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Steady flow of a simple fluid around a rotating sphere

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With 8 figures

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1. Introduction

This paper studies the flow around a steadily rotating sphere which is submerged in an infinite vat containing an incompressible, homogeneous simple fluid. While the problem has been studied previously, both analytically and experimentally, for several special non-Newtonian fluids (5, 6, 7, 16, 17, 18), there yet remains certain disagreements between the existing theoretical predictions and the experimental observations; in particular, the shape of many observed secondary flow cells has neither been accurately predicted, nor even adequately approximated. This secondary flow region is of potential importance to the understanding of mixing, and, of course, the general question of the secondary flow of non-Newtonian fluids is clearly relevant to the processing of polymeric materials.

The development of non-Newtonian fluid mechanics has been largely motivated by the observed curious behavior of polymeric materials. Many observed phenomena have been modeled and understood satisfactorily by applying the principles of non-Newtonian fluid mechanics, and, conversely, the theory of non-Newtonian fluid mechanics has been used to predict certain realistic and unusual flow behavior which had not previously been observed. An example of this latter situation is the well-known secondary steady swirl flow phenomenon which takes place down straight pipes of non-circular cross section (1, 2, 3, 4) under a given pressure head.

A general analysis of the secondary flow field around a steadily rotating submerged sphere is complicated by geometric considerations: this may be responsible for the fact that the works of *Giesekus* (5), *Thomas* and *Walters* (6) and *Langlois* (7) have been concerned with at most second order perturbation solutions, the perturbation parameter being the angular velocity. However, at this order the theoretical prediction of the shape of the cells of secondary flow apparently is not always confirmed by experiment[1]). Since the interesting secondary flow phenomenon in the flow down rectilinear pipes is first present at the fourth order perturbation solution (3) (the perturbation parameter here being pressure head), we are inclined to believe

[1]) See, however, the work of *Walters* and *Savins* (16) and *Giesekus* (17).

that in the present problem the disagreement between certain observed secondary flow and the independent analytical predictions will be explained by carrying out the perturbation analysis to orders greater than two. We confirm this belief with our computations at fourth order.

In this work we apply, essentially, the approximation theory of *Coleman* and *Noll* (8), and develop a perturbation scheme for the stress which takes as its central parameter the angular velocity, Ω, of the sphere. For this purpose, we tacitly assume that the stress response functional of the simple fluid is sufficiently differentiable at the zero history in an appropriate function space of fading memories. Specifically, our approach follows that used by *Joseph* and *Fosdick* (9) in their study of the shape of the free surface of a simple fluid which is contained between concentric rotating cylinders.

After a brief preliminary discusion of some pertinent results from the theory of simple fluids in Section 2, we outline, in Section 3, the general problem of concern in this paper, introduce our series procedure for its approximate solution, and set up for study an ordered sequence or related but simpler problems, ordered according to integral powers of the angular velocity of the turning sphere. In Section 4, we develop the solutions to this ordered sequence of problems through the fourth order, and in Section 5 we give a brief summary and discussion of our results. In figures 2 – 8, we show, by way of some explicit computations, that at fourth order the cells of secondary flow take on the shape of an equatorial torus and/or a polar cap-shape not unlike those which have been observed experimentally. Reference to these figures is included in our discussions in Section 5. While we make no claim that all of the material constants used in preforming the computations for the figures have any physical significance, nevertheless, we do believe that these figures support the view that many of the non-trivial secondary flow regions that are observed in the laboratory around immersed and rotating spheres (5, 17, 18) are well within theoretical description.

2. Preliminaries

A simple fluid is defined (10) as a material with no preferred reference configuration whose stress response is determined by the past history of the relative deformation gradient. To formulate this concept, the relative fluid motion is first represented as a 3-dimensional (Euclidean) point valued function

$$\xi(\tau) = \chi(\boldsymbol{x}, t; \tau), \quad -\infty < \tau \leqslant t, \qquad [2.1]$$

where $\xi(\tau)$ denotes the location at time τ of a typical fluid particle which occupies $\boldsymbol{x}$ at the present time t. For any given $\boldsymbol{x}$ and t, [2.1] describes the path of a typical particle.

The relative deformation gradient is defined as the spatial gradient of $\xi(\tau)$ with respect to $\boldsymbol{x}$, i.e.,

$$\boldsymbol{F}_t(\tau) = \boldsymbol{F}(\boldsymbol{x}, t; \tau) = \operatorname{grad}\chi(\boldsymbol{x}, t; \tau). \qquad [2.2]$$

In this paper, we are concerned with an incompressible fluid for which only isochoric motion is possible. Hence, $\boldsymbol{F}_t(\tau)$ satisfies

$$\det \boldsymbol{F}_t(\tau) = 1. \qquad [2.3]$$

Let $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x}, t)$ be the (symmetric) Cauchy stress tensor field for the fluid. In view of the assumption of incompressibility, $\boldsymbol{T}$ is then constitutively determined by the fluid motion up to an arbitrary added hydrostatic pressure p. It is therefore convenient to introduce the extra stress tensor

$$\boldsymbol{S}(\boldsymbol{x}, t) \equiv \boldsymbol{T}(\boldsymbol{x}, t) + p(\boldsymbol{x}, t)\,\mathbf{1}, \qquad [2.4]$$

and to normalize $\boldsymbol{S}(\boldsymbol{x}, t)$ by the convention

$$\operatorname{tr}\boldsymbol{S}(\boldsymbol{x}, t) = 0. \qquad [2.5]$$

Thus, the constitutive assumption for an incompressible simple fluid requires that the extra stress $\boldsymbol{S}$ at $(\boldsymbol{x}, t)$ be determined by the time history of the relative deformation gradient $\boldsymbol{F}_t(\tau)$. Mathematically, this idea is expressed by

$$\boldsymbol{S} = \mathop{\mathscr{H}}_{\tau=-\infty}^{\tau=t}(\boldsymbol{F}_t(\tau)), \qquad [2.6]$$

where $\mathscr{H}$ is a mapping from the space of (proper) unimodular tensor valued functions on $(-\infty, t]$ to the vector space of traceless, symmetric second order tensors. The principle of material frame indifference then allows [2.6] to be written in the form

$$\boldsymbol{S}(t) = \mathop{\mathscr{I}}_{\tau=-\infty}^{\tau=t}(\boldsymbol{C}_t - \mathbf{1}), \qquad [2.7\text{a}]$$

where the extra stress response functional $\mathscr{I}$ is symmetric and satisfies the invariance condition

$$\boldsymbol{Q} \mathop{\mathscr{I}}_{\tau=-\infty}^{\tau=t}(\boldsymbol{C}_t - \mathbf{1})\, \boldsymbol{Q}^{\mathsf{T}} = \mathop{\mathscr{I}}_{\tau=-\infty}^{\tau=t}(\boldsymbol{Q}\boldsymbol{C}_t(\tau)\boldsymbol{Q}^{\mathsf{T}} - \mathbf{1}) \qquad [2.7\text{b}]$$

for all fixed orthogonal tensors $\boldsymbol{Q}$, and where $\boldsymbol{C}_t(\tau)$ is the relative right Cauchy-Green strain tensor defined by

$$\boldsymbol{C}_t(\tau) \equiv \boldsymbol{F}_t^{\mathrm{T}}(\tau)\boldsymbol{F}_t(\tau). \qquad [2.8]$$

Now, for *slow steady motions* and with certain technical smoothness assumptions on the response functional $\mathscr{I}$ at the zero history when defined on a Banach space of fading memories, the retardation approximation theorem of *Coleman* and *Noll* (8) may be applied to yield the following approximate formula complete up to and including the fourth order in the velocity magnitude, cf. (11, p. 494):

$$\boldsymbol{S} = \sum_{i=1}^{4} \boldsymbol{S}_i,$$

where

$$\begin{aligned}
\boldsymbol{S}_1 &= \mu \boldsymbol{A}_1, \\
\boldsymbol{S}_2 &= \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2, \\
\boldsymbol{S}_3 &= \beta_1 \boldsymbol{A}_3 + \beta_2(\boldsymbol{A}_2\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{A}_2) \\
&\quad + \beta_3(\mathrm{tr}\boldsymbol{A}_2)\boldsymbol{A}_1, \qquad [2.9] \\
\boldsymbol{S}_4 &= \gamma_1 \boldsymbol{A}_4 + \gamma_2(\boldsymbol{A}_3\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{A}_3) + \gamma_3 \boldsymbol{A}_2^2 \\
&\quad + \gamma_4(\boldsymbol{A}_2\boldsymbol{A}_1^2 + \boldsymbol{A}_1^2\boldsymbol{A}_2) + \gamma_5(\mathrm{tr}\boldsymbol{A}_2)\boldsymbol{A}_2 \\
&\quad + \gamma_6(\mathrm{tr}\boldsymbol{A}_2)\boldsymbol{A}_1^2 + \gamma_7(\mathrm{tr}\boldsymbol{A}_3)\boldsymbol{A}_1 \\
&\quad + \gamma_8(\mathrm{tr}\boldsymbol{A}_2\boldsymbol{A}_1)\boldsymbol{A}_1,
\end{aligned}$$

and where $\boldsymbol{A}_k$ is the k-th Rivlin-Ericksen tensor defined by

$$\boldsymbol{A}_k \equiv \frac{d^k}{d\tau^k}\boldsymbol{C}_t(\tau)\bigg|_{\tau=t}, k \geqslant 1, \qquad [2.10]$$

$$\boldsymbol{A}_0 \equiv \boldsymbol{C}_t(t) = \mathbf{1}.$$

In these formulae, $\mu > 0$ is the viscosity, and $\alpha_1, \alpha_2, \beta_1, \ldots, \gamma_8$ are material constants. In addition, it should be remarked that a constant multiple of $\mathbf{1}$ would have to be included in the first of [2.9] in order to satisfy the convention [2.5]. We have absorbed this spherical stress into the constitutively indeterminate pressure $-p\mathbf{1}$ of $\boldsymbol{T}$.

For steady motion, the present velocity $\boldsymbol{u}(\boldsymbol{x}) = \dfrac{d\xi(\tau)}{d\tau}$ depends solely on the spatial location $\boldsymbol{x}$ and it follows that the Rivlin-Ericksen tensors must satisfy the recursion relation

$$\boldsymbol{A}_{k+1} = (\mathrm{grad}\boldsymbol{A}_k)\boldsymbol{u} + \boldsymbol{A}_k \mathrm{grad}\boldsymbol{u} + (\boldsymbol{A}_k \mathrm{grad}\boldsymbol{u})^{\mathrm{T}}, \qquad [2.11]$$

$$\boldsymbol{A}_0 = \mathbf{1}.$$

In addition, the incompressibility condition [2.3] requires that

$$\mathrm{div}\,\boldsymbol{u}(\boldsymbol{x}) = 0, \qquad [2.12]$$

and the balance of linear momentum has the form

$$\rho(\mathrm{grad}\boldsymbol{u})\boldsymbol{u} = -\mathrm{grad}p + \mathrm{div}\boldsymbol{S}, \qquad [2.13]$$

where ρ is the (constant) density of the fluid, and where body force has been neglected. Clearly, a conservative body force field could be included as part of the pressure field p.

3. Problem statement and governing equations

The physical problem which we shall consider is to determine the steady flow field and possible secondary flow cells in a large vat of incompressible simple fluid; the motion being caused solely by the steady rotation about a diameter of a submerged rigid sphere. As an idealization, the container of the fluid will be assumed to be infinite in extent.

The prescribed constant angular velocity of the sphere, Ω, and the related steady motion internal to the fluid will be assumed to be small enough that the approximate constitutive relation [2.9] applies, at least to within order Ω^4 accuracy. Of course, at the surface of the sphere the conventional no-slip boundary condition will be employed.

Naturally, it is convenient to adopt a right-handed spherical coordinate system (r, θ, ϕ) with the origin of the coordinates located at the

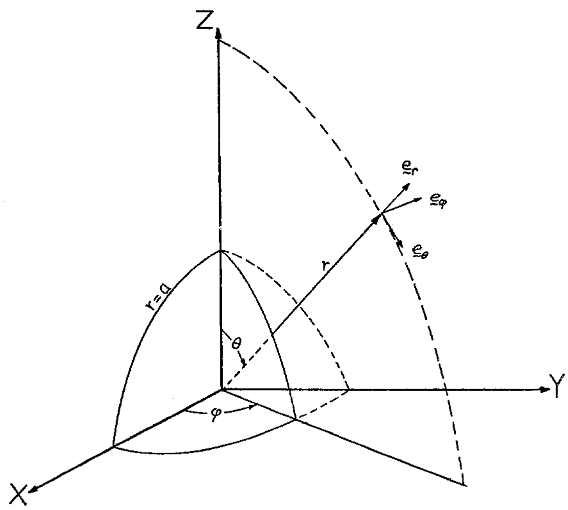


Fig. 1. Positive quadrant of the rotating sphere $r = a$

center of the sphere, and to take the polar axis from which θ is measured as the axis about which the sphere is rotating (cf., fig. 1). The geometric symmetry of the problem implies that the longitudinal plane $\phi = 0$ can be chosen arbitrarily. If we let $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ be the natural orthonormal basis of the coordinate system at any spatial point $\boldsymbol{x}$ in the fluid domain, then the position vector $\boldsymbol{r}$ from the origin $\boldsymbol{0}$ to $\boldsymbol{x}$ is given by $\boldsymbol{r} = r\boldsymbol{e}_r$, where r is the associated radial distance.

