# Long wavelength approximation to peristaltic motion of a power law fluid

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*Abstract:* Peristaltic motion of a power law fluid in a two-dimensional channel is studied. Assuming that the wavelength of the peristaltic wave is large in comparison to the mean half-width of the channel, a solution for the stream function is obtained as an asymptotic expansion in terms of slope parameter. Expressions for axial pressure gradient and shear stress are derived. The effect of flow behaviour index $n$ on the streamline pattern and shear stress is studied and the phenomenon of trapping is discussed.

*Key words:* Peristaltic motion, power law fluid, asymptotic expansion, trapping

## 1. Introduction

The fluid mechanics of peristaltic motion has been extensively studied for some years as it is known to be one of the main mechanism for fluid transport in biological systems. From the point of view of fluid mechanics, peristaltic pumping is characterized by the dynamic interaction of fluid flow with the movement of a flexible boundary. In fact peristalsis is the major mechanism for the transport of urine from kidney to bladder, food mixing in the intestines etc. It is also speculated that peristalsis is involved in the vasomotion of small blood vessels. Also mechanical devices like finger pumps and roller pumps use peristalsis to pump blood, slurries, corrosive fluids and so on.

Several authors [1 – 5] have studied the fluid mechanics of peristalsis and a review of much of the literature is given by Jaffrin and Shapiro [6]. Most of the investigations were carried out for better understanding of urine transport from kidney to bladder. However, Nicoll and Webb [7] and Nicoll [8] suggested that peristalsis plays a major role in blood circulation. The arterioles and venules are seen to change their diameters periodically. Although the spatial waveform of such a vasomotion has not been ascertained it is conceivable that peristalsis is involved in the vasomotion of small blood vessels [2]. Some theoretical and experimental studies [9 – 13] have been made on the peristaltic motion of blood considering blood as a non-Newtonian fluid and also as suspension of solid particles in Newtonian fluid. Raju and Devanathan [9] studied peristaltic motion of blood considering blood as a power law fluid. They obtained the solution for the stream function as a power series in terms of the amplitude of deformation and evaluated the stream function and velocity components by solving numerically two point boundary value problems with a singular point at the origin.

In this paper we study the peristaltic transport of blood, modelled by a power law fluid, under long wavelength approximation for better understanding of the role of peristalsis in blood circulation. The solution for stream function is obtained as an asymptotic series in terms of slope parameter and the expressions for axial pressure gradient and shear stress are derived. The effect of flow behaviour index $n$ on the streamline pattern and shear stress is studied and the phenomenon of trapping is discussed.

## 2. Formulation of the problem

We choose non-Newtonian fluid of a power law model which is characterized by the constitutive equation

$$T_{ij} = -p\delta_{ij} + \mu_p \theta e_{ij} \tag{1}$$

where $T_{ij}$ and $e_{ij}$ are the stress and rate of deformation tensors, $p$ is the pressure, $\mu_p$ is the flow consistency index, $n$ is the flow behaviour index and

$$\theta = \left|\frac{1}{2} e_{ij} e_{ij}\right|^{(n-1)/2}.$$

This model represents dilatant, Newtonian and pseudoplastic fluids $n > 1$, $n = 1$, $n < 1$, respectively.

We consider laminar flow of a power law fluid, characterized by eq. (1), through a two-dimensional channel with flexible walls on which are imposed travelling sinusoidal waves of long wavelength. A rectangular cartesian coordinate system $(x, y)$ is chosen with the $x$ axis aligned with the centre line of the channel. The travelling waves are represented by

$$\eta(x, t) = d + a \sin \frac{2\pi}{\lambda}(x - ct) \tag{2}$$

where $d$ is the mean half width of the channel, $a$ is the amplitude of the wave, $\lambda$ is the wavelength and $c$ is the wave speed. The equations of momentum and continuity are

