Rheol. Acta 14, 850–857 (1975)

Department of Applied Mathematics, Indian Institute of Science, Bangalore (India)

Similar solutions for the incompressible laminar boundary layer with pressure gradient in micropolar fluids

G. Nath

With 8 figures and 1 table

(Received July 15, 1974)

1. Introduction

The theory of microfluids was introduced by *Eringen* (1, 2) to study a class of fluids which exhibit certain microscopic effects arising from the local structure and micro-motions of the fluid elements. A subclass of these fluids which shows the microrotational effects and micro-rotational inertia was named the micropolar fluids by *Eringen* (3). The micropolar fluids can support couple stresses and body couples only. Physically, they may represent fluids consisting of bar-like or sphere-like elements (3, 4). The mathematical model underlying micropolar fluids may be considered to represent the behaviour of certain polymeric fluids, animal blood and liquid crystals (3, 4, 5, 6).

The flow problems of the micropolar fluids through various configurations have been studied by *Eringen* (3), *Ariman* et al. (5), *Ariman* (7), *Willson* (8, 9) etc. The stability problem for micropolar fluids for the plane *Poiseuille* type of flow has been considered by *Liu* (4). The boundary layer concept in micropolar fluids was introduced by *Willson* (6) to obtain the solution in the stagnation region of a two-dimensional body using *Kármán-Pohlhausen* technique. He found that the standard length of the micropolar fluid plays an important role in determining the structure of the boundary layer.

It has been well established that similar solutions of laminar boundary layer equations for *Newton*ian fluids provide a relatively simple and exact method of solving a number of viscous problems. As far as the author is aware, the similarity solution of the boundary layer equations for micropolar fluids has not been considered in the available literature. Therefore, the aim of this investigation is to obtain the similar solution of the steady laminar incompressible boundary layer equations of micropolar fluid past an infinite wedge. The governing coupled equations have been solved numerically using fourth order *Runge-Kutta-Gill* algorithm. The velocity and microrotation profiles and the surface shear stress for a range of parameters have been presented. The significant role played by the standard length of the micropolar fluid in determining the structure of the boundary layer has been discussed.

2. Governing equations

We consider the steady incompressible two-dimensional flow of a micropolar fluid in the neighbourhood of the stagnation point of an infinite wedge. We assume that all the physical properties of the fluid are constant. It is also assumed that the microscopic inertia term involving J (where J is the square of the characteristic length of the microstructure) can be neglected for steady two-dimensional boundary layer flows in a micropolar fluid without introducing any appreciable error in the solution (6).

Under these assumptions, the governing boundary layer equations for micropolar fluids (neglecting the effect of the transverse curvature etc.) and their boundary conditions can be expressed as (6)

$$u u_x + v u_y = U U_x + \nu u_{yy} + K_1 \sigma_y \qquad [1]$$

$$G_1 \sigma_{yy} - 2\sigma - u_y = 0 \qquad [2]$$

$$u_x + v_y = 0 \qquad [3]$$

$$\left.\begin{aligned} &u(0) = v(0) = 0,\ u(\infty) \to U(x) \\ &\sigma(0) = 0,\ \sigma(\infty) \to 0 \end{aligned}\right\} \qquad [4]$$

where x is the distance measured along the surface from the stagnation point of the body, y is the distance measured perpendicular to the body, u and v are the velocity components along x and y directions respectively, U is the velocity of the potential flow, $\nu = (\mu + S)/\rho$ is the apparent kinematic viscosity of the fluid, μ is the coefficient of viscosity, S is the constant characteristic of the particular fluid, ρ is the density, σ is the microrotation component, $K_1 = S/\rho (K_1 > 0)$ is the coupling constant and $G_1 (G_1 > 0)$ is the microrotation parameter, and suffixes x and y denote partial derivatives with respect to x and y respectively.