If $\boldsymbol{u}(\boldsymbol{r})$ denotes the velocity field in the fluid domain, then the no-slip boundary condition and the condition that the fluid be at rest at infinity requires that

$$\boldsymbol{u} = \begin{cases} \Omega a \sin\theta \boldsymbol{e}_\phi & on \quad r = a, \\ o(1) & as \quad r \to \infty, \end{cases} \qquad [3.1]$$

where a is the radius of the sphere, and $o(1)$ denotes the conventional "small order" function. We shall assume that the velocity field and the induced pressure field in the fluid domain are of the form

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}(\boldsymbol{r}; \Omega), \\ p &= p(\boldsymbol{r}; \Omega), \end{aligned} \qquad [3.2]$$

and, moreover, that these functions are sufficiently differentiable with respect to Ω to warrent the series expansions

$$\begin{aligned} \boldsymbol{u} &= \sum_{k=1}^{n} \frac{1}{k!} \boldsymbol{u}^{(k)} \Omega^k + o(\Omega^n), \\ p &= \sum_{k=0}^{n} \frac{1}{k!} p^{(k)} \Omega^k + o(\Omega^n), \end{aligned} \qquad [3.3]$$

where $(\,)^{(k)} \equiv \left.\dfrac{\partial^k}{\partial \Omega^k}(\,)\right|_{\Omega=0}$, and where $\boldsymbol{u}^{(k)}$ and $p^{(k)}$ are functions of position. Because of symmetry about the axis of rotation of the sphere it follows that

$$\boldsymbol{u}^{(k)} = u_r^{(k)}(r, \theta)\boldsymbol{e}_r + u_\theta^{(k)}(r, \theta)\boldsymbol{e}_\theta + u_\phi^{(k)}(r, \theta)\boldsymbol{e}_\phi,$$

and that $p^{(k)} = p^{(k)}(r, \theta)$. Of course, [3.3] tacitly assumes that when $\Omega = 0$, then $\boldsymbol{u} = \boldsymbol{0}$ and $p = p^{(0)}$, where $p^{(0)}$ is the uniform static pressure field of the fluid, which could be set to zero without loss of generality.

The governing differential equations for $\boldsymbol{u}^{(k)}$ and $p^{(k)}$ now follow from [2.12] and [2.13], with the aid of [2.9], [2.11] and [3.3]. Thus, we obtain

$$\rho[(\operatorname{grad}\boldsymbol{u})\boldsymbol{u}]^{(k)} = -\operatorname{grad} p^{(k)} + \operatorname{div} \boldsymbol{S}^{(k)}, r > a, \qquad [3.4]$$

and

$$\operatorname{div}\boldsymbol{u}^{(k)} = 0, r > a, \qquad [3.5]$$

where for $\boldsymbol{S}^{(k)}$ we must call upon [2.9] and [2.11]. We note that by introducing a streamfunction $\psi^{(k)}(r, \theta)$ such that

$$\boldsymbol{u}^{(k)} = -\operatorname{curl}\left(\frac{\psi^{(k)}}{r\sin\theta}\boldsymbol{e}_\phi\right) + u_\phi^{(k)}(r, \theta)\boldsymbol{e}_\phi, \qquad [3.6]$$

then [3.5] is satisfied identically.

From [3.3] and [3.1], the boundary conditions for $\boldsymbol{u}^{(k)}$ have the form

$$\begin{aligned} &\boldsymbol{u}^{(1)} = a\sin\theta \boldsymbol{e}_\phi, \\ &\boldsymbol{u}^{(k)} = 0, (k > 1), on\, r = a, \\ &\boldsymbol{u}^{(k)} = o(1), (k \geqslant 1), as\, r \to \infty, \end{aligned}$$

which, with the aid of [3.6], may be stated as follows:

$$\left.\begin{aligned} &u_\phi^{(1)} = a\sin\phi,\ u_\phi^{(k)} = 0, (k > 1), \\ &\psi^{(k)} = c^{(k)}, \frac{\partial\psi^{(k)}}{\partial r} = 0, (k \geqslant 1), \end{aligned}\right\} on\, r = a, \qquad [3.7]$$

$$\begin{aligned} &u_\phi^{(k)} = o(1),\ \psi^{(k)} = o(r^2), \\ &\frac{\partial\psi^{(k)}}{\partial r} = o(r), (k \geqslant 1), as\, r \to \infty, \end{aligned}$$

where $c^{(k)}$ is a constant.

4. Perturbation solution

We now seek to solve, at each order $k = 1, 2, 3$ and 4, the sequence of problems outlined in Section 3, and to thereby generate the series solution [3.3]. Specifically, we shall use [3.6], [2.9], [2.11] and [3.4] to develop, for each k, the fundamental field equations for $p^{(k)}$, $\psi^{(k)}$ and $u_\phi^{(k)}$, and to determine these fields subject to the boundary conditions [3.7]. Since [2.9] already represents a 4th order retardation approximation of the response function of a simple fluid, the analysis given here cannot be extended beyond $k = 4$. Fortunately, the solution which we develop at order Ω^4 exhibits the presence of bulging equatorial and/or polar circulating cells, and this agrees favorably with the interesting photographs of *Giesekus* (5, p. 260) concerning this phenomenon. These equatorial and polar cells are not predicted at orders less than four.

4.1. First order problem

Here, the problem basically is the same as that for Newtonian fluids without inertial effects. Even so, we shall carry out the discussion of this first order problem fairly completely since the higher order problems will require similar operations.

For $k = 1$, [3.4] has the form

$$-\operatorname{grad} p^{(1)} + \operatorname{div} \boldsymbol{S}^{(1)} = 0, r > a, \qquad [4.1]$$

where $\boldsymbol{S}^{(1)}$ is obtained from [2.9] by essentially differentiating once with respect to Ω and then setting $\Omega = 0$. Specifically, with the aid of $[3.3]_1$, [2.11] and [2.9], it follows that

$$\begin{aligned} \boldsymbol{A}_1^{(1)} &= \operatorname{grad} \boldsymbol{u}^{(1)} + (\operatorname{grad} \boldsymbol{u}^{(1)}) , \\ \boldsymbol{A}_i^{(1)} &= 0, i \geqslant 2, \end{aligned} \qquad [4.2]$$

and that

$$\begin{aligned} \boldsymbol{S}^{(1)} &= \boldsymbol{S}_1^{(1)} + \boldsymbol{S}_2^{(1)} + \boldsymbol{S}_3^{(1)} + \boldsymbol{S}_4^{(1)}, \\ \boldsymbol{S}_1^{(1)} &= \mu \boldsymbol{A}_1^{(1)}, \\ \boldsymbol{S}_i^{(1)} &= 0, i \geqslant 2. \end{aligned} \qquad [4.3]$$

Of course, for $\boldsymbol{u}^{(1)}$ we have the representation [3.6] in terms of the streamfunction $\psi^{(1)}$ and $u_\phi^{(1)}$.

It is now straightforward to see, by substituting [4.2] and [4.3] into [4.1], that the ϕ-component of [4.1] reduces to

$$\mu \left(\Delta u_\phi^{(1)} - \frac{1}{r^2 \sin^2 \theta} u_\phi^{(1)} \right) = 0, \qquad [4.4]$$

where Δ denotes the Laplacian operator, and where we have taken $p^{(1)}$, $\psi^{(1)}$ and $u_\phi^{(1)}$ independent of ϕ, as noted earlier. Thus, since $\mu > 0$, the boundary conditions on $u_\phi^{(1)}$ in [3.7] imply that

$$u_\phi^{(1)} = \frac{a^3}{r^2} \sin \theta. \qquad [4.5]$$

The remaining r- and θ-components of [4.1] are, of course, generated in a similar fashion, and it follows that the equation governing them can be reduced to the form

$$\operatorname{grad} p^{(1)} + \mu \Delta \left[\operatorname{curl} \left(\frac{\psi^{(1)}}{r \sin \theta} \boldsymbol{e}_\phi \right) \right] = 0. \qquad [4.6]$$

By applying the curl operation to this equation in order to eliminate $p^{(1)}$, we readily obtain

$$\mathscr{L}^2 \psi^{(1)} = 0, \qquad [4.7]$$

where $\mathscr{L}$ is a second order linear differential operator defined by

$$\mathscr{L} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta}. \qquad [4.8]$$

Thus, by applying the boundary conditions for $\psi^{(1)}$ as given in [3.7], we see that $\psi^{(1)} \equiv c^{(1)}$ for all $r \geqslant a$, and consequently from [4.6] it follows that $p^{(1)} \equiv$ constant. Whence, at first order we conclude that

$$\begin{aligned} \boldsymbol{u}^{(1)} &= \frac{a^3}{r^2} \sin \theta \boldsymbol{e}_\phi, \\ p^{(1)} &= \text{constant}. \end{aligned} \qquad [4.9]$$

4.2. Second order problem

For $k = 2$, the governing equations, i.e., [3.3], [3.4], [2.9], and [2.11], yield

$$-\operatorname{grad} p^{(2)} + \operatorname{div} \boldsymbol{S}^{(2)} = 2\rho (\operatorname{grad} \boldsymbol{u}^{(1)}) \boldsymbol{u}^{(1)}, \qquad [4.10]$$

where

$$\begin{aligned} \boldsymbol{A}_1^{(2)} &= \operatorname{grad} \boldsymbol{u}^{(2)} + (\operatorname{grad} \boldsymbol{u}^{(2)})^{\mathrm{T}}, \\ \boldsymbol{A}_2^{(2)} &= (\operatorname{grad} \boldsymbol{A}_1^{(1)}) \boldsymbol{u}^{(1)} + \boldsymbol{A}_1^{(1)} \operatorname{grad} \boldsymbol{u}^{(1)} \\ &\quad + (\boldsymbol{A}_1^{(1)} \operatorname{grad} \boldsymbol{u}^{(1)})^{\mathrm{T}}, \\ \boldsymbol{A}_i^{(2)} &= 0, i \geqslant 3, \end{aligned} \qquad [4.11]$$

with $\boldsymbol{A}_1^{(1)}$ given by $[4.2]_1$ and [4.9], and where

$$\boldsymbol{S}^{(2)} = \boldsymbol{S}_1^{(2)} + \boldsymbol{S}_2^{(2)} + \boldsymbol{S}_3^{(2)} + \boldsymbol{S}_4^{(2)}, \qquad [4.12]$$

with

$$\begin{aligned} \boldsymbol{S}_1^{(2)} &= \mu \boldsymbol{A}_1^{(2)}, \\ \boldsymbol{S}_2^{(2)} &= \alpha_1 \boldsymbol{A}_2^{(2)} + 2\alpha_2 (\boldsymbol{A}_1^{(1)})^2, \\ \boldsymbol{S}_i^{(2)} &= 0, i \geqslant 3. \end{aligned} \qquad [4.13]$$

From [4.13] and [4.11] it is clear that $\boldsymbol{S}_2^{(2)}$ is determined solely by $\boldsymbol{u}^{(1)}$ and, hence, the field $\boldsymbol{u}^{(2)}$ enters $\boldsymbol{S}^{(2)}$ only through $\boldsymbol{S}_1^{(2)}$ in $[4.13]_1$. Therefore, with the representation [3.6] for $k = 2$, and $\boldsymbol{u}^{(1)}$ as given in [4.9], [4.10] may be reduced to a set of linear partial differential equations of the form

$$-\operatorname{grad} p^{(2)} + \mu \Delta \left[-\operatorname{curl}\left(\frac{\psi^{(2)}}{r \sin\theta} e_\phi\right) + u_\phi^{(2)} e_\phi \right]$$

$$= \left[18(8\alpha_1 + 5\alpha_2)\frac{a^6}{r^7} - 2\rho\frac{a^6}{r^5}\right]\sin^2\theta e_r$$

$$+ \left[18\,\alpha_2\frac{a^6}{r^7} - 2\rho\frac{a^6}{r^5}\right]\sin\theta\cos\theta e_\theta. \qquad [4.14]$$

Note that since $p^{(2)}$, $\psi^{(2)}$ and $u_\phi^{(2)}$ are independent of ϕ then the ϕ-component of [4.14] is identical in form with [4.4] with $p^{(1)}$ and $u_\phi^{(1)}$ replaced by $p^{(2)}$ and $u_\phi^{(2)}$. Since the boundary conditions for $u_\phi^{(2)}$ in [3.7] are zero, both at $r = a$ and as $r \to \infty$, it follows that $u_\phi^{(2)} \equiv 0$ for all $r \geqslant a$.

If we now take the curl operation on the remaining part of [4.14] we reach

$$\mathscr{L}^2\psi^{(2)} = \frac{12}{\mu}\left[\rho\frac{a^6}{r^5} - 24\frac{a^6}{r^7}\left(\alpha_1 + \alpha_2\right)\right]$$

$$\cdot \sin^2\theta\cos\theta,\ r > a, \qquad [4.15]$$

where $\mathscr{L}$ is given by [4.8]. Thus, using the boundary conditions for $\psi^{(2)}$ as given in [3.7], it follows that, aside from an added constant $c^{(2)}$, $\psi^{(2)}$ has the form

$$\psi^{(2)} = F(r)\sin^2\theta\cos\theta,$$

$$F(r) \equiv \frac{a^3}{\mu}\left[\rho\frac{a^2}{4} - (\alpha_1 + \alpha_2)\left(1 + 2\frac{a}{r}\right)\right]$$

$$\cdot \left(1 - \frac{a}{r}\right)^2. \qquad [4.16]$$

The pressure field $p^{(2)}$ now can be found by integrating [4.14] with $\psi^{(2)}$ given by [4.16]. Since $p^{(2)}$ does not have an explicit contribution in the higher order problems which we shall consider, it is not necessary to record this result.