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}(T_{xx}) + \frac{\partial}{\partial y}(T_{xy}), \tag{3}$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = \frac{\partial}{\partial x}(T_{yx}) + \frac{\partial}{\partial y}(T_{yy}), \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

where $u$, $v$ are the velocity components along $x$ and $y$ directions, respectively, and $\rho$ is the density of the fluid. The boundary conditions on the velocity components are

$$u = 0 \quad \text{at} \quad y = \eta, \tag{6}$$

$$v = \frac{\partial \eta}{\partial t} \quad \text{at} \quad y = \eta, \tag{7}$$

together with the regularity condition

$$v = 0 = \frac{\partial u}{\partial y} \quad \text{at} \quad y = 0. \tag{8}$$

Using the transformation

$$\frac{\partial \Psi}{\partial y} = u - c, \quad \frac{\partial \Psi}{\partial x} = -v,$$

$$x' = \xi - t', \quad \xi = \frac{x}{\lambda}, \quad t' = \frac{ct}{\lambda} \tag{9}$$

from a stationary to a moving frame of reference and introducing the following non-dimensional quantities

$$x' = \xi - t', \quad y' = \frac{y}{d}, \quad \Psi' = \frac{\Psi}{cd}, \quad \varepsilon = \frac{a}{d},$$

$$\alpha = \frac{d}{\lambda}, \quad p' = \frac{pd}{\rho c^2 \lambda}, \quad R = \frac{\rho d^n}{\mu_p c^{n-2}}, \quad \eta' = \frac{\eta}{d}$$

eqs. (3) and (4) can be written in non-dimensional form, after dropping the primes, as

$$\alpha\left[\frac{\partial \Psi}{\partial y}\nabla^2\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi}{\partial x}\nabla^2\frac{\partial \Psi}{\partial y}\right] = \frac{1}{R} \cdot \left[4\alpha^2\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 \Psi}{\partial x \partial y}\phi\right) + \left(\frac{\partial^2}{\partial y^2} - \alpha^2\frac{\partial^2}{\partial x^2}\right) \cdot \left\{\left(\frac{\partial^2 \Psi}{\partial y^2} - \alpha^2\frac{\partial^2 \Psi}{\partial x^2}\right)\phi\right\}\right] \tag{10}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial y^2} + \alpha^2\frac{\partial^2}{\partial x^2}$$

and

$$\phi = \left|\left(2\alpha\frac{\partial^2 \Psi}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 \Psi}{\partial y^2} - \alpha^2\frac{\partial^2 \Psi}{\partial x^2}\right)^2\right|^{(n-1)/2}$$

is the non-dimensional form of $\theta$.

The corresponding boundary conditions are

$$\frac{\partial \Psi}{\partial y} = -1 \quad \text{at} \quad y = \eta, \tag{11}$$

$$\Psi = q \quad \text{at} \quad y = \eta, \tag{12}$$

$$\Psi = \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad \text{at} \quad y = 0, \tag{13}$$

where $q$ is half the flux in the negative axial direction in the moving frame of reference. However, by accounting for Galilean transformation, there will be a net positive flux in the stationary frame of reference.

### 3. Method of solution

We now consider long wavelength approximation and seek an asymptotic solution for the stream function in terms of the parameter $\alpha$ ($\alpha \ll 1$) as

$$\Psi = \Psi_0 + \alpha \Psi_1 + \cdots . \tag{14}$$

Substituting (14) in (10)–(13) and collecting coefficients of various powers of $\alpha$, we get the following set of equations:

Zeroth order:

$$\frac{\partial^2}{\partial y^2}\left[\frac{\partial^2 \Psi_0}{\partial y^2}\right]^n = 0 , \tag{15}$$

$$\frac{\partial \Psi_0}{\partial y} = -1 \quad \text{at} \quad y = \eta , \tag{16}$$

$$\Psi_0 = q \quad \text{at} \quad y = \eta , \tag{17}$$

$$\Psi_0 = \frac{\partial^2 \Psi_0}{\partial y^2} = 0 \quad \text{at} \quad y = 0 . \tag{18}$$