We know that for similar solutions the velocity of the potential flow in the neighbourhood of the stagnation point of the body is given by

$$U(x) = u_1 x^m$$

and that the equation of continuity [3] is integrated by the introduction of a stream function $\psi(x,y)$, so that

$$u = \partial\psi/\partial y;\ v = -\partial\psi/\partial x$$

where u_1 and m are constants. Eqs. [1] and [2] can be reduced to ordinary differential equations by writing

$$\eta = y\left(\frac{m+1}{2}\frac{U}{\nu x}\right)^{\frac{1}{2}} = y\left(\frac{m+1}{2}\frac{u_1}{\nu}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}}$$

$$\psi = \left(\frac{2\nu u_1}{m+1}\right)^{\frac{1}{2}} x^{\frac{m+1}{2}} f(\eta) \qquad [5]$$

$$\sigma = \left(\frac{m+1}{2\nu}\right)^{\frac{1}{2}} u_1^{\frac{3}{2}} x^{\frac{(3m-1)}{2}} g(\eta)$$

so that

$$u = u_1 x^m f'(\eta) = U f'(\eta)$$

$$v = -\left(\frac{m+1}{2}\nu u_1 x^{m-1}\right)^{\frac{1}{2}}\left[f + \frac{m-1}{m+1}\eta f'\right]. \qquad [6]$$

Substituting for u, v, σ and y from eqs. [5]–[6] in eqs. [1]–[2], we get

$$f''' + ff'' + \beta(1 - f'^2) + Kg' = 0 \qquad [7]$$

$$Gg'' - (2g + f'')(2 - \beta) = 0 \qquad [8]$$

where $K = K_1/\nu$ and $G = G_1 U/\nu x$ are non-dimensional coupling and microrotation parameters respectively, $\beta = 2m/(m+1)$ is the pressure gradient parameter and prime denotes differentiation with respect to η. The boundary conditions for f and g can be expressed as

$$f(0) = f'(0) = 0;\ f'(\infty) \to 1$$
$$g(0) = 0;\ g(\infty) \to 0\,. \qquad [9]$$

From the relation between m and β (i.e. $m = \beta/(2-\beta)$), it can be observed that $m \to \infty$ as $\beta \to 2$. Hence the magnitude of β should be less than 2.

The shear stress function τ can be expressed as

$$\tau = (\mu + S)u_y + S\sigma = \rho[\nu u_y + K_1\sigma]\,. \qquad [10]$$

Hence the surface skin friction coefficient c_f is given by

$$c_f = [2\tau/\rho U^2]_{\eta=0} = [2/\{(2-\beta)\mathrm{Re}\}^{\frac{1}{2}}] \cdot [f''(\eta) + Kg(\eta)]_{\eta=0} = [2/\{(2-\beta)\mathrm{Re}\}^{\frac{1}{2}}] f''(0) \qquad [11]$$

where $\mathrm{Re} = Ux/\nu$ is the *Reynolds* number. It may be remarked that for micropolar fluids, the surface skin friction coefficient c_f does not contain microrotation term explicitly because $g(0) = 0$.

It may be noted that the two-dimensional stagnation flow and the boundary layer on a flat plate at zero angle of incidence are the particular cases of the present solution. The former for $\beta = 1$ and the latter for $\beta = 0$. As in *Newton*ian fluids, it can be easily shown that the governing equations for the axisymmetric stagnation point flow can be obtained from eqs. [7] and [8] by putting $\beta = 1/2$ and using the transformations

$$\eta = (2)^{\frac{1}{2}}\eta_1,\ f(\eta) = (2)^{\frac{1}{2}}\phi(\eta_1),\ g(\eta) = (\tfrac{1}{2})^{\frac{1}{2}} g_1(\eta_1). \qquad [12]$$

3. Structure of boundary layer

It can be easily shown (6) that the structure of the boundary layer in micropolar fluids depends upon the magnitude of the non-dimensional standard length

$$L(L = G/[(2-\beta)(2-K)]\,;\ K = \frac{S}{\mu + S};\ \beta \neq 2)\,. \qquad [13]$$

It may be remarked (3) that for micropolar fluids $\mu > 0$, $S \geq 0$, hence K is always less than one ($0 \leq K < 1$) and the case $K = 1$ corresponds to limiting (but unphysical) situation of infinite S. For a fixed β and K, $L \ll 1$ implies $G \ll 1$. It is important to note that the case $G \to 0$ is a classical singular perturbation problem as the highest derivative in g in eq. [8] is dropped and then the problem can be solved by well known singular perturbation techniques (10). It can be conjectured (6) that for $L \ll 1$, the boundary layer splits into two regions: the contact layer adjacent to the wall and an outer layer. It has been shown that (6, 11)

$$\mu_{\mathrm{app}} = \mu + \frac{S}{2} \quad \text{as} \quad L \to 0;$$
$$\mu_{\mathrm{app}} = \mu + S \quad \text{as} \quad L \to \infty\,. \qquad [14]$$