We have found that at second order,

$$u^{(2)} = -\operatorname{curl}\left(\frac{\psi^{(2)}}{r\sin\theta}e_\phi\right), \qquad [4.17]$$

where $\psi^{(2)}$ is given in [4.16]. The problem at this order has been considered previously by *Langlois* (7), *Giesekus* (5), and *Thomas* and *Walters* (6). We believe that one of their most interesting conclusion concerns the existence of an "inertial radius", i.e., a spherical surface of radius $r^* > a$ concentric with the solid turning sphere $r = a$ and inside of which the same fluid particles always remain while circulating both longitudinally and from poles to equator and back again. The existence of this "secondary flow" and enveloping spherical surface is guaranteed if the material constants ρ, α_1, and α_2, and the given spherical radius $r = a$ satisfy

$$4 < \frac{\rho a^2}{\alpha_1 + \alpha_2} < 12, \qquad [4.18]$$

which, in effect, shows that there is a spherical level surface $\psi^{(2)} = 0$ of [4.16] at $r = r^* > a$ in the fluid domain. In this case, the secondary flow in the domain $a < r < r^*$ is dominated by normal stress effects, while that outside $r > r^*$ is influenced more heavily by centrifugal force. If $\rho a^2/(\alpha_1 + \alpha_2) > 12$, then centrifugal force dominates everywhere, while if $\rho a^2/(\alpha_1 + \alpha_2) < 4$, the normal stress effects will dominate in the whole fluid domain.

Motivated by the experimental observations of *Giesekus* (5), which illustrate that the secondary flow domain generally is not housed within a whole spherical envelope but is rather more concentrated in an equatorial torus or perhaps in polar caps, we shall go now to investigate the effects of higher order in Ω, and show that, indeed, a toroidal and/or polar domain emerges at order Ω^4.

4.3. *Third order problem*

For $k = 3$, the governing equations, i.e., [3.3], [3.4], [2.9], and [2.11], yield

$$-\operatorname{grad} p^{(3)} + \operatorname{div} S^{(3)} = 3\rho[(\operatorname{grad} u^{(2)})u^{(1)} + (\operatorname{grad} u^{(1)})u^{(2)}], \qquad [4.19]$$

where

$$A_1^{(3)} = \operatorname{grad} u^{(3)} + (\operatorname{grad} u^{(3)})^{\mathrm{T}},$$

$$A_2^{(3)} = (\operatorname{grad} A_1^{(2)})u^{(1)} + A_1^{(2)}\operatorname{grad} u^{(1)} + (A_1^{(2)}\operatorname{grad} u^{(1)})^{\mathrm{T}},$$

$$A_3^{(3)} = (\operatorname{grad} A_2^{(2)})u^{(1)} + A_2^{(2)}\operatorname{grad} u^{(1)} + (A_2^{(2)}\operatorname{grad} u^{(1)})^{\mathrm{T}} + (\operatorname{grad} A_2^{(1)})u^{(2)} + A_2^{(1)}\operatorname{grad} u^{(2)} + (A_2^{(1)}\operatorname{grad} u^{(2)})^{\mathrm{T}},$$

$$A_i^{(3)} = 0,\ i \geqslant 4, \qquad [4.20]$$

with $A_1^{(1)}$, $A_1^{(2)}$, $A_2^{(2)}$, $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ given by [4.2]$_1$, [4.11], [4.9], [4.17] and [4.16], and where

$$S^{(3)} = S_1^{(3)} + S_2^{(3)} + S_3^{(3)} + S_4^{(3)}, \qquad [4.21]$$

with

$$\begin{aligned} S_1^{(3)} &= \mu A_1^{(3)}, \\ S_2^{(3)} &= \alpha_1 A_2^{(3)} + \alpha_2(3A_1^{(2)}A_1^{(1)} + 3A_1^{(1)}A_1^{(2)}), \\ S_3^{(3)} &= \beta_1 A_3^{(3)} + \beta_2(3A_2^{(2)}A_1^{(1)} + 3A_1^{(1)}A_2^{(2)}) \\ &\quad + 3\beta_3(\mathrm{tr}A_2^{(2)})A_1^{(1)}, \\ S_i^{(3)} &= 0, i \geqslant 4. \end{aligned} \qquad [4.22]$$

It can be seen, analogous to earlier orders, that the field $\boldsymbol{u}^{(3)}$ enters $S^{(3)}$ only through $S_1^{(3)}$ in [4.22]$_1$. Therefore, with the representation [3.6] for $k = 3$, and $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ as given in [4.9], [4.17] and [4.16], the system [4.19] may be reduced to the form

$$\begin{aligned} &-\mathrm{grad}\, p^{(3)} + \mu\Delta\left[-\mathrm{curl}\left(\frac{\psi^{(3)}}{r\sin\theta}\boldsymbol{e}_\phi\right) + u_\phi^{(3)}\boldsymbol{e}_\phi\right] \\ &= \Bigg\{\Bigg[-3\rho a^3\frac{F}{r^5} + 9(\alpha_1+\alpha_2)a^3 \\ &\quad\cdot\left(\frac{F''}{r^5} - 4\frac{F'}{r^6}\right) - 1944(\beta_2+\beta_3)\frac{a^9}{r^{10}}\Bigg] \\ &\quad\cdot\sin\theta + \Bigg[3\rho a^3\left(2\frac{F'}{r^4} + 3\frac{F}{r^5}\right) \\ &\quad + 9(\alpha_1+\alpha_2)a^3\left(\frac{F''}{r^5} + 4\frac{F'}{r^6} + 12\frac{F}{r^7}\right) \\ &\quad + 1944(\beta_2+\beta_3)\frac{a^9}{r^{10}}\Bigg]\sin\theta\cos^2\theta\Bigg\}\boldsymbol{e}_\phi, \end{aligned} \qquad [4.23]$$

where F' and F'' denotes differentiations of F given in [4.16].

We shall first concentrate on the ϕ-component of [4.23]. Since $p^{(3)}$, $\psi^{(3)}$ and $u_\phi^{(3)}$ are independent of ϕ, this component of [4.23] contains the single unknown field $u_\phi^{(3)}$ which, because of the right hand side of [4.23], we assume to be of the form

$$u_\phi^{(3)} = H(r)\sin\theta + K(r)\sin\theta\cos^2\theta, \qquad [4.24]$$

where $H(r)$ and $K(r)$ are to be determined. It is clear from [3.7] that the boundary conditions for $H(r)$ and $K(r)$ are,

$$\begin{aligned} &H(a) = K(a) = 0, \\ &H(r) = K(r) = o(1), \text{as } r \to \infty. \end{aligned} \qquad [4.25]$$

In addition, upon substituting [4.24] into the ϕ-component of [4.23] we find that

$$\begin{aligned} &\left(H'' + \frac{2}{r}H' - \frac{2}{r^2}H + \frac{2}{r^2}K\right)\sin\theta \\ &\quad + \left(K'' + 2\frac{K'}{r} - 12\frac{K}{r^2}\right)\sin\theta\cos^2\theta \\ &= \Bigg[-3\rho a^3\frac{F}{r^5} + 9(\alpha_1+\alpha_2)a^3\left(\frac{F''}{r^5} - 4\frac{F'}{r^6}\right) \\ &\quad - 1944(\beta_2+\beta_3)\frac{a^9}{r^{10}}\Bigg]\sin\theta \\ &\quad + \Bigg[3\rho a^3\left(2\frac{F'}{r^4} + 3\frac{F}{r^5}\right) + 9(\alpha_1+\alpha_2)a^3 \\ &\quad\cdot\left(\frac{F''}{r^5} + 4\frac{F'}{r^6} + 12\frac{F}{r^7}\right) \\ &\quad + 1944(\beta_2+\beta_3)\frac{a^9}{r^{10}}\Bigg]\sin\theta\cos^2\theta, \end{aligned} \qquad [4.26]$$

which clearly yields two linear ordinary differential equations for $H(r)$ and $K(r)$, whose solutions are subject to the boundary conditions [4.25]. Since one of the equations is independent of $H(r)$, it can be integrated first to yield $K(r)$ and then $H(r)$ can be integrated from the second equation. Upon integration and satisfaction of the boundary conditions [4.25], we find that $K(r)$ and $H(r)$ have the following forms:

$$\begin{aligned} K(r) &= \sum_{i=3}^{8}\frac{k_i}{r^i} + \frac{k_4'}{r^4}\ln\frac{r}{a}, \\ H(r) &= \sum_{i=2}^{8}\frac{h_i}{r^i} + \frac{h_4'}{r^4}\ln\frac{r}{a}, \end{aligned} \qquad [4.27]$$

where the constants k_i, h_i, k_4' and h_4' are given by

$$k_3 = -\frac{3}{2}\frac{\rho a^6}{\mu^2}\left(\frac{\rho a^2}{4} - \alpha_1 - \alpha_2\right),$$

$$k_4 = \frac{15}{32}\frac{\rho^2 a^9}{\mu^2} - 3\frac{\rho a^7}{\mu^2}(\alpha_1 + \alpha_2) + \frac{207}{22}\frac{a^5}{\mu^2} \cdot (\alpha_1 + \alpha_2)^2 - \frac{486}{11}\frac{a^5}{\mu}(\beta_2 + \beta_3),$$

$$k_5 = -\frac{3}{32}\frac{a^6}{\mu^2}[\rho a^2 - 12(\alpha_1 + \alpha_2)]^2,$$

$$k_6 = -\frac{3}{2}\frac{\rho a^9}{\mu^2}(\alpha_1 + \alpha_2),$$

$$k_7 = \frac{3}{4}\frac{a^8}{\mu^2}(\alpha_1 + \alpha_2)[\rho a^2 + 12(\alpha_1 + \alpha_2)],$$

$$k_8 = \frac{54}{11}\frac{a^9}{\mu}\left[9(\beta_2 + \beta_3) - \frac{(\alpha_1 + \alpha_2)^2}{\mu}\right],$$

$$k_4' = \frac{3}{14}\frac{\rho^2 a^9}{\mu^2},$$

$$h_2 = \frac{\rho^2 a^7}{200\mu^2} + \frac{3}{70}\frac{\rho a^5}{\mu^2}(\alpha_1 + \alpha_2) - \frac{4}{5}\frac{a^3}{\mu^2} \cdot (\alpha_1 + \alpha_2)^2 + \frac{144}{5}\frac{a^3}{\mu}(\beta_2 + \beta_3),$$

$$h_3 = 0,$$

$$h_4 = \frac{21}{800}\frac{\rho^2 a^9}{\mu^2} + \frac{3}{5}\frac{\rho a^7}{\mu^2}(\alpha_1 + \alpha_2) - \frac{207}{110} \cdot \frac{a^5}{\mu^2}(\alpha_1 + \alpha_2)^2 + \frac{486}{55}\frac{a^5}{\mu}(\beta_2 + \beta_3),$$

$$h_5 = -\frac{1}{32}\frac{\rho^2 a^{10}}{\mu^2} - \frac{3}{4}\frac{\rho a^8}{\mu^2}(\alpha_1 + \alpha_2) + \frac{3}{2}\frac{a^6}{\mu^2}(\alpha_1 + \alpha_2)^2,$$

$$h_6 = -\frac{9}{14}\frac{\rho a^9}{\mu^2}(\alpha_1 + \alpha_2),$$

$$h_7 = \frac{3}{4}\frac{\rho a^{10}}{\mu^2}(\alpha_1 + \alpha_2) + 9\frac{a^8}{\mu^2}(\alpha_1 + \alpha_2)^2,$$

$$h_8 = -\frac{86}{11}\frac{a^9}{\mu^2}(\alpha_1 + \alpha_2)^2 - \frac{414}{11}\frac{a^9}{\mu} \cdot (\beta_2 + \beta_3),$$

$$h_4' = -\frac{3}{70}\frac{\rho^2 a^9}{\mu^2}. \qquad [4.28]$$

Now, the r- and θ-components of [4.23] lead to an equation of the same form as [4.6] with $p^{(1)}$ and $\psi^{(1)}$ replaced by $p^{(3)}$ and $\psi^{(3)}$, respectively. Thus, since the problems governing ($p^{(1)}$, $\psi^{(1)}$) and ($p^{(3)}$, $\psi^{(3)}$) are the same, we see by analogy that $\psi^{(3)} \equiv c^{(3)}$ and $p^{(3)}$ = constant for all $r \geqslant a$, where $c^{(3)}$ was introduced in [3.7]. Therefore, at third order the complete solution for $\boldsymbol{u}^{(3)}$ is

$$\boldsymbol{u}^{(3)} = (H(r)\sin\theta + K(r)\sin\theta\cos^2\theta)\,\boldsymbol{e}_\phi, \quad [4.29]$$

with H and K given by [4.27] and [4.28].

It follows from [4.29] that $\boldsymbol{u}^{(3)}$ has no contribution to the secondary flow cell found at second order, but rather provides for an adjustment in the tangential ϕ-directional flow. It is clear, therefore, that at least a fourth order problem must be considered in order to study the equatorial toroidal and polar cap secondary flow that *Giesekus* observed.

