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On a classification scheme for flow fields

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1. Introduction

An attractive rheological concept is the idea of a flow classification scheme. Ideally, such a scheme could be used to give an indication of how a given real or model fluid would behave in a given flow field. *Giesekus* (1, 2) had investigated in 1962 in a thorough manner how various solid particles of a general dumbbell nature behaved in homogeneous flow fields, and the current contribution may be regarded as an extension of his work. The main difference is that here we permit the suspended "test" particles to change their configurations as they are transported by the fluid, instead of regarding them as fixed in their shape. This gives rise to a different flow classification scheme, as we shall discuss below. The idea of flow classification is latent in the book of *Lodge* (3), who contrasted the response of a simple rubberlike liquid in steady elongational and shearing flows. *Noll* (4) has introduced a classification scheme applicable to steady, homogeneous, isochoric flows, but it is not satisfactory for many purposes as we shall mention again below. Based on the response of a dilute solution of linear-spring dumbbell molecules a more realistic classification scheme (5–7) has been proposed for the motions with constant stretch history (MWCSH) of the steady, homogeneous incompressible type mentioned above, and the present paper is motivated by similar considerations, although the results are ultimately not dependent on any special molecular model. Note that none of the above-cited or the present material seeks to discuss the problem of flow classification when the stress state of a particle varies in time; in short, we are not concerned with how to define a *Deborah* number (8).

Since steady, homogeneous velocity fields give rise to motions with constant stretch history (4, 9) which include viscometric flows (4, 10), elongational flows (11), *Maxwell* orthogonal rheometer flows (12) and others (9), we shall discuss the concept of flow classification within the theory of motions with constant stretch history.

In our set of motions with constant stretch history (MWCSH) let $\boldsymbol{L}$ be the velocity gradient of a steady, homogeneous velocity field. Then the strain history $C_t(t-s)$, $0 \leq s < \infty$, has the form (4, 9)

$$C_t(t-s) = e^{-s\boldsymbol{L}^T} e^{-s\boldsymbol{L}} \qquad [1]$$

where $\boldsymbol{L}^T$ is the transpose of $\boldsymbol{L}$.

Under the classification originally introduced by *Noll* (4), one has three possibilities to consider. Since the space we consider is 3-dimensional, one has that either

(i) $\boldsymbol{L}^2 = 0$; or
(ii) $\boldsymbol{L}^2 \neq 0$, $\boldsymbol{L}^3 = 0$; or
(iii) $\boldsymbol{L}^n \neq 0$, $n = 1, 2, 3, \ldots$

Note that (i) gives rise to viscometric flows (4, 10), (ii) to the types considered in (9); the case (iii) includes both extensional flows ($\boldsymbol{L}$ is symmetric) and the flow in the orthogonal rheometer ($\boldsymbol{L}$ is not symmetric). It is this aspect of including the last two flows in one category which is not very satisfactory, because in the extensional flow, one has an exponentially increasing strain component while in the rheometer flow, one has strain components which are sinusoidal. In many cases, the fluid behaviour in these two fields is very different. We thus feel that a criterion ought to be developed to distinguish between these two flows, over and beyond the statement that in one case the velocity gradient is symmetric while in the other it is not. To make this more precise, consider the following velocity field (in a *Cartesian* frame).