4. Results and discussion

The governing set of coupled eqs. [7]–[8] subject to the boundary conditions [9] have been

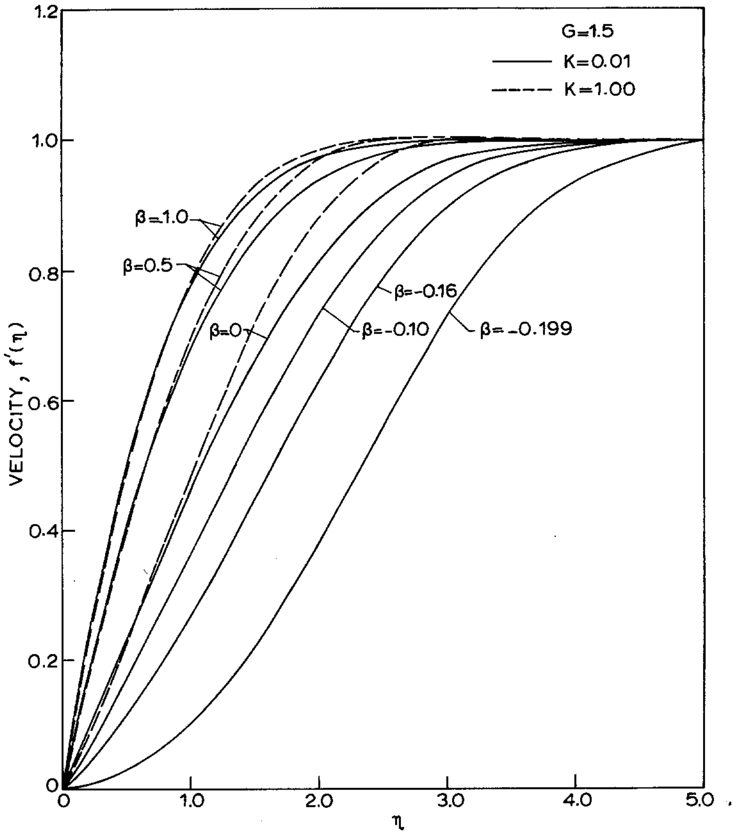

Fig. 1. Velocity distribution

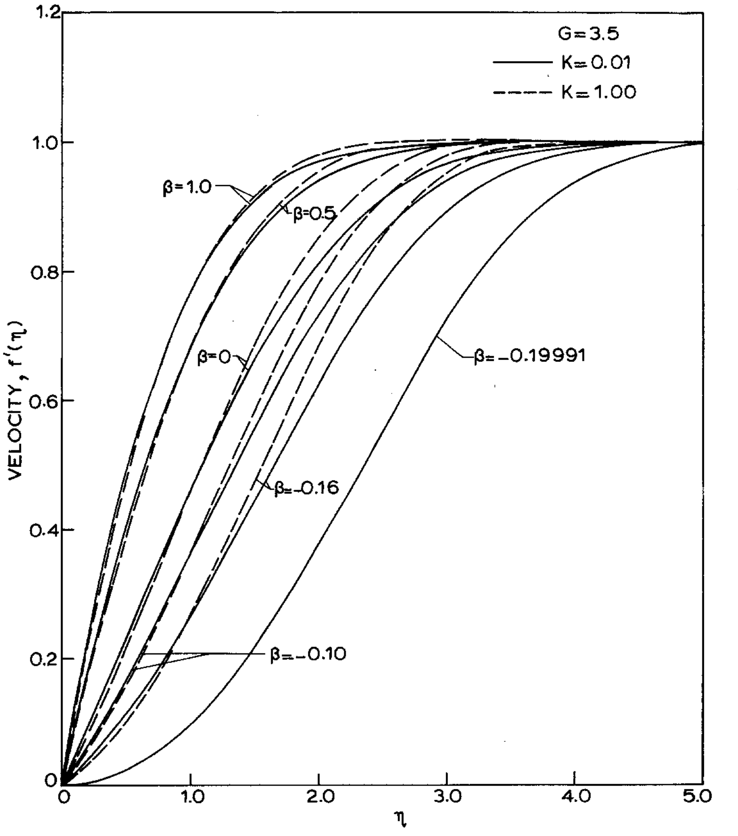

Fig. 2. Velocity distribution

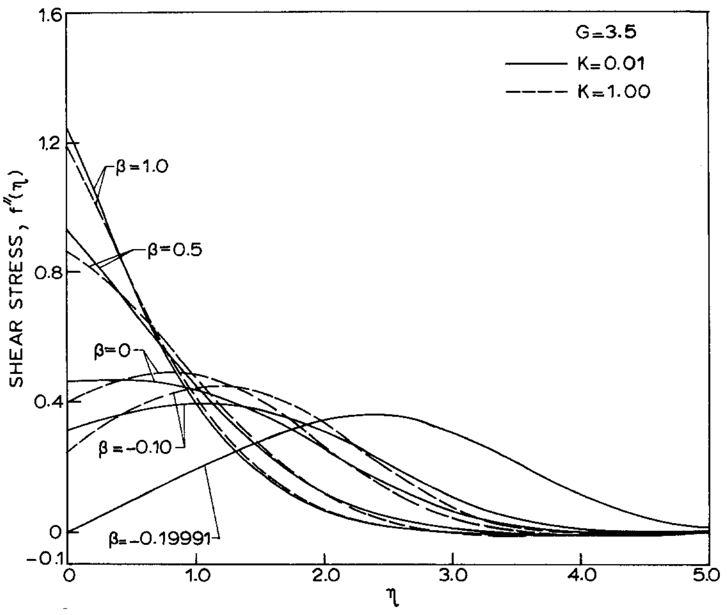

Fig. 3. Shear stress distribution

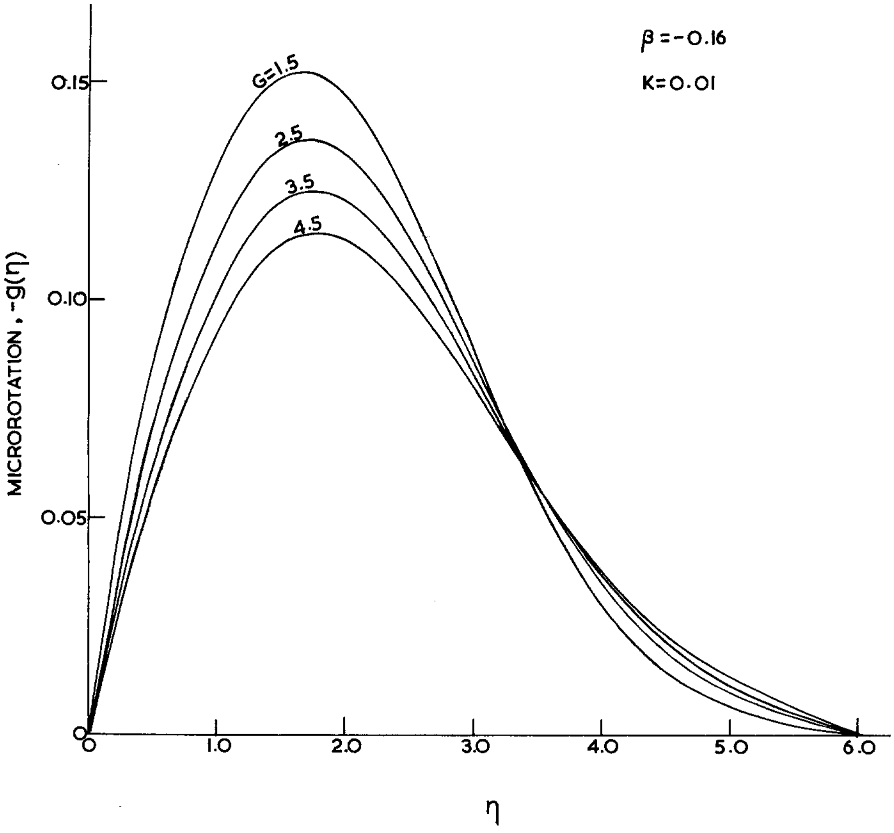

Fig. 4. Microrotation distribution

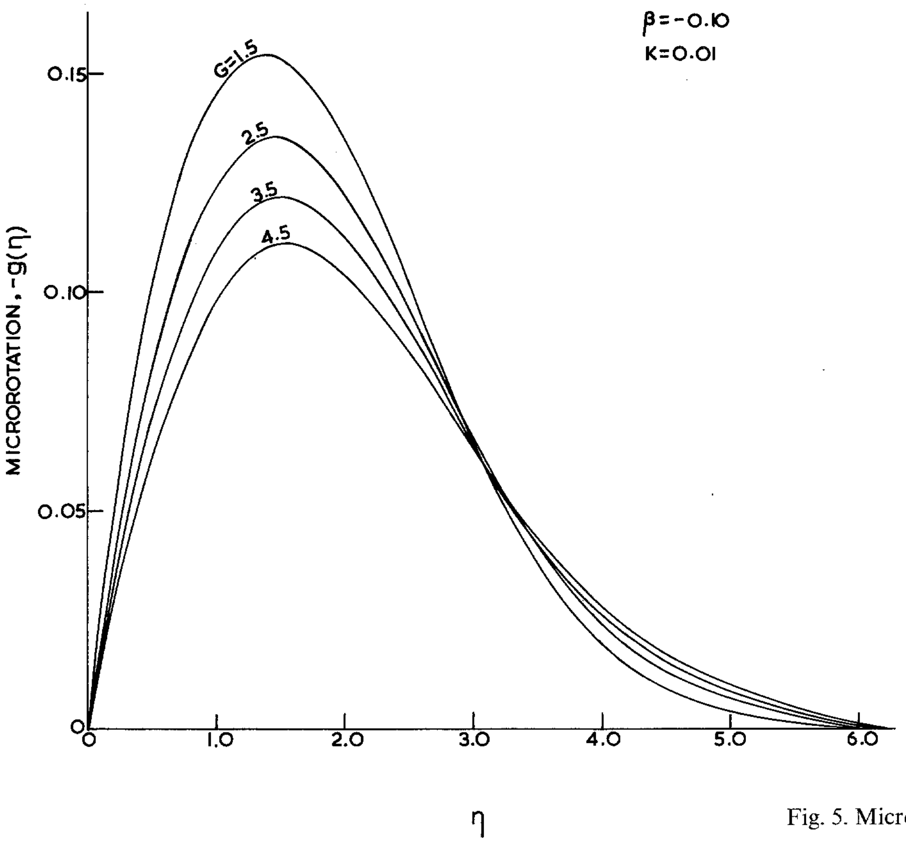

Fig. 5. Microrotation distribution

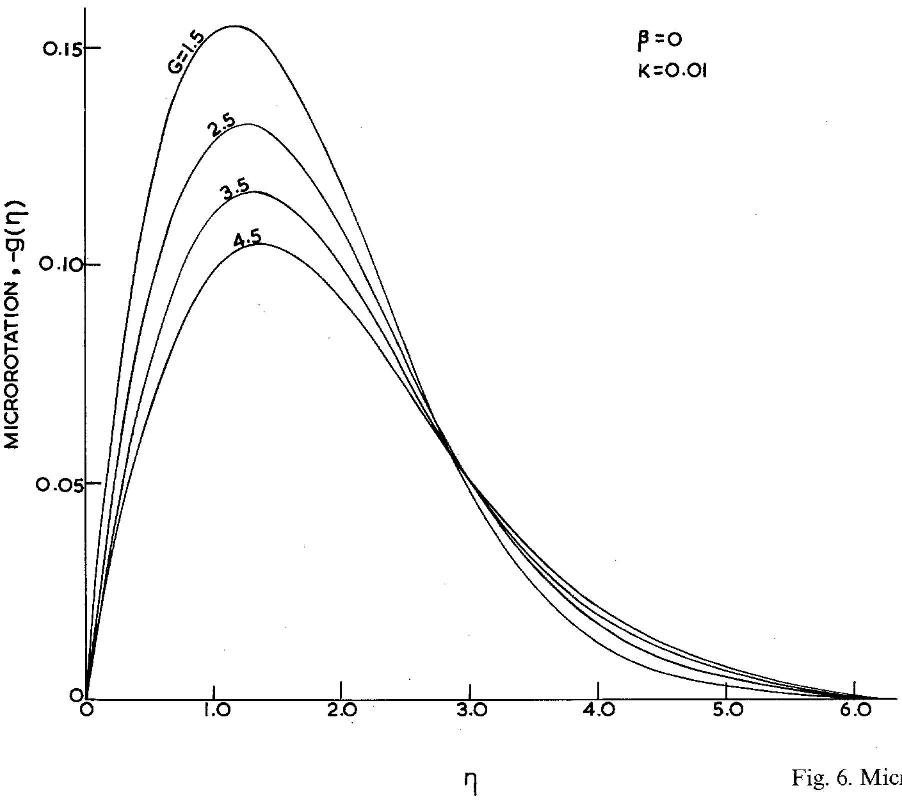

Fig. 6. Microrotation distribution

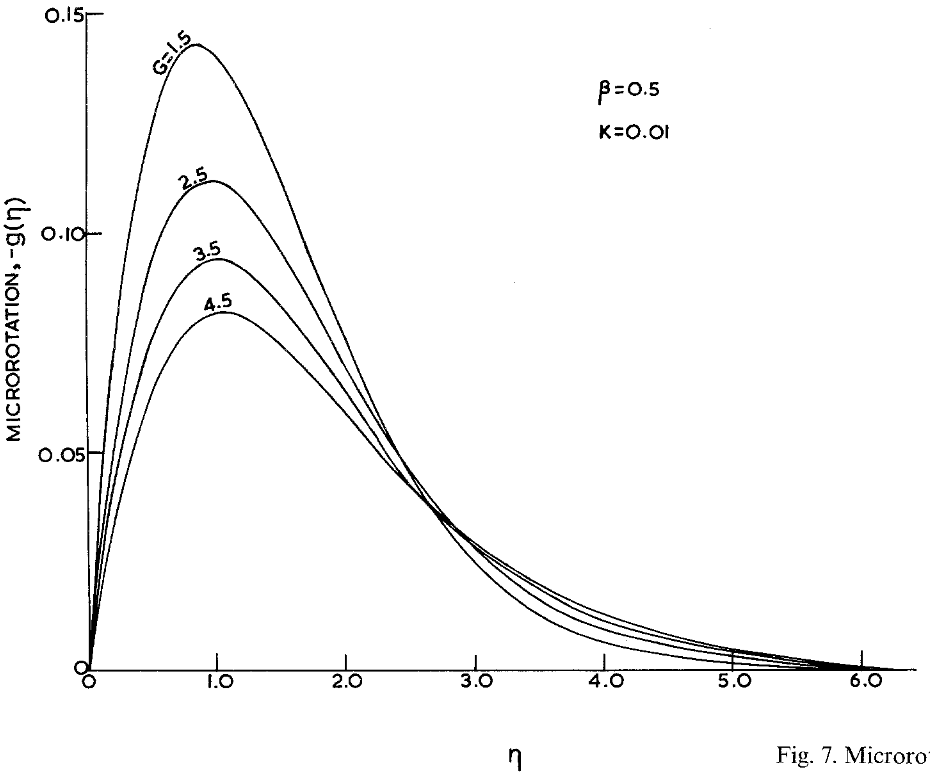

Fig. 7. Microrotation distribution

