can conclude that the determination of the critical values of k_0 may serve as a diagnostic tool to understand the peristaltic transport of visco-elastic fluids.

A special feature of the present analysis is that we can discuss the streamline pattern for $k_0 = 0$, which was not possible in Part I. However, in this case also the streamlines form closed loops as described above. The behaviour of the axial velocity profiles for $k_0 = 0$ and $k_0 = 1$ is shown in fig. 2. We notice that w takes positive values to negative values as we pass from $k_0 = 0$ to $k_0 = 1$. The critical value of k_0 which changes the behaviour of w markedly, happens to be the same k_{0c} as described in the case of stream-line pattern.

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