4.4. Fourth order problem

For $k = 4$, the governing equations, i.e., [3.3], [3.4], [2.9], and [2.11], yield

$$-\operatorname{grad} p^{(4)} + \operatorname{div} S^{(4)} = \rho[4(\operatorname{grad}\boldsymbol{u}^{(3)})\boldsymbol{u}^{(1)} + 4(\operatorname{grad}\boldsymbol{u}^{(1)})\boldsymbol{u}^{(3)} + 6(\operatorname{grad}\boldsymbol{u}^{(2)})\boldsymbol{u}^{(2)}], \quad [4.30]$$

where

$$A_1^{(4)} = \operatorname{grad}\boldsymbol{u}^{(4)} + (\operatorname{grad}\boldsymbol{u}^{(4)})^{\mathrm{T}},$$

$$A_2^{(4)} = 4(\operatorname{grad}A_1^{(3)})\boldsymbol{u}^{(1)} + 4[A_1^{(3)}\operatorname{grad}\boldsymbol{u}^{(1)} + (A_1^{(3)}\operatorname{grad}\boldsymbol{u}^{(1)})^{\mathrm{T}}] + 6(\operatorname{grad}A_1^{(2)})\boldsymbol{u}^{(2)} + 6[A_1^{(2)}\operatorname{grad}\boldsymbol{u}^{(2)} + (A_1^{(2)}\operatorname{grad}\boldsymbol{u}^{(2)})^{\mathrm{T}}] + 4(\operatorname{grad}A_1^{(1)})\boldsymbol{u}^{(3)} + 4[A_1^{(1)}\operatorname{grad}\boldsymbol{u}^{(3)} + (A_1^{(1)}\operatorname{grad}\boldsymbol{u}^{(3)})^{\mathrm{T}}],$$

$$A_3^{(4)} = 4(\operatorname{grad}A_2^{(3)})\boldsymbol{u}^{(1)} + 4[A_2^{(3)}\operatorname{grad}\boldsymbol{u}^{(1)} + (A_2^{(3)}\operatorname{grad}\boldsymbol{u}^{(1)})^{\mathrm{T}}] + 6(\operatorname{grad}A_2^{(2)})\boldsymbol{u}^{(2)} + 6[A_2^{(2)}\operatorname{grad}\boldsymbol{u}^{(2)} + (A_2^{(2)}\operatorname{grad}\boldsymbol{u}^{(2)})^{\mathrm{T}}],$$

$$A_4^{(4)} = 4(\operatorname{grad}A_3^{(3)})\boldsymbol{u}^{(1)} + 4[A_3^{(3)}\operatorname{grad}\boldsymbol{u}^{(1)} + (A_3^{(3)}\operatorname{grad}\boldsymbol{u}^{(1)})^{\mathrm{T}}], \quad [4.31]$$

with $A_1^{(1)}$, $A_i^{(2)}$ $(i = 1, 2)$, $A_i^{(3)}$ $(i = 1, 2, 3)$, $\boldsymbol{u}^{(1)}$, $\boldsymbol{u}^{(2)}$, and $\boldsymbol{u}^{(3)}$ given by [4.2]$_1$, [4.11], [4.20], [4.9], [4.17], [4.16], [4.29], [4.27] and [4.28], and where

$$S^{(4)} = S_1^{(4)} + S_2^{(4)} + S_3^{(4)} + S_4^{(4)}, \quad [4.32]$$

with

$$S_1^{(4)} = \mu A_1^{(4)},$$
$$S_2^{(4)} = \alpha_1 A_2^{(4)} + \alpha_2(4A_1^{(3)}A_1^{(1)} + 6A_1^{(2)}A_1^{(2)} + 4A_1^{(1)}A_1^{(3)}),$$
$$S_3^{(4)} = \beta_1 A_3^{(4)} + \beta_2(4A_2^{(3)}A_1^{(1)} + 4A_1^{(1)}A_2^{(3)} + 6A_2^{(2)}A_1^{(2)} + 6A_1^{(2)}A_2^{(2)}) + \beta_3[4(\mathrm{tr}A_2^{(3)})A_1^{(1)} + 6(\mathrm{tr}A_2^{(2)})A_1^{(2)}],$$
$$S_4^{(4)} = \gamma_1 A_4^{(4)} + \gamma_2(A_3^{(3)}A_1^{(1)} + A_1^{(1)}A_3^{(3)}) + 6\gamma_3 A_2^{(2)}A_2^{(2)} + 24\gamma_4 A_2^{(2)}A_1^{(1)}A_1^{(1)} + 6\gamma_5(\mathrm{tr}A_2^{(2)})A_2^{(2)} + 12\gamma_6(\mathrm{tr}A_2^{(2)})A_1^{(1)}A_1^{(1)}. \qquad [4.33]$$

If we now follow the procedure outlined at the first three orders, and introduce the representation [3.6] for $k = 4$ and the results already found for $\boldsymbol{u}^{(1)}$, $\boldsymbol{u}^{(2)}$, and $\boldsymbol{u}^{(3)}$ in [4.9], [4.17] and [4.16], and [4.29], [4.27] and [4.28] into the system of equations [4.30 – 33], we will obtain a linear system of partial differential equations for $p^{(4)}$, $\psi^{(4)}$ and $u_\phi^{(4)}$ similar to that given in [4.23] at third order but with a right hand side having only r- and θ-components. Since the generation of this system requires a fair amount of computational effort and time, we shall not record the intermediate details here. It does follow, however, since $p^{(4)}$, $\psi^{(4)}$, and $u_\phi^{(4)}$ are independent of ϕ, that the ϕ-component of the generated system contains only $u_\phi^{(4)}$ and is identical in form to [4.4]. Thus, with the homogeneous boundary conditions for $u_\phi^{(4)}$ given in [3.7], it follows that $u_\phi^{(4)} \equiv 0$ for all $r \geqslant a$.

Again, following a procedure outlined earlier, if we now eliminate $p^{(4)}$ from the generated system by applying the curl operator to it, we shall obtain the following partial differential equation for $\psi^{(4)}$:

$$\frac{\mu}{r\sin\theta}\mathscr{L}^2\psi^{(4)} = g_1(r)\sin\theta\cos\theta + g_2(r)\sin\theta\cos^3\theta, \qquad [4.34]$$

where $g_1(r)$ and $g_2(r)$ are given by

$$g_1(r) = \rho\left[-8a^3\left(\frac{H'}{r^3} - 4\frac{H}{r^4} + 2\frac{K}{r^4}\right) + 6\left(\frac{FF'''}{r^3} - \frac{F'F''}{r^3} - 2\frac{FF''}{r^4} + \frac{F^2}{r^6}\right)\right]$$
$$+ [311040(\gamma_3 + \gamma_4 + \gamma_5) + 217728\gamma_6]\frac{a^{13}}{r^{14}} - 24a^3\alpha_1\left(-2\frac{K''}{r^4} + 24\frac{K}{r^6} + 4\frac{H''}{r^4}\right.$$
$$\left. - 8\frac{H'}{r^5} + 8\frac{H}{r^6}\right) - 24a^3\alpha_2\left(-\frac{K''}{r^4} + 2\frac{K'}{r^5} + 10\frac{K}{r^6} + 3\frac{H''}{r^4} - 10\frac{H'}{r^5} + 10\frac{H}{r^6}\right)$$
$$-6\alpha_1\left(\frac{FF^{\mathrm{v}}}{r^3} - \frac{F'F^{\mathrm{iv}}}{r^3} - 4\frac{FF^{\mathrm{iv}}}{r^4} - 4\frac{F'F'''}{r^4} + 12\frac{F'F''}{r^3} + 96\frac{FF''}{r^6} - 288\frac{FF'}{r^7}\right.$$
$$\left. + 144\frac{F^2}{r^8}\right) - 6\alpha_2\left(-2\frac{FF^{\mathrm{iv}}}{r^4} - 4\frac{F'F'''}{r^4} + 12\frac{FF'''}{r^5} + 24\frac{FF''}{r^6} + 24\frac{F'^2}{r^6} - 168\frac{FF'}{r^7}\right.$$
$$\left. + 144\frac{F^2}{r^8}\right) + 216a^6\beta_1\left(3\frac{F''}{r^9} - 42\frac{F'}{r^{10}} + 78\frac{F}{r^{11}}\right) - 216a^6\beta_2\left(\frac{F^{\mathrm{iv}}}{r^7} - 12\frac{F'''}{r^8}\right.$$
$$\left. + 40\frac{F''}{r^9} + 52\frac{F'}{r^{10}}\right) - 216a^6\beta_3\left(\frac{F^{\mathrm{iv}}}{r^7} - 12\frac{F'''}{r^8} + 40\frac{F''}{r^9} + 52\frac{F'}{r^{10}} + 60\frac{F}{r^{11}}\right), \qquad [4.35a]$$

$$g_2(r) = \rho\left[8a^3\left(-\frac{K'}{r^3} + 6\frac{K}{r^4}\right) - 6\left(3\frac{FF'''}{r^3} - \frac{F'F''}{r^3} - 6\frac{FF''}{r^4} - 12\frac{FF'}{r^5} + 72\frac{F^2}{r^6}\right)\right]$$

$$- [311040(\gamma_3 + \gamma_4 + \gamma_5) + 217728\,\gamma_6]\frac{a^{13}}{r^{14}} - 48a^3\alpha_1\left(3\frac{K''}{r^4} - 4\frac{K'}{r^5} - 16\frac{K}{r^6}\right)$$

$$- 48a^3\alpha_2\left(2\frac{K''}{r^4} - 6\frac{K'}{r^5} - 4\frac{K}{r^6}\right) + 6\alpha_1\left(3\frac{FF^{\text{v}}}{r^3} + \frac{F'F^{\text{iv}}}{r^3} + 4\frac{F''F'''}{r^3} - 12\frac{FF^{\text{iv}}}{r^4}\right.$$

$$\left. - 20\frac{F'F'''}{r^4} - 8\frac{F''^2}{r^4} - 4\frac{F'F''}{r^5} + 336\frac{FF''}{r^6} + 144\frac{F'^2}{r^6} - 1152\frac{FF'}{r^7} + 720\frac{F^2}{r^8}\right)$$

$$+ 6\alpha_2\left(2\frac{F'F^{\text{iv}}}{r^3} + 4\frac{F''F'''}{r^3} - 6\frac{FF^{\text{iv}}}{r^4} + 20\frac{F'F'''}{r^4} - 8\frac{F''^2}{r^4} + 36\frac{FF'''}{r^5} - 16\frac{F'F''}{r^5}\right.$$

$$\left. + 120\frac{FF''}{r^6} + 168\frac{F'^2}{r^6} - 792\frac{FF'}{r^7} + 720\frac{F^2}{r^8}\right) - 216a^6\beta_1\left(3\frac{F''}{r^9} - 90\frac{F'}{r^{10}} + 342\frac{F}{r^{11}}\right)$$

$$+ 216a^6\beta_2\left(\frac{F^{\text{iv}}}{r^7} - 12\frac{F'''}{r^8} + 38\frac{F''}{r^9} + 132\frac{F'}{r^{10}} - 252\frac{F}{r^{11}}\right)$$

$$+ 216a^6\beta_3\left(\frac{F^{\text{iv}}}{r^7} - 12\frac{F'''}{r^8} + 36\frac{F''}{r^9} + 132\frac{F'}{r^{10}} - 60\frac{F}{r^{11}}\right), \qquad [4.35b]$$

where F, H and K are given by [4.16], [4.27], and [4.28].

Due to the particular structure of the right hand side of [4.34] we shall assume that the solution $\psi^{(4)}$ has the form

$$\psi^{(4)} = M(r)\sin^2\theta\cos^3\theta + N(r)\sin^2\theta\cos\theta + \text{const.}, \qquad [4.36]$$

where $M(r)$ and $N(r)$ are yet to be determined. While the boundary conditions [3.7] for $\psi^{(4)}$ require $M(r)$ and $N(r)$ to satisfy

$$M(a) = M'(a) = N(a) = N'(a) = 0, M(r) = N(r) = o(r^2), \text{as } r \to \infty, \qquad [4.37]$$

and the added constant term in [4.36] to be taken as $c^{(4)}$, we also find, by substituting [4.36] into [4.34] and making use of [4.8], that $M(r)$ and $N(r)$ must satisfy the following ordinary differential equations:

$$M^{\text{iv}} - \frac{40}{r^2}M'' + \frac{80}{r^3}M' + \frac{280}{r^4}M = rg_2(r),$$

$$N^{\text{iv}} - 12\frac{N''}{r^2} + 24\frac{N'}{r^3} + 12\frac{M''}{r^2} - 24\frac{M'}{r^3} - 120\frac{M}{r^4} = rg_1(r). \qquad [4.38]$$