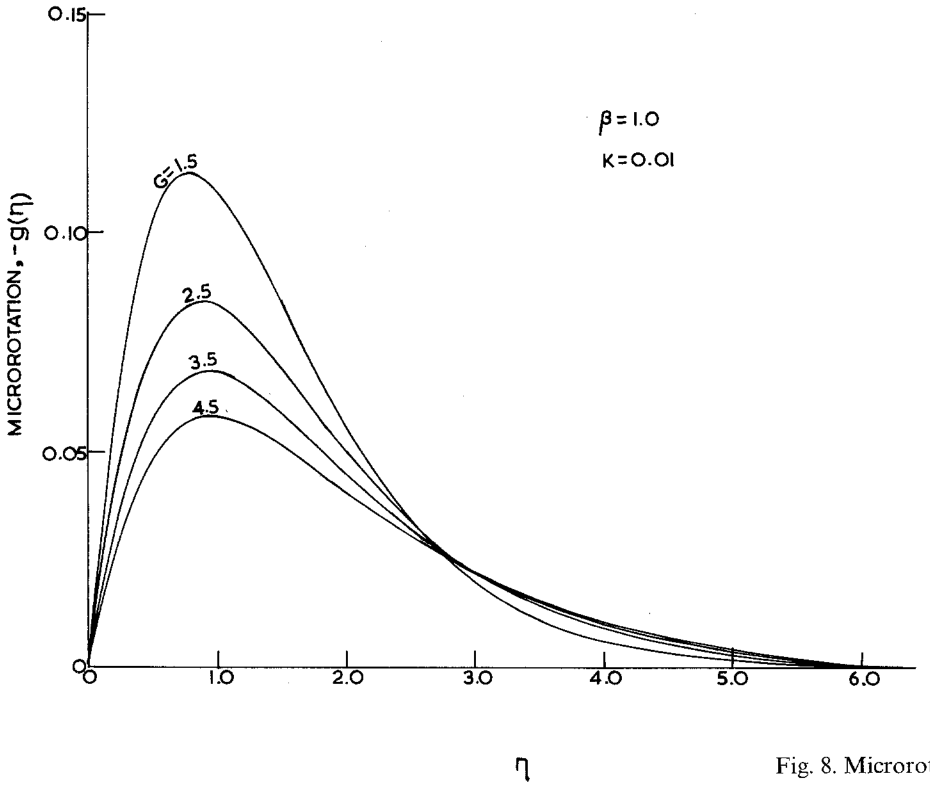

Fig. 8. Microrotation distribution

solved numerically using fourth-order *Runge-Kutta-Gill* algorithm on IBM 360 digital computer for a range of values ($0.01 \leq K \leq 1$) of the dimensionless coupling parameter and a range of values ($1.5 \leq G \leq 4.5$) of the dimensionless microrotation parameter. The pressure gradient parameter β was varied from -0.19991 to 1.0.

Figs. 1–2 depict the velocity profiles $f'(\eta)$ for different values of K, G and β. It is seen that the velocity profiles are more sensitive to pressure gradient parameter β and the coupling parameter K, but they are rather little affected by microrotation parameter G. The