First order:

$$\frac{n}{R}\frac{\partial^2}{\partial y^2}\left[\frac{\partial^2 \Psi_1}{\partial y^2}\left(-\frac{\partial^2 \Psi_0}{\partial y^2}\right)^{n-1}\right] = \frac{\partial \Psi_0}{\partial y}\frac{\partial^3 \Psi_0}{\partial x \partial y^2} - \frac{\partial \Psi_0}{\partial x}\frac{\partial^3 \Psi_0}{\partial y^3} , \tag{19}$$

$$\frac{\partial \Psi_1}{\partial y} = 0 \quad \text{at} \quad y = \eta , \tag{20}$$

$$\Psi_1 = 0 \quad \text{at} \quad y = \eta , \tag{21}$$

$$\Psi_1 = \frac{\partial^2 \Psi_1}{\partial y^2} = 0 \quad \text{at} \quad y = 0 . \tag{22}$$

Solving eq. (15) under the corresponding boundary conditions, we get

$$\Psi_0 = \frac{n}{n+1}\left[\frac{2n+1}{n}\frac{y}{\eta} - \left(\frac{y}{\eta}\right)^{2+(1/n)}\right](q+\eta) - y . \tag{23}$$

The solution of eq. (19) under the boundary conditions (20)–(22) and using (23) can be obtained as

$$\begin{aligned}\Psi_1 = \frac{RB_1}{n}\Bigg[ & A_1 A_{1x}\frac{2n+1}{3n+2}\,\frac{n}{4n+3}\,\frac{n}{3n+3} \\ & \cdot\{y^{4+(3/n)} - 3y^{2+(1/n)}\eta^{2+(2/n)} + 2y\eta^{3+(3/n)}\} \\ & + \left(A_2 A_{1x} - \frac{1}{n}A_1 A_{2x}\right)\frac{n}{3n+2}\,\frac{n}{2n+2} \\ & \cdot\{y^{3+(2/n)} - 2y^{2+(1/n)}\eta^{1+(1/n)} + y\eta^{2+(2/n)}\}\Bigg] ,\end{aligned} \tag{24}$$

where

$$A_1 = -\frac{n}{n+1}\,\frac{q+\eta}{\eta^{2+(1/n)}} ,$$

$$A_2 = \frac{2n+1}{n+1}\,\frac{q+\eta}{\eta} - 1 ,$$

and

$$B_1 = \left[\frac{2n+1}{n}\,\frac{q+\eta}{\eta^{2+(1/n)}}\right]^{(1-n)/n} .$$

The axial pressure gradient can be obtained from eq. (3) using eqs. (14), (23) and (24) in the following form:

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial p_0}{\partial x} + \alpha\frac{\partial p_1}{\partial x} + \cdots \\ &= -\frac{1}{R}\left[\frac{2n+1}{n}\,\frac{q+\eta}{\eta^{2+(1/n)}}\right]^n \\ &\quad + \alpha(K_1 - A_2 A_{2x}) + o(\alpha^2)\end{aligned} \tag{25}$$

where

$$K_1 = -\left[\frac{(2n+1)^2}{(3n+2)(4n+3)}A_1 A_{1x}\eta^{2+(2/n)} + \frac{2n+1}{3n+2}\left(A_2 A_{1x} - \frac{1}{n}A_1 A_{2x}\right)\eta^{1+(1/n)}\right] .$$

The shear stress acting on the wall is defined as

$$\tau = \frac{\sigma_{xy}\left\{1 - \left(\frac{d\eta}{dx}\right)^2\right\} + (\sigma_{yy} - \sigma_{xx})\frac{d\eta}{dx}}{1 + \left(\frac{d\eta}{dx}\right)^2} \quad \text{at } y = \eta(x) , \tag{26}$$

where $\sigma_{xy}$, $\sigma_{yy}$ and $\sigma_{xx}$ are the usual stress components. The non-dimensional shear stress can now be obtained as