Thus, since F, H and K are known functions of r, then so too are g_1 and g_2, and it is a straightforward, albeit lengthy, matter to construct $M(r)$ and $N(r)$ which meet [4.38] and [4.37]. We obtain

$$M(r) = \sum_{i=1}^{9}\frac{m_i}{r^i} + \left(\ln\frac{r}{a}\right)\sum_{i=2}^{5}\frac{m_i'}{r^i},$$

$$N(r) = \sum_{i=0}^{9}\frac{n_i}{r^i} + \left(\ln\frac{r}{a}\right)\sum_{i=2}^{5}\frac{n_i'}{r^i}, \qquad [4.39]$$

where the constants m_i, m_i', n_i and n_i' are given by

$$m_1 = -\frac{3}{16}\frac{\rho^3 a^{10}}{\mu^3} + \frac{3}{2}\frac{\rho^2 a^8}{\mu^3}(\alpha_1+\alpha_2) - 3\frac{\rho a^6}{\mu^3}(\alpha_1+\alpha_2)^2,$$

$$m_3 = -\frac{2043}{1120}\frac{\rho^3 a^{12}}{\mu^3} - \frac{21}{8}\frac{\rho^2 a^{10}}{\mu^3}(\alpha_1+\alpha_2) + \frac{342}{11}\frac{\rho a^8}{\mu^3}(\alpha_1+\alpha_2)^2 - 54\frac{a^6}{\mu^3}(\alpha_1+\alpha_2)^3 + \frac{486}{11}\frac{\rho a^8}{\mu^2}(\beta_2+\beta_3),$$

$$m_5 = \frac{4}{5}\frac{\rho^2 a^{12}}{\mu^3}(\alpha_1+\alpha_2) + \frac{307}{70}\frac{\rho^2 a^{12}}{\mu^3}\alpha_1 + 42\frac{\rho a^{10}}{\mu^3}(\alpha_1+\alpha_2)\alpha_1 - \frac{4016}{11}\frac{a^8}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_1 + \frac{3888}{11}\frac{a^8}{\mu^2}(\beta_2+\beta_3)(\alpha_1+\alpha_2) + \frac{517}{70}\frac{\rho^2 a^{12}}{\mu^3}\alpha_2 + 6\frac{\rho a^{10}}{\mu^3}(\alpha_1+\alpha_2)\alpha_2 + \frac{360}{11}\frac{a^8}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_2,$$

$$m_6 = -\frac{3}{13}\frac{\rho^2 a^{13}}{\mu^3}(\alpha_1+\alpha_2) - \frac{36}{13}\frac{\rho a^{11}}{\mu^3}(\alpha_1+\alpha_2)^2 - \frac{1479}{1144}\frac{\rho^2 a^{13}}{\mu^3}\alpha_1 - \frac{3015}{143}\frac{\rho a^{11}}{\mu^3}\cdot(\alpha_1+\alpha_2)\alpha_1 + \frac{3186}{11}\frac{a^9}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_1 - \frac{2793}{1144}\frac{\rho^2 a^{13}}{\mu^3}\alpha_2 - \frac{7902}{143}\frac{\rho a^{11}}{\mu^3}(\alpha_1+\alpha_2)\alpha_2 + \frac{1350}{13}\frac{a^9}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_2 - \frac{4617}{286}\frac{\rho a^{11}}{\mu^2}\beta_1 + \frac{9234}{143}\frac{a^9}{\mu^2}(\alpha_1+\alpha_2)\beta_1 - \frac{1701}{143}\frac{\rho a^{11}}{\mu^2}\beta_2 + \frac{6804}{143}\frac{a^9}{\mu^2}(\alpha_1+\alpha_2)\beta_2 - \frac{405}{143}\frac{\rho a^{11}}{\mu^2}\beta_3 + \frac{1620}{143}\frac{a^9}{\mu^2}(\alpha_1+\alpha_2)\beta_3,$$

$$m_7 = \frac{312}{385}\frac{\rho a^{12}}{\mu^3}(\alpha_1+\alpha_2)^2 + \frac{108}{55}\frac{\rho a^{12}}{\mu^2}(\beta_2+\beta_3) + \frac{3}{14}\frac{\rho^2 a^{14}}{\mu^3}\alpha_1 + \frac{278}{35}\frac{\rho a^{12}}{\mu^3}(\alpha_1+\alpha_2)\alpha_1 + \frac{216}{7}\frac{a^{10}}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_1 + \frac{12}{7}\frac{\rho a^{12}}{\mu^3}(\alpha_1+\alpha_2)\alpha_2 + \frac{657}{35}\frac{\rho a^{12}}{\mu^2}\beta_1 + \frac{318}{35}\frac{\rho a^{12}}{\mu^2}\beta_2 + \frac{144}{35}\frac{\rho a^{12}}{\mu^2}\beta_3,$$

$$m_8 = \frac{282}{13}\frac{\rho a^{13}}{\mu^3}(\alpha_1+\alpha_2)\alpha_1 + \frac{684}{13}\frac{a^{11}}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_1 + \frac{30}{13}\frac{\rho a^{13}}{\mu^3}(\alpha_1+\alpha_2)\alpha_2 + \frac{360}{13}\frac{a^{11}}{\mu^3}\cdot(\alpha_1+\alpha_2)^2\alpha_2 - \frac{81}{13}\frac{\rho a^{13}}{\mu^2}\beta_1 - \frac{972}{13}\frac{a^{11}}{\mu^2}\beta_1(\alpha_1+\alpha_2) + \frac{27}{13}\frac{\rho a^{13}}{\mu^2}\beta_2 + \frac{324}{13}\frac{a^{11}}{\mu^2}\cdot\beta_2(\alpha_l+\alpha_2) + \frac{45}{52}\frac{\rho a^{13}}{\mu^2}\beta_3 + \frac{540}{13}\frac{a^{11}}{\mu^2}\beta_3(\alpha_1+\alpha_2),$$

$$m_9 = -\left[\frac{1944}{49}(\gamma_3+\gamma_4+\gamma_5)+\frac{972}{35}\gamma_6\right]\frac{a^{12}}{\mu} - \frac{71496}{2695}\frac{a^{12}}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_1 + \frac{169128}{2695}\frac{a^{12}}{\mu^2}$$

$$\cdot\,\alpha_1(\beta_2+\beta_3) - \frac{16848}{2695}\frac{a^{12}}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_2 - \frac{137052}{2695}\frac{a^{12}}{\mu^2}\alpha_2(\beta_2+\beta_3) + \frac{8748}{245}\frac{a^{12}}{\mu^2}$$

$$\cdot\,(\alpha_1+\alpha_2)\beta_1 - \frac{11988}{245}\frac{a^{12}}{\mu^2}(\alpha_1+\alpha_2)\beta_2 - \frac{14256}{245}\frac{a^{12}}{\mu^2}(\alpha_1+\alpha_2)\beta_3,$$

$$m_2' = \frac{15}{28}\frac{\rho^3a^{11}}{\mu^3} - \frac{15}{7}\frac{\rho^2a^9}{\mu^3}(\alpha_1+\alpha_2),$$

$$m_3' = -\frac{3}{14}\frac{\rho^3a^{12}}{\mu^3},$$

$$m_4' = -\frac{41}{264}\frac{\rho^3a^{13}}{\mu^3} - \frac{25}{11}\frac{\rho^2a^{11}}{\mu^3}(\alpha_1+\alpha_2) + \frac{18}{11}\frac{\rho a^9}{\mu^3}(\alpha_1+\alpha_2)^2 + 7\frac{\rho^2a^{11}}{\mu^3}\alpha_1$$

$$-\frac{308}{11}\frac{\rho a^9}{\mu^3}(\alpha_1+\alpha_2)\alpha_1 + \frac{40}{11}\frac{\rho^2a^{11}}{\mu^3}\alpha_2 - \frac{160}{11}\frac{\rho a^9}{\mu^3}\alpha_2(\alpha_1+\alpha_2),$$

$$m_5' = -\frac{12}{7}\frac{\rho^2a^{12}}{\mu^3}(\alpha_1+\alpha_2),$$

$$m_2 = -\frac{3}{2}m_1a - \frac{m_2'}{2} - \frac{m_3+m_3'}{2a} - \frac{m_4'}{2a^2} + \frac{m_5-m_5'}{2a^3} + \frac{m_6}{a^4} + \frac{3}{2}\frac{m_7}{a^5} + 2\frac{m_8}{a^6} + \frac{5}{2}\frac{m_9}{a^7},$$

$$m_4 = \frac{1}{2}m_1a^3 + \frac{1}{2}m_2'a^2 - \frac{m_3'-m_3}{2}a + \frac{m_4'}{2} + \frac{m_5'-3m_5}{2a} - 2\frac{m_6}{a^2} - \frac{5}{2}\frac{m_7}{a^3} - 3\frac{m_8}{a^4} - \frac{7}{2}\frac{m_9}{a^5},$$

$$n_1 = -3m_1 - \frac{19}{50}\frac{\rho^3a^{10}}{\mu^3} + \frac{102}{35}\frac{\rho^2a^8}{\mu^3}(\alpha_1+\alpha_2) - \frac{22}{5}\frac{\rho a^6}{\mu^3}(\alpha_1+\alpha_2)^2$$

$$-\frac{288}{5}\frac{\rho a^6}{\mu^2}(\beta_2+\beta_3),$$

$$n_3 = -\frac{1}{3}m_3' - \frac{2}{3}m_3 + \frac{493}{5600}\frac{\rho^3a^{12}}{\mu^3} + \frac{17}{20}\frac{\rho^2a^{10}}{\mu^3}(\alpha_1+\alpha_2) - \frac{186}{55}\frac{\rho a^8}{\mu^3}(\alpha_1+\alpha_2)^2$$

$$-\frac{2187}{220}\frac{\rho a^8}{\mu^2}(\beta_2+\beta_3) - \frac{83}{200}\frac{\rho^2a^{10}}{\mu^3}\alpha_1 + \frac{101}{35}\frac{\rho a^8}{\mu^3}(\alpha_1+\alpha_2)\alpha_1 + \frac{2}{5}\frac{a^6}{\mu^3}$$

$$\cdot\,(\alpha_1+\alpha_2)^2\alpha_1 - \frac{1152}{5}\frac{a^6}{\mu}(\beta_2+\beta_3)\alpha_1 - \frac{267}{17500}\frac{\rho^2a^{10}}{\mu^3}\alpha_2 - \frac{8}{105}\frac{\rho a^8}{\mu^3}(\alpha_1+\alpha_2)\alpha_2$$

$$+\frac{156}{875}\frac{a^6}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_2 - \frac{1728}{125}\frac{a^6}{\mu^2}(\beta_2+\beta_3)\alpha_2,$$

$$n_4 = -\frac{33}{196}m_4' - \frac{3}{7}m_4 - \frac{11}{672}\frac{\rho^3a^{13}}{\mu^3} - \frac{1}{4}\frac{\rho^2a^{11}}{\mu^3}(\alpha_1+\alpha_2) + \frac{3}{14}\frac{\rho a^9}{\mu^3}(\alpha_1+\alpha_2)^2$$

$$+\frac{23}{28}\frac{\rho^2a^{11}}{\mu^3}\alpha_1 - \frac{23}{7}\frac{\rho a^9}{\mu^3}\alpha_1(\alpha_1+\alpha_2) + \frac{3}{7}\frac{\rho^2a^{11}}{\mu^3}\alpha_2 - \frac{12}{7}\frac{\rho a^9}{\mu^3}(\alpha_1+\alpha_2)\alpha_2,$$

$$n_5 = -\frac{39}{400}m_5' - \frac{3}{10}m_5 - \frac{1}{7}\frac{\rho^2 a^{12}}{\mu^3}(\alpha_1+\alpha_2) - \frac{53}{175}\frac{\rho^2 a^{12}}{\mu^3}\alpha_1 - \frac{33}{25}\frac{\rho a^{10}}{\mu^3}(\alpha_1+\alpha_2)\alpha_1$$
$$+\frac{612}{55}\frac{a^8}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_1 - \frac{13608}{55}\frac{a^8}{\mu^2}\alpha_1(\beta_2+\beta_3) + \frac{227}{1400}\frac{\rho^2 a^{12}}{\mu^3}\alpha_2 - \frac{12}{5}\frac{\rho a^{10}}{\mu^3}$$
$$\cdot\alpha_2(\alpha_1+\alpha_2) + \frac{414}{55}\frac{a^8}{\mu^3}\alpha_2(\alpha_1+\alpha_2)^2 - \frac{1944}{55}\frac{a^8\alpha_2}{\mu^2}(\beta_2+\beta_3),$$

$$n_6 = -\frac{2}{9}m_6 + \frac{13}{88}\frac{\rho^2 a^{13}}{\mu^3}\alpha_1 + \frac{50}{33}\frac{\rho a^{11}}{\mu^3}(\alpha_1+\alpha_2)\alpha_1 + \frac{10}{11}\frac{a^9}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_1 + \frac{29}{264}$$
$$\cdot\frac{\rho^2 a^{13}}{\mu^3}\alpha_2 + \frac{27}{11}\frac{\rho a^{11}}{\mu^3}\alpha_2(\alpha_1+\alpha_2) - \frac{50}{11}\frac{a^9}{\mu^3}\alpha_2(\alpha_1+\alpha_2)^2 + \frac{39}{22}\frac{\rho a^{11}}{\mu^2}\beta_1$$
$$-\frac{78}{11}\frac{a^9}{\mu^2}\beta_1(\alpha_1+\alpha_2) - \frac{46}{11}\frac{\rho a^{11}}{\mu^2}\beta_3 + \frac{184}{11}\frac{a^9}{\mu^2}\beta_3(\alpha_1+\alpha_2),$$

$$n_7 = -\frac{6}{35}m_7 - \frac{536}{1925}\frac{\rho a^{12}}{\mu^3}(\alpha_1+\alpha_2)^2 - \frac{2304}{1925}\frac{\rho a^{12}}{\mu^2}(\beta_2+\beta_3) + \frac{426}{175}\frac{\rho a^{12}}{\mu^3}\alpha_1(\alpha_1+\alpha_2)$$
$$+\frac{24}{25}\frac{\rho a^{12}}{\mu^3}(\alpha_1+\alpha_2)\alpha_2 - \frac{567}{175}\frac{\rho a^{12}}{\mu^2}\beta_1 + \frac{558}{175}\frac{\rho a^{12}}{\mu^2}\beta_2 + \frac{828}{175}\frac{\rho a^{12}}{\mu^2}\beta_3,$$