shear stress profiles $f''(\eta)$ for some representative values of β, K and G are shown in fig. 3. The microrotation profiles $-g(\eta)$ have been plotted graphically in figs. 4–8. It is evident from these figures that $-g(\eta)$ attains a maximum value at certain η and for a given K, the maximum value decreases as G increases. For a prescribed G, it is rather insensitive to change in K. The effect of pressure gradient parameter β is to shift the location where $-g(\eta)$ attains the maximum value and also to decrease its magnitude. It can be observed from figs. 1–2 that like classical boundary layer theory, the velocity profiles for accelerated flow ($\beta > 0$) have no point of inflexion, whereas in the case of decelerated flow ($\beta < 0$) they exhibit a point of inflexion. It may be remarked that the surface shear stress parameter $f''(0)$ depends on three parameters β, K and G where as in the classical boundary layer (*Newton*ian fluids), it depends only on β. Figs. 1–3 show that $f''(0) = 0$ for $\beta = -0.19991$ when $K = 0.01$, $G = 3.5$ and $f''(0) = 0$ for $\beta = -0.199$ when $K = 0.01$ and $G = 1.5$ where as in the classical boundary layer also $f''(0) = 0$ for $\beta = -0.199$ ($\beta = -0.19884$ (correct upto 5-decimal place)). These results indicate that separation occurs for almost the same value of β as in *Newton*ian fluids when $K \ll 1$.

Table 1 presents values of the surface shear stress $f''(0)$ and the gradient of microrotation at the surface $g'(0)$ for various values of K, G. and β. It is evident from Table 1 that $f''(0)$ changes very little with G and K provided K is small (i.e. $K \leq 0.1$). However, for large K (i.e. $K = 1$), $f''(0)$ slightly increases with G, but for prescribed G, it decreases with K. It is important to note that $f''(0)$ is largely influenced by the pressure gradient parameter β and it increases with β. On the other hand, $-g'(0)$ decreases as G increases, but it changes very little with K, while other parameters are kept constant.