$$\tau_w = \tau \Big/ \frac{\mu_p c^n}{d^n}$$
$$= -\left[\frac{2n+1}{n}\,\frac{q+\eta}{\eta^{2+(1/n)}}\right]^n \eta L_1$$
$$+ \alpha\left[R\left\{A_1 A_{1x}\frac{2n+1}{3n+2}\,\frac{2n+2}{4n+3}\,\eta^{3+(2/n)}\right.\right.$$
$$\left.+\left(A_2 A_{1x} - \frac{1}{n}A_1 A_{2x}\right)\frac{n+1}{3n+2}\,\eta^{2+(1/n)}\right\}L_1$$
$$\left.- 4\left\{\left(\frac{2n+1}{n}\,\frac{q+\eta}{\eta^{2+(1/n)}}\right)^n \eta\frac{d\eta}{dx}\right\}L_2\right] + o(\alpha^2) \tag{27}$$

where

$$L_1 = \frac{1 - \left(\frac{d\eta}{dx}\right)^2}{1 + \left(\frac{d\eta}{dx}\right)^2}$$

and

$$L_2 = \frac{\frac{d\eta}{dx}}{1 + \left(\frac{d\eta}{dx}\right)^2}.$$

## 4. Discussion

We have presented approximate solution for the stream function as an asymptotic series in terms of slope parameter $\alpha$. The leading term of the expansion (14) gives the limiting solution for very long waves in

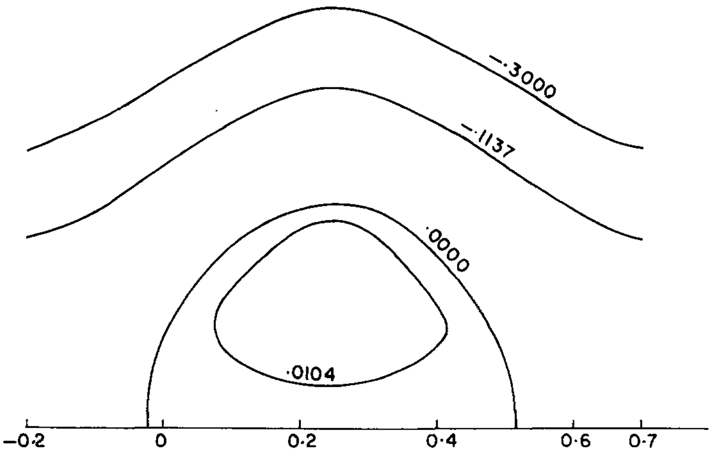


Fig. 1. Streamlines in waveframe when a trapped bolus exists in laboratory frame for $n = 0.8$

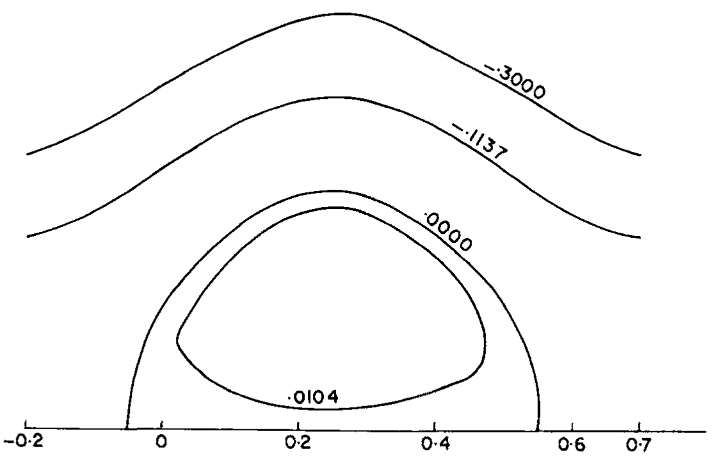


Fig. 2. Streamlines in waveframe when a trapped bolus exists in laboratory frame for $n = 1.0$