$$n_8 = \frac{3}{22}m_8 - \frac{387}{286}\frac{\rho a^{13}}{\mu^3}\alpha_1(\alpha_1+\alpha_2) - \frac{2322}{143}\frac{a^{11}}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_1 - \frac{126}{143}\frac{\rho a^{13}}{\mu^3}\alpha_2(\alpha_1+\alpha_2)$$
$$+\frac{1512}{143}\frac{a^{11}}{\mu^3}\alpha_2(\alpha_1+\alpha_2)^2 + \frac{405}{286}\frac{\rho a^{13}}{\mu^2}\beta_1 + \frac{2430}{143}\frac{a^{11}}{\mu^2}\beta_1(\alpha_1+\alpha_2) - \frac{612}{143}\frac{\rho a^{13}}{\mu^2}\beta_2$$
$$-\frac{7344}{143}\frac{a^{11}}{\mu^2}\beta_2(\alpha_1+\alpha_2) - \frac{1359}{286}\frac{\rho a^{13}}{\mu^2}\beta_3 - \frac{8154}{143}\frac{a^{11}}{\mu^2}(\alpha_1+\alpha_2)\beta_3,$$

$$n_9 = -\frac{m_9}{9} + \left[\frac{1440}{49}(\gamma_3+\gamma_4+\gamma_5) + \frac{144}{7}\gamma_6\right]\frac{a^{12}}{\mu} + \frac{5008}{539}\frac{a^{12}}{\mu^3}\alpha_1(\alpha_1+\alpha_2)^2 + \frac{23040}{539}\frac{a^{12}}{\mu^2}$$
$$\cdot\alpha_1(\beta_2+\beta_3) + \frac{3688}{539}\frac{a^{12}}{\mu^3}(\alpha_1+\alpha_2)^2\alpha_2 + \frac{18288}{539}\frac{a^{12}}{\mu^2}\alpha_2(\beta_2+\beta_3) - \frac{480}{49}\frac{a^{12}}{\mu^2}$$
$$\cdot(\alpha_1+\alpha_2)\beta_1 + \frac{936}{49}\frac{a^{12}}{\mu^2}\beta_2(\alpha_1+\alpha_2) + \frac{2928}{49}\frac{a^{12}}{\mu^2}(\alpha_1+\alpha_2)\beta_3,$$

$$n_2' = -\frac{6}{5}m_2' + \frac{9}{28}\frac{\rho^3 a^{11}}{\mu^3} - \frac{9}{7}\frac{\rho^2 a^9}{\mu^3}(\alpha_1+\alpha_2),$$

$$n_3' = -\frac{2}{3}m_3' - \frac{3}{70}\frac{\rho^3 a^{12}}{\mu^3},$$

$$n_4' = -\frac{3}{7}m_4',$$

$$n_5' = -\frac{3}{10}m_5' + \frac{6}{35}\frac{\rho^2 a^{12}}{\mu^3}(\alpha_1+\alpha_2),$$

$$n_0 = -\frac{n_1}{2a} - \frac{n_2'}{2a^2} + \frac{n_3 - n_3'}{2a^3} + \frac{2n_4 - n_4'}{2a^4} + \frac{3n_5 - n_5'}{2a^5} + 2\frac{n_6}{a^6} + \frac{5}{2}\frac{n_7}{a^7} + 3\frac{n_8}{a^8} + \frac{7n_9}{2a^9},$$

$$n_2 = -\frac{an_1}{2} + \frac{n_2'}{2} - \frac{3n_3 - n_3'}{2a} - \frac{4n_4 - n_4'}{2a^2} - \frac{5n_5 - n_5'}{2a^3} - 3\frac{n_6}{a^4} - \frac{7n_7}{2a^5} - 4\frac{n_8}{a^6} - \frac{9n_9}{2a^7}. \quad [4.40]$$

Finally, by returning to the representation [3.6] for $k = 4$, we have shown that at fourth order the complete solution for $\boldsymbol{u}^{(4)}$ is

$$\boldsymbol{u}^{(4)} = -\operatorname{curl}\left(\frac{\psi^{(4)}}{r\sin\theta}\boldsymbol{e}_\phi\right), \quad [4.41]$$

where $\psi^{(4)}$ is given in [4.36], [4.39] and [4.40]. We shall not give the computation of $p^{(4)}$ here.

5. Discussion

In the special theory of steady viscometric flows of incompressible simple fluids it is well known, see p. 495 of (11), that the fourteen material constants which appear in the fourth order approximation [2.9] can occur only in the six combinations μ, $2\alpha_1 + \alpha_2$, α_2, $\beta_2 + \beta_3$, $\gamma_3 + \gamma_4 + \gamma_5 + \frac{1}{2}\gamma_6$, and γ_6. Thus, experiments concerning viscometric flows can at best determine these six material properties among the fourteen possibilities. Since our solution for $\boldsymbol{u}^{(4)}$ exhibits a dependence on material constants outside of this set, it may be possible to appraise the significance of the non-viscometric material constants in an experimental program by comparing certain characteristics of the computed flow field, assuming that all non-viscometric material constants are zero, to the flow field which actually is observed. In particular, we shall consider here only one feature of the flow field; the shape of the cells of secondary flow. As mentioned earlier, *Giesekus* (5) has shown convincing experimental evidence that when the cells are present they have the shape of an equatorial torus or perhaps polar caps, possibly attached to the turning sphere.

Recalling $[3.3]_1$ and [3.6], we have shown that

$$\boldsymbol{u} = \Omega u_\phi^{(1)}\boldsymbol{e}_\phi - \frac{1}{2}\Omega^2\operatorname{curl}\left(\frac{\psi^{(2)}}{r\sin\theta}\boldsymbol{e}_\phi\right) + \frac{1}{6}\Omega^3 u_\phi^{(3)}\boldsymbol{e}_\phi - \frac{1}{24}\Omega^4\operatorname{curl}\left(\frac{\psi^{(4)}}{r\sin\theta}\boldsymbol{e}_\phi\right) + o(\Omega^4), \quad [5.1]$$

where $u_\phi^{(1)}$ is given in [4.9], $\psi^{(2)}$ is given in [4.16], $u_\phi^{(3)}$ is given in [4.29], [4.27], and [4.28], and where $\psi^{(4)}$ is given in [4.36], [4,39] and [4.40]. Thus, it is clear that the r- and θ-components of the fluid motion, i.e., those components which associate with the possible cells of secondary flow, are solely dependent upon the stream function

$$\psi = \frac{1}{2}\Omega^2\left(\psi^{(2)} + \frac{1}{12}\Omega^2\psi^{(4)}\right) + o(\Omega^4), \quad [5.2]$$

which, by [4.16] and [4.36], may be written as

$$\psi = \frac{1}{2}\Omega^2\left[F(r) + \frac{1}{12}\Omega^2(N(r) + M(r)\cos^2\theta)\right] \cdot \sin^2\theta\cos\theta + o(\Omega^4). \quad [5.3]$$

Thus, if we set $\psi \equiv 0$ in [5.3] and neglect the higher order terms $o(\Omega^4)$, we then obtain an approximation to the boundary of the possible secondary flow cell (or cells), i.e.,

$$F(r) + \frac{N(r)}{12}\Omega^2 + \frac{M(r)}{12}\Omega^2\cos^2\theta = 0. \quad [5.4]$$

Clearly, since $F(a) = M(a) = N(a) = 0$, this relation is satisfied for all θ at $r = a$, i.e., at the boundary of the turning sphere. Also, for $M(r) \neq 0$ and Ω prescribed, whenever $\theta(r; \Omega)$ exists it is given by

$$\cos^2\theta = -\left\{\frac{12F(r) + \Omega^2 N(r)}{\Omega^2 M(r)}\right\}, \quad [5.5]$$

where for existence we must have

$$0 \leqslant -\left\{\frac{12F(r) + \Omega^2 N(r)}{\Omega^2 M(r)}\right\} \leqslant 1. \quad [5.6]$$

In this case we see that the boundary of the region of secondary flow is symmetric with respect to the equatorial plane of the turning sphere (i.e., $\theta = \frac{\pi}{2}$) and, depending upon the functions F, N and M, and the angular velocity Ω, can even take on a form similar to that reported by *Giesekus* (5, cf., fig. 9 and 18, cf., fig. 13).

Because of our rather complicated expressions for M and N in [4.39] and [4.40], we shall not here attempt to extract any explicit general information concerning, for example, the apogee or perigee of the boundary of the secondary flow region. This information would, however, be of interest and we believe it would even be of practical significance if the solution which has been generated were to be used in combination with an experimental program which is directed toward determining information concerning the material constants. Instead, we give, in figures 2–8, some results of a few specific numerical computations. Since there is a fairly large set of material constants to specify for the purpose of computation, we shall for the most part assume

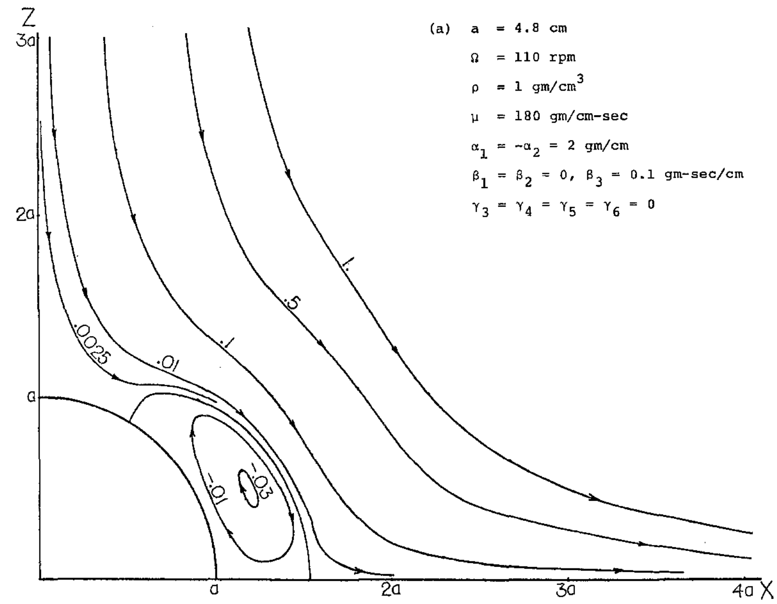


Fig. 2a

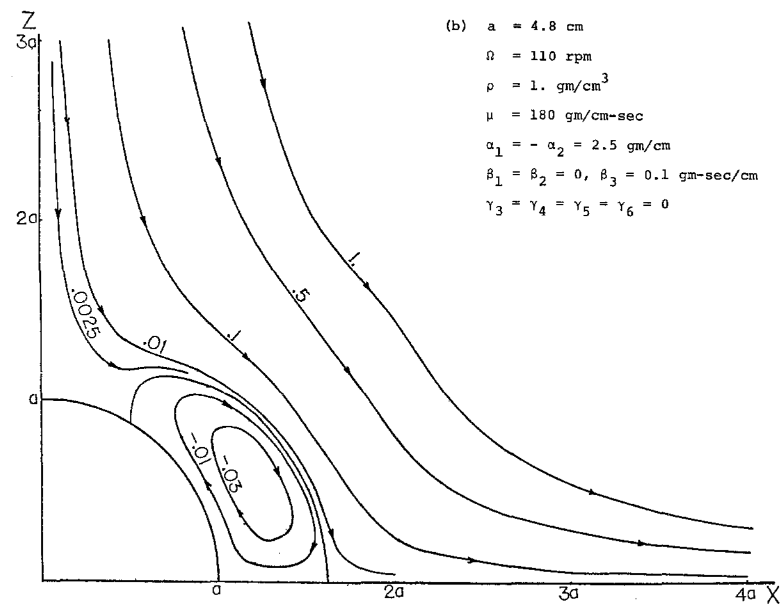


Fig. 2b

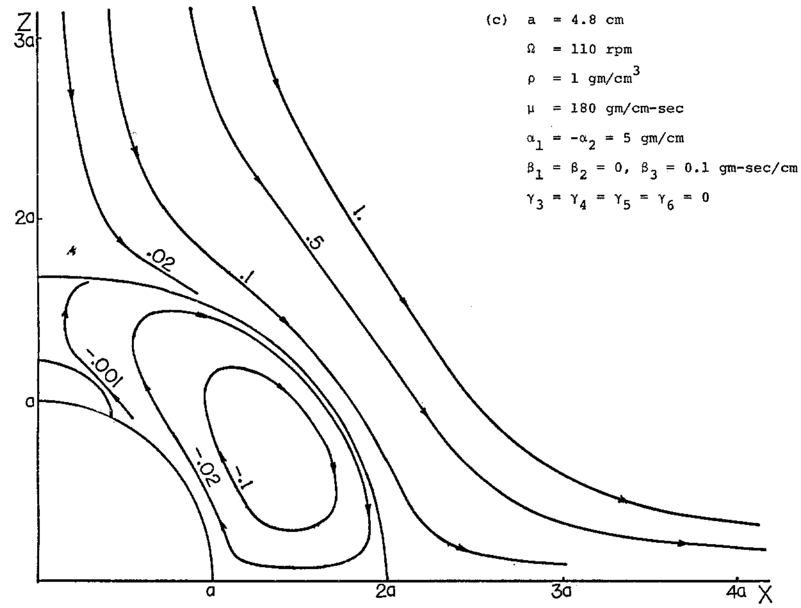


Fig. 2a – c. Level curves of ψ/Ω^2 from [5.3] for Ω = 110 rpm and three values of $\alpha_1 > 0$ with $\alpha_1 + \alpha_2 = 0$

that the fluid is essentially a fluid of grade 3, cf., (11, p. 494) or (12), and choose the material constants such that the thermodynamical constraints of *Fosdick* and *Rajagopal* (12) are satisfied. Thus, in the computations leading to figures 2 – 4 we consider particular examples in which the following conditions are met: [2])

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geqslant 0,$$
$$|\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta_3}, \quad \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 0. \qquad [5.7]$$

Since the fourth order material constants γ_1, γ_2, γ_7 and γ_8 do not enter our formulas for *F*, *N* and *M*, we need not make a specific commitment on these. For figure 5, we assume, for some variety, that $\gamma_3 \neq 0$, but we continue to satisfy the remaining conditions in [5.7]. In all of the above computations we have taken a = 4.8 cm, and Ω = 110 rpm.