Table 1. Values of shear stress and the gradient of microrotation on the surface

β		$f''(0)$			$-g'(0)$		
	G \ K	0.01	0.1	1.0	0.01	0.1	1.0
−0.16	1.5	0.1902	–	–	0.2242	–	–
−0.16	2.5	0.1902	0.1845	0.1242	0.1782	0.1786	0.1839
−0.16	3.5	0.1902	0.1849	0.1268	0.1516	0.1518	0.1548
−0.16	4.5	0.1902	0.1851	0.1297	0.1333	0.1335	0.1354
−0.1	1.5	0.3185	–	–	0.2939	–	–
−0.1	2.5	0.3186	0.3122	0.2449	0.2244	0.2247	0.2287
−0.1	3.5	0.3186	0.3127	0.2510	0.1858	0.1859	0.1885
−0.1	4.5	0.3187	0.3133	0.2552	0.1603	0.1605	0.1619
0	1.5	0.4687	0.4604	0.3769	0.3585	0.3585	0.3638
0	2.5	0.4688	0.4620	0.3892	0.2651	0.2653	0.2678
0	3.5	0.4689	0.4628	0.3975	0.2149	0.2151	0.2164
0	4.5	0.4690	0.4634	0.4045	0.1826	0.1827	0.1838
0.5	1.5	0.9267	0.9182	0.8296	0.4353	0.4354	0.4362
0.5	2.5	0.9269	0.9202	0.8502	0.3023	0.3023	0.3026
0.5	3.5	0.9270	0.9213	0.8627	0.2352	0.2352	0.2354
0.5	4.5	0.9271	0.9221	0.8712	0.1939	0.1939	0.1940
1.0	1.5	1.2318	1.2247	1.1520	0.3688	0.3688	0.3687
1.0	2.5	1.2320	1.2267	1.1727	0.2473	0.2473	0.2472
1.0	3.5	1.2321	1.2278	1.1842	0.1882	0.1882	0.1882
1.0	4.5	1.2322	1.2285	1.1917	0.1527	0.1527	0.1527

5. Conclusions

The main contribution of the present analysis is to show that a similarity solution exists for the two-dimensional flow of a micropolar fluid in the stagnation region of an infinite wedge. The results indicate that the velocity and the surface shear stress are little affected by the microrotation parameter provided coupling parameter is small. On the other hand, the microrotation profiles are appreciably affected by the microrotation parameter for a given coupling parameter, but they change very little with the coupling parameter. It is to be noted that the pressure gradient parameter has considerable effect on the velocity and the microrotation profiles and also on the surface shear stress.

Zusammenfassung

Die vorliegende Untersuchung liefert Ähnlichkeitslösungen für die stationäre inkompressible laminare Grenzschichtströmung einer mikro-polaren Flüssigkeit längs einem unendlich ausgedehnten Keil. Die zugrundeliegenden Gleichungen werden unter Verwendung einer *Runge-Kutta-Gill*-Methode vierter Ordnung numerisch gelöst. Die Ergebnisse zeigen den Grad, bis zu welchem die Profile von Geschwindigkeit und Mikrorotation, sowie die Wandschubspannung durch die Parameter von Drehmoment, Mikrorotation und Druckgradient beeinflußt werden. Die wesentliche Rolle, welche die Standard-Länge der mikro-polaren Flüssigkeit für die Struktur der Grenzschicht spielt, wird ebenfalls diskutiert.

Summary

This paper presents the similarity solution for the steady incompressible laminar boundary layer flow of a micropolar fluid past an infinite wedge. The governing equations have been solved numerically using fourth order *Runge-Kutta-Gill* method. The results indicate the extent to which the velocity and microrotation profiles, and the surface shear stress are influenced by coupling, microrotation, and pressure gradient parameters. The important role played by the standard length of the micropolar fluid in determining the structure of the boundary layer has also been discussed.

References

1) *Eringen, A. C.*, Int. J. Engr. Sci. **2**, 205 (1964).
2) *Eringen, A. C.*, Proc. XI Intern. Congress of Appl. Mech. p. 131, (Berlin-Heidelberg-New York 1965).
3) *Eringen, A. C.*, J. Math. Mech. **16**, 1 (1966).
4) *Liu, C. Y.*, Phys. Fluids **14**, 1808 (1971).
5) *Ariman, T.*, *Cakmak, A. S.*, and *L. R. Hill*, Phys. Fluids **10**, 2545 (1967).
6) *Willson, A. J.*, Proc. Camb. Phil. Soc. **67**, 469 (1970).
7) *Ariman, T.*, Int. J. Engr. Sci. **6**, 1 (1968).
8) *Willson, A. J.*, Proc. Camb. Phil. Soc. **64**, 513 (1968).
9) *Willson, A. J.*, J. Appl. Sci. Res. **20 A**, 338 (1969).
10) *Cole, J. D.*, Perturbation Methods in Applied Mathematics (Waltham, Mass. 1968).
11) *Cowin, S.*, Phys. Fluids **11**, 1919 (1968).

Author's address:

Girishwar Nath
Department of Applied Mathematics
Indian Institute of Science
Bangalore 560012 (India)