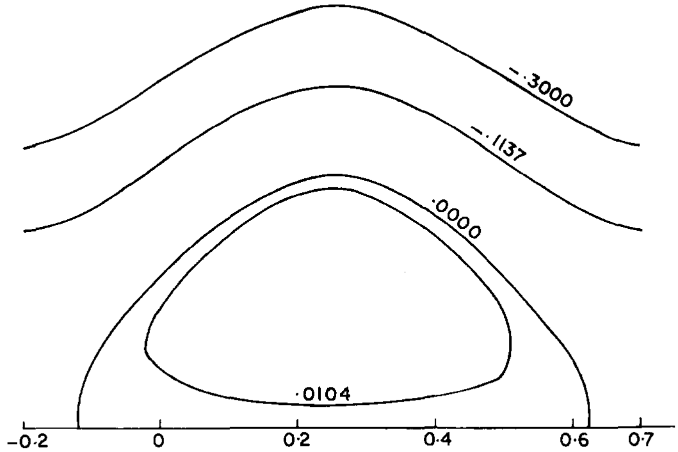


Fig. 3. Streamlines in waveframe when a trapped bolus exists in laboratory frame for $n = 1.2$

the limit $\alpha \to 0$. Further it may be observed that the zeroth order solution $\Psi_0$ for $n = 1$ (Newtonian fluid case) reduces to the solution of Shapiro et al. [3]. The velocity components can be obtained by using the solution for stream function $\Psi$. To study the effect of flow behaviour index $n$ on the streamline pattern and shear stress, we have carried out numerical computation and the results are graphically depicted.

The streamlines have been drawn for $R = 5$, $\varepsilon = 0.2$, $\alpha = 0.1$, $q = -0.3$ and they are presented in figures 1–3 for pseudoplastic ($n = 0.8$), Newtonian ($n = 1$) and dilatant ($n = 1.2$) fluids respectively. It can be clearly observed from the figures 1–3 that "trapping" occurs in all the cases. When trapping occurs, the centre streamline splits and there is a region of recirculating, closed streamlines and it comprises of a bolus of fluid. In the stationary frame of reference the bolus of fluid is trapped with the wave and it advances as a whole with the wave speed. We can notice from figures 1–3 that the streamline pattern remains the same as $n$ varies but the area of the trapped bolus increases as $n$ increases. Further, it may be pointed out that the streamline pattern is similar to the one obtained by Shapiro et al. [3] for Newtonian fluid ($n = 1$). Because of the presence of $n$, it may not be possible to analytically obtain a range for trapping.

Table 1. Shear stress acting on the wall ($R = 5$, $\alpha = 0.1$, $\varepsilon = 0.2$, $q = -0.3$)

| $x$ | $n = 0.8$ | 1.0 | 1.2 |
|---|---|---|---|
| 0.0 | 0.0408 | 0.0438 | 0.0470 |
| 0.1 | 0.3418 | 0.3668 | 0.3920 |
| 0.2 | 1.3978 | 1.4889 | 1.5787 |
| 0.3 | 1.4007 | 1.4910 | 1.5799 |
| 0.4 | 0.3416 | 0.3667 | 0.3919 |
| 0.5 | 0.0377 | 0.0416 | 0.0456 |
| 0.6 | 0.3800 | 0.4190 | 0.4599 |
| 0.7 | 1.6610 | 1.8449 | 2.0400 |

The shear stress acting on the wall $\tau_w$ is presented in table 1 for $n = 0.8, 1.0$ and $1.2$. We can observe from this table that there is no qualitative change in the behaviour of shear stress as $n$ varies. But the magnitude of the wall shear increases as $n$ increases.

Using eq. (25) we can obtain the pressure rise over one wavelength as

$$\Delta p_\lambda = \int_0^1 \frac{\partial p}{\partial x} dx = \int_0^1 \left[ -\frac{1}{R} \left\{ \frac{2n+1}{n} \frac{q+\eta}{\eta^{2+(1/n)}} \right\}^n + \alpha(K_1 - A_2 A_{2x}) \right] dx + o(\alpha^2) . \quad (28)$$

Because of the non-linearity caused by $n$ it is not possible to carry out analytical integration and obtain explicit pressure-flow relationship.

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