In several of our example computations we have taken $\alpha_1 + \alpha_2 = 0$ which, if we applied our result [4.18] at second order, should correspond to a flow without a secondary flow region. However, it will be noted from our figures that in all cases a secondary flow region was found to be present. Thus, at second order the result [4.18] concerning the existence of an "inertial radius" must be considered, at best, suggestive, and should be applied cautiously, if at all, in drawing conclusions regarding the second order material constants α_1 and α_2. It is of interest to observe from figures 2 – 5 that the secondary flow in the toroidal cells appears to be dominated by normal stress effects while the flow outside the cells is more strongly influenced by the effects of inertia.

In figure 6 we show the pattern of secondary flow at Ω = 300 rpm for the same choice of material constants and spherical radius as was used for figure 2a where Ω = 110 rpm. Clearly, the effect of increasing the angular velocity is to cause the equatorial toroidal cell to grow and to eventually enclose the spinning sphere. A computation made at Ω = 1450 rpm shows that this trend continues as Ω increases.

In figures 7 and 8 we show the pattern of the secondary flow for choices of material constants that correspond roughly to those that could be appropriate for STP motor oil (fig. 7) and TL 227 oil (fig. 8). While the values for the viscosity μ and the normal stress coefficients α_1 and α_2 for these figures are based upon the sources (13, 14, 15), there are no rheological measurements to guide us in the choice of β_1,

[2]) Actually, only the material constants μ, α_1, α_2, β_1, β_2 and β_3 enter the constitutive assumption of an incompressible fluid of grade 3, and just those conditions in [5.7] which relate to these coefficients are shown in (12) to have a thermodynamical significance. It should be emphasized that these conditions necessarily are based on the premise that the material is *exactly* a fluid of grade 3. Therefore, they need not hold if the fluid is more complex.

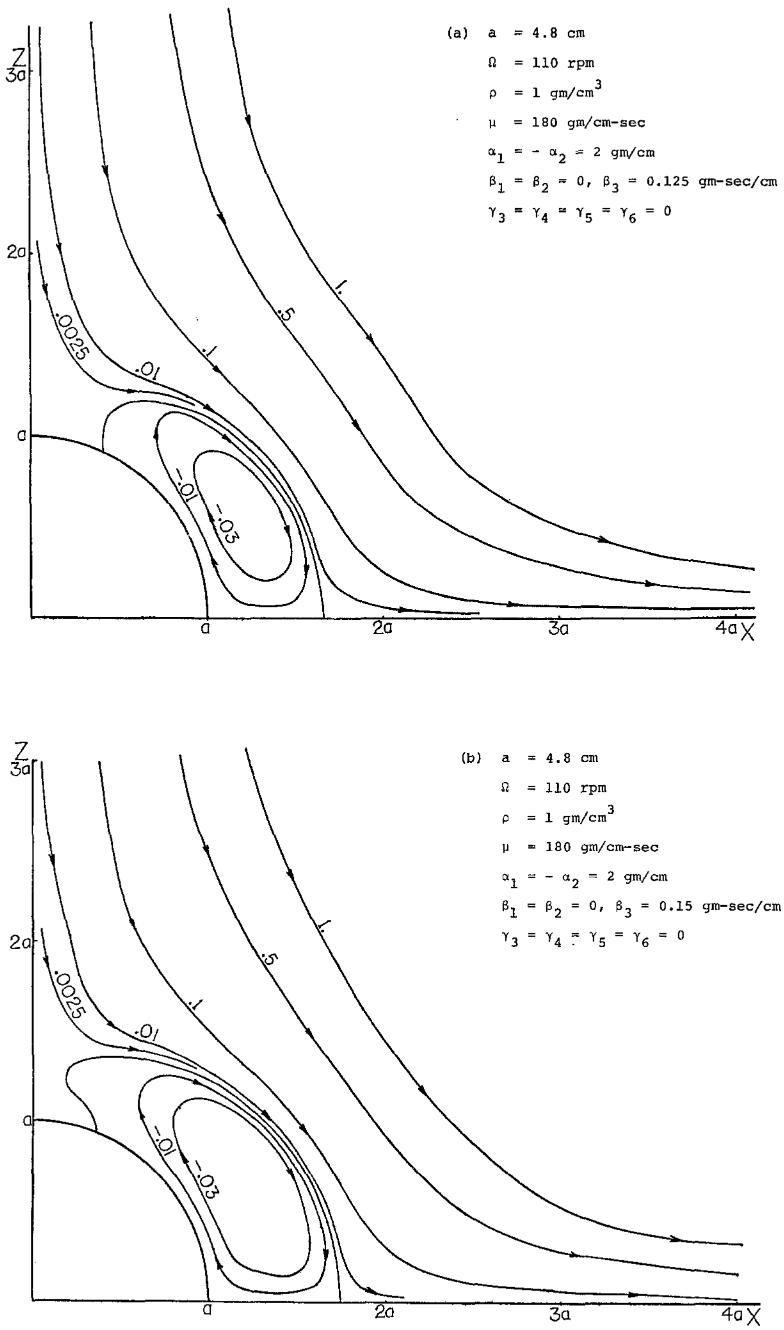


Fig. 3. Level curves of ψ/Ω^2 from [5.3] for Ω = 110 rpm and two values of β_3 with $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 > 0$

β_2, β_3, γ_3, γ_4, γ_5 and γ_6; we set all of these to zero except for β_3.

Figure 7a shows the same characteristics for the pattern of flow as was noted in figures 2–6. Since the quantity $\rho a^2/(\alpha_1 + \alpha_2)$, cf. [4.18], is large ($>12$) for STP motor oil, we expect the flow to be dominated by centrifugal force far away from the spherical surface and this is the case everywhere outside the equatorial cell. In figure 7b we show the effect of increasing the angular velocity from Ω = 110 rpm to Ω = 1450 rpm. The equatorial toroidal cell, in which the flow is effected strongly by normal stresses, has now become large and is enveloped by an ellipsoidal shaped surface outside of which the flow is dominated by inertial effects. This ellipsoidal shaped surface is not shown in figure 7b because in the scale of the figure it is too far away from the turning sphere. It should be noted that an additional secondary flow in

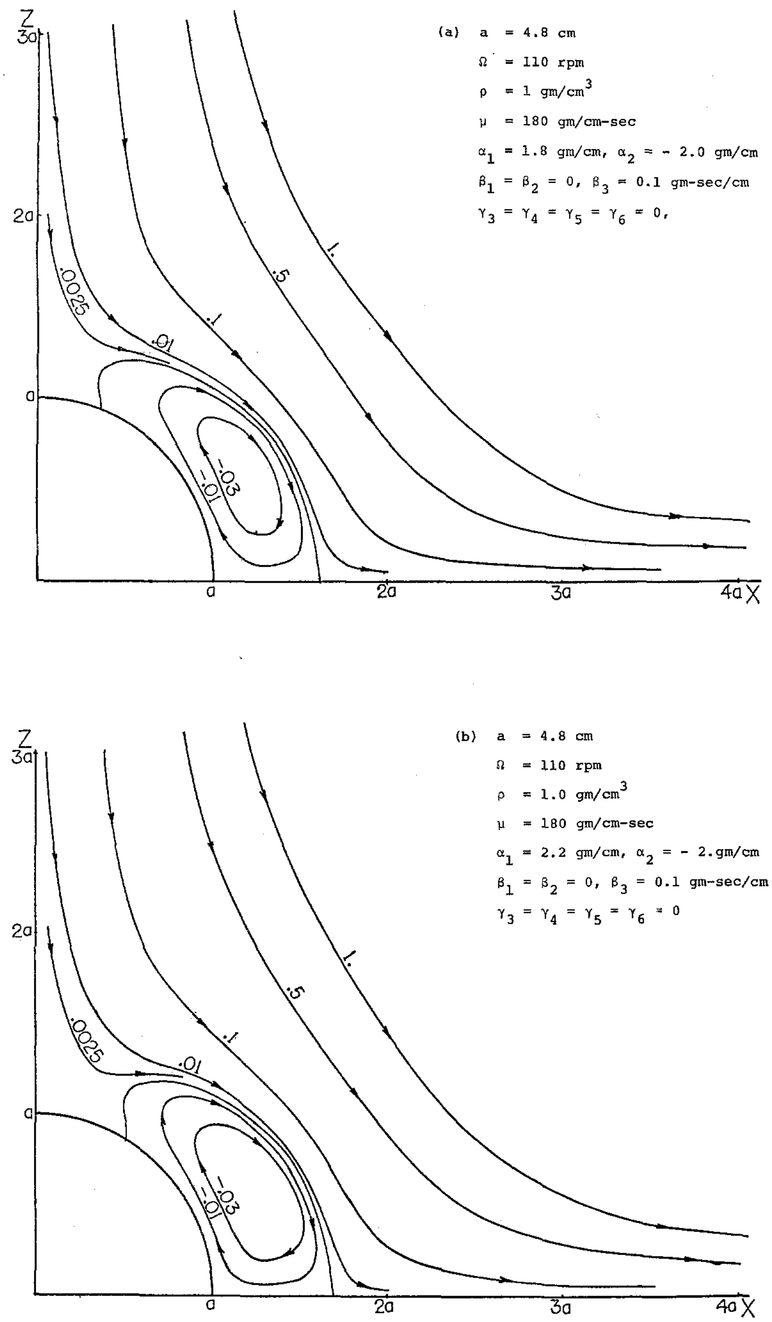


Fig. 4. Level curves of ψ/Ω^2 from [5.3] for Ω = 110 rpm and two values of $\alpha_1 > 0$ and $\alpha_2 < 0$ with $\alpha_1 + \alpha_2 \neq 0$

polar caps is now evident and that this flow also appears to be dominated by inertial effects. The general features illustrated in figure 7b have been experimentally observed by *Giesekus* (5, p. 260).

For TL 227 oil the quantity $\rho a^2/(\alpha_1 + \alpha_2)$, cf. [4.18], is small ($<4$) so we expect that far away from the spinning sphere the secondary flow should be dominated by normal stress effects. The streamline pattern for TL 227 in figure 8 has this property and the numerical computations far away from the spherical surface do not indicate that the streamline pattern has an envelope (contrary to the case in figure 7b). In addition, a secondary flow which is dominated by inertia is found in polar cap regions. Moreover, the polar caps are found to be fairly uniform in size and shape over a wide

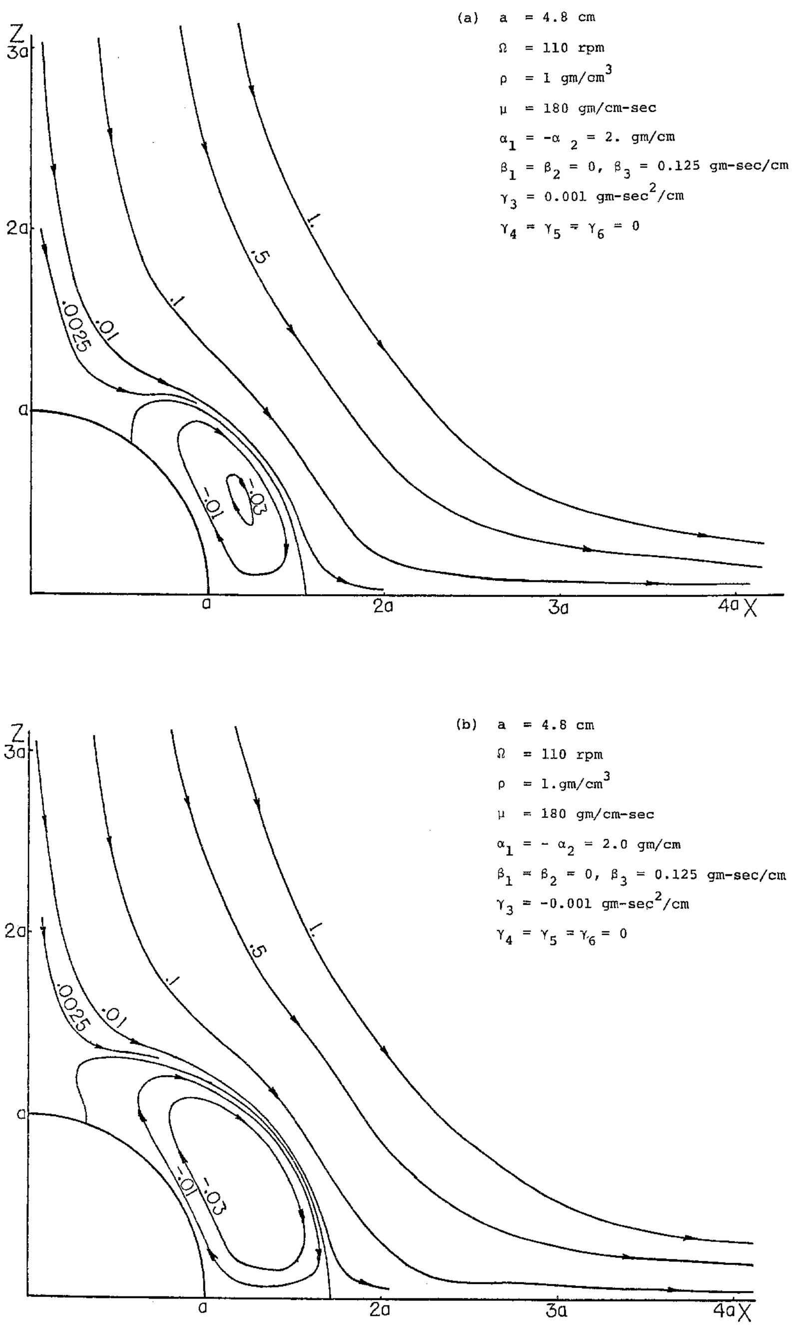


Fig. 5. Level curves of ψ/Ω^2 from [5.3] for Ω = 110 rpm and two values of γ_3 with $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 > 0$

range of angular velocities. Again, we note that *Giesekus* (5) has made general experimental observations of this nature.

Finally, we emphasize that the results given in figures 2 – 8 are intended only as examples to illustrate the type of secondary flow phenomenon that is predictable using the general incompressible simple fluid model. We make no claim as to whether or not the particular material constants used in our computations represent an existing non-Newtonian liquid. It should again be emphasized that in constructing figures 7 and 8 only the three material constants μ, α_1 and α_2 that were used had some experimental justification for STP motor oil and TL 227 oil, respectively. For all of the figures the constants were chosen and the computations were performed without special concern for the

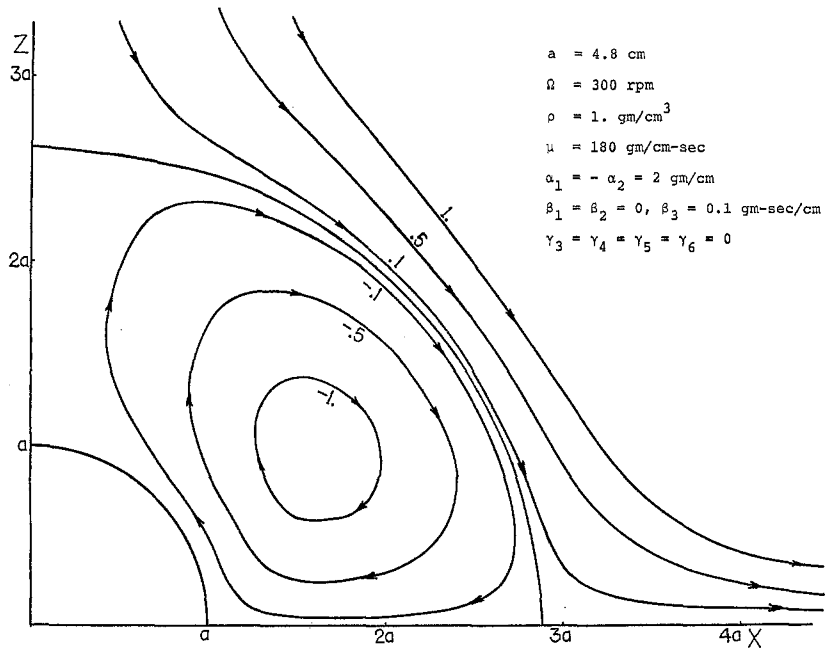


Fig. 6. Level curves of ψ/Ω^2 from [5.3] for the same material constants as in figure 2a but with higher angular velocity Ω = 300 rpm

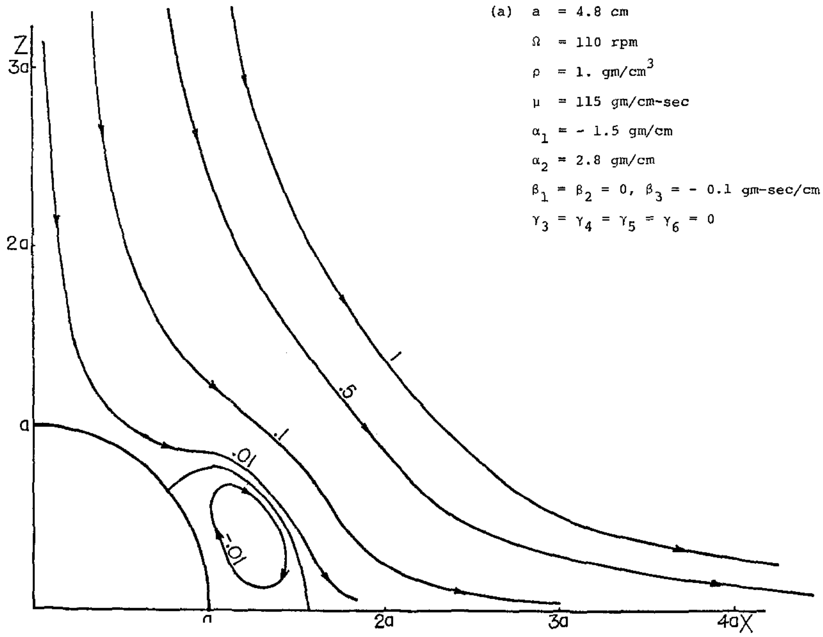
outcome, and we obtained at least in the cases of figures 7b and 8a, b, a somewhat favorable qualitative comparison with the experimental observations of *Giesekus* (5, cf., fig. 9; 17, cf., fig. 2). While our figures illustrate polar cap regions of secondary flow in which inertial forces are dominant, we do not also obtain equatorial toroidal cells that are inertial dominated. In a private communication, *Giesekus* has suggested that this omission may be due to our particular choice of material constants. While figures 2–6 may at first appear to conform to published experimental observations, we emphasize, again, that in these figures the equatorial cells are dominated by normal stresses and this is opposite to what has been observed (5, 17, 18). More detailed studies yet need to be made on the comparison between the theoretical predictions and the experimental observations. It seems likely to us that such a programm would lead to a deeper understanding of the material properties and the phenomenological behavior of non-Newtonian liquids.

Acknowledgement

This work was carried out under the support of the U.S. National Science Foundation. We thank Professor *Giesekus* for his comments on an earlier draft and for his help in directing us to some of the experimental literature.

Summary

We give a solution to the problem of the motion of a homogeneous incompressible simple fluid around a submerged sphere which is in steady rotation about a fixed axis. The solution is complete up to fourth order in the angular velocity. By way of some explicit computations we exhibit, in a series of figures, some possible streamline fields that show secondary flow regions in the shapes of equatorial tori and/or polar caps. The aim of this study is to give support to the view that many of the non-trivial secondary flow regions that are observed in the laboratory for this problem are well within theoretical description. In our computations there are several material constants to be chosen and we make no claim that all of those we have used have any correspondence with a particular fluid substance. However, certain of those in figures 7 and 8 have been taken from the experimental literature. A discussion of our results is contained in Section 5.



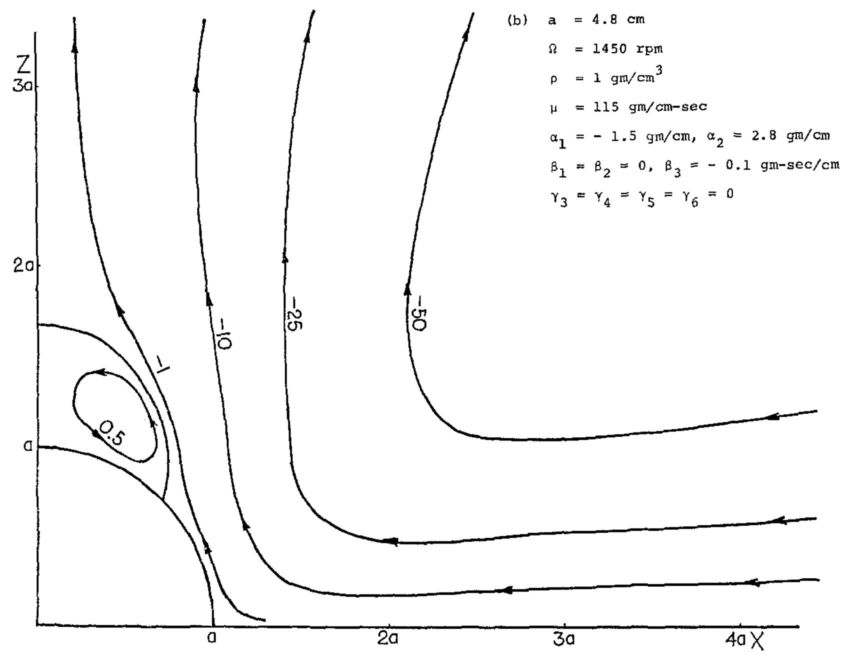
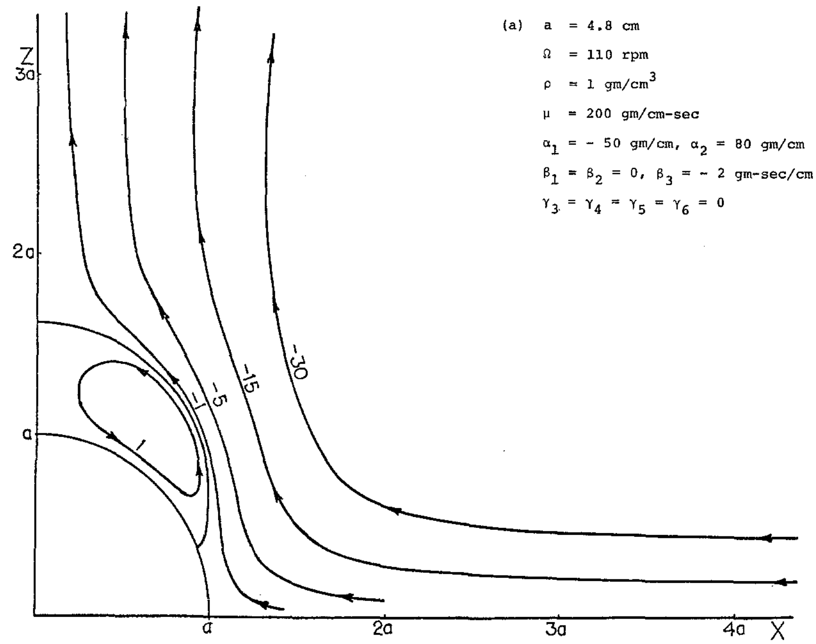


Fig. 7. Level curves of ψ/Ω^2 from [5.3] for Ω = 110 rpm and 1450 rpm. Values for μ, α_1 and α_2 are as measured for STP motor oil at room temperature



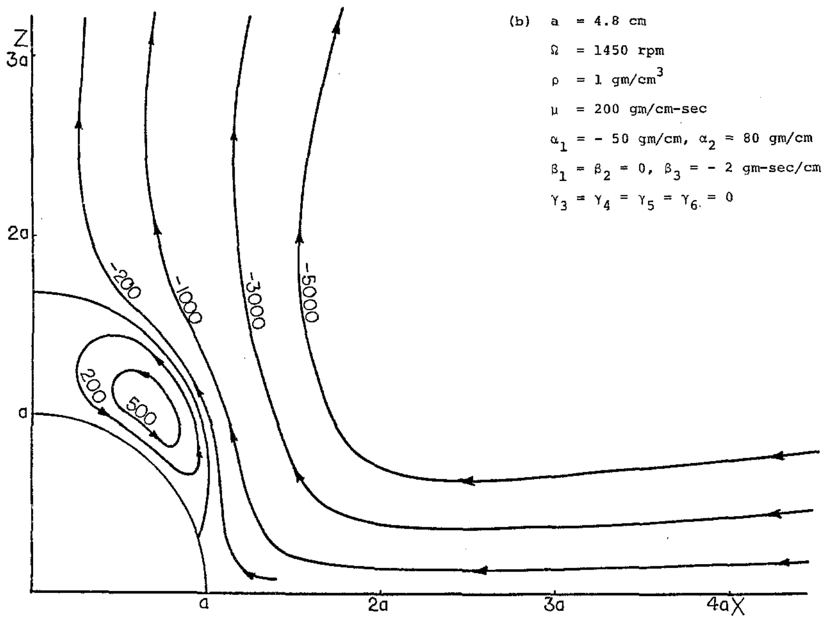


Fig. 8. Level curves of ψ/Ω^2 from [5.3] for Ω = 110 rpm and Ω = 1450 rpm. Values for μ, α_1 and α_2 are as measured for TL 227 oil at room temperature

Zusammenfassung

Es wird eine Lösung angegeben für das Problem der Strömung einer homogenen inkompressiblen einfachen Flüssigkeit um eine eingetauchte Kugel, die um eine feste Achse stationär rotiert. Die Lösung ist vollständig bis zu Gliedern vierter Ordnung in der Winkelgeschwindigkeit. Durch explizite Ausrechnung werden in einer Folge von Abbildungen Beispiele von Stromlinienfeldern gezeigt, die Sekundärströmungszonen in der Form von äquatorialen Doppelringen und/oder Polkappen aufweisen. Das Ziel dieser Untersuchung ist aufzuzeigen, daß viele der nichttrivialen Sekundärströmungserscheinungen einer theoretischen Beschreibung fähig sind. In den numerischen Rechnungen mußten verschiedene Stoffkonstanten eingeführt werden, von denen nicht durchweg behauptet werden kann, daß sie einer bestimmten realen Flüssigkeit zugeordnet sind. Einige derselben (in Abb. 7 und 8) sind allerdings der experimentellen Literatur entnommen. Eine Diskussion der Ergebnisse ist in Abschnitt 5 enthalten.

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