$$v_1 = a_1 x_1 - \Omega x_2,$$

$$v_2 = \alpha_2 x_2 + \Omega x_1,$$

$$v_3 = a_3 x_3,$$

$$a_1 + a_2 + a_3 = 0, \; a_1 \neq a_2,$$

$$a_i, \; i = 1, 2, 3, \text{ and } \Omega \text{ are const.} \qquad [2]$$

It gives rise to a MWCSH, but the velocity gradient $\boldsymbol{L}$ does not commute with $\boldsymbol{L}^T$ and thus this flow is not an extensional flow [1]). But it is clear that the strain tensor will have exponentially increasing components like $e^{\lambda s}$ (λ real) along with sinusoidal ones. So a second criterion is necessary to deal more precisely with MWCSH under (iii) above. This concept, to be introduced here, is based on the idea, hinted at above and originally proposed from consideration of molecular models (5, 6), that a flow is *strong* if the strain history contains a component which grows sufficiently fast exponentially in time. (Note: increasing time corresponds to decreasing s.) Such flows may then give rise to unbounded stresses. On the other hand the flow is *weak* if it does not contain any exponentially growing strain terms. Note that in weak flows the strain thus admits polynomial dependence on s. In this note we would like to recast this definition in terms of the eigenvalues of the velocity gradient $\boldsymbol{L}$ and we shall use the *Jordan* form of $\boldsymbol{L}$ to arrive at our conclusions. *Giesekus* (1, 2) also based his classification scheme on the eigenvalues of $\boldsymbol{L}$, and gave names to the various flow categories depending on the shape of the particle trajectories. There were several categories, and our present classification is much simpler, containing only two flow categories (strong and weak), which we believe to be adequate for many purposes.

Previously (5–7) we have taken into account the time constant θ of our molecular material when dividing the flow fields into "strong" and "weak" cases, and found criteria of the following form for determining strong fields

$$f(\boldsymbol{L}) > 1/\theta. \qquad [3]$$

The present treatment is equivalent to letting θ approach very large values; which, in turn, is equivalent to assuming that hydrodynamic forces completely dominate spring forces in the dumbbell models (5–7). The treatment can be altered to include the time constant terms, but in that case one might perhaps object that our criteria are not purely kinematic, as we wish them to be here. Therefore no time constant terms will be considered now.

2. Analysis

Suppose, given a flow field, it is determined that $\boldsymbol{L}$ falls under the case (iii) above, that is $\boldsymbol{L}^n \neq \boldsymbol{0}$, $n = 1, 2, 3, \ldots$. Then the eigenvalues of $\boldsymbol{L}$ cannot all be zero. If all the eigenvalues of $\boldsymbol{L}$ are zero, then the *Jordan* form of the matrix of $\boldsymbol{L}$ will have the form

$$[\boldsymbol{L}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad [\boldsymbol{L}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad [4]$$

and these obey $\boldsymbol{L}^2 = \boldsymbol{0}$ or $\boldsymbol{L}^2 \neq \boldsymbol{0}$, $\boldsymbol{L}^3 = \boldsymbol{0}$ respectively. Thus MWCSH of types (i) and (ii) above are weak by our definition.

Hence we see that if $\boldsymbol{L}^n \neq \boldsymbol{0}$, $n = 1, 2, 3, \ldots$, then the eigenvalues $\lambda_i, i = 1, 2, 3$ of $\boldsymbol{L}$ are not all zero. We now have three more cases, under the condition of incompressibility of the material, i.e., div $\boldsymbol{v} = 0$, and that $\boldsymbol{L}$ is real. Thus,

(a) the eigenvalues are real and unequal, i.e.,

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1, \; \lambda_1 + \lambda_2 + \lambda_3 = 0;$$

(b) the eigenvalues are unequal, but two occur in conjugate pairs, i.e.,

$$\lambda_1 = \alpha + i\beta, \; \lambda_2 = \alpha - i\beta, \; \lambda_3 = -2\alpha;$$

(c) the eigenvalues are real, but two are equal, i.e.,

$$\lambda_1 = \lambda_2 = \alpha, \; \lambda_3 = -2\alpha.$$

(The case of all equal real eigenvalues leads to $\lambda_i \equiv 0$, $i = 1, 2, 3$, and so cannot be included here; it was treated above.) Corresponding to the above three cases, one can find three non-singular transformations $\boldsymbol{M}_i$, $i = 1,2,3$ such that

[1]) If $\boldsymbol{L}$ commutes with $\boldsymbol{L}^T$, then $e^{-s\boldsymbol{L}^T} e^{-s\boldsymbol{L}} = e^{-s(\boldsymbol{L}^T + \boldsymbol{L})}$ and we get elongational flow. See (13) for other cases.

the matrix of L is put into the respective *Jordan* forms:

$$\text{(a)}\ [M_1 L M_1^{-1}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = [\Lambda_1]. \quad [5]$$

$$\text{(b)}\ [M_2 L M_2^{-1}] = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & -2\alpha \end{bmatrix}$$

$$= [\Lambda_2] + [W] \quad [6]$$

where $[\Lambda_2]$ is diagonal and $[W]$ is skew;

$$\text{(c)}\ [M_3 L M_3^{-1}] = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{bmatrix}$$

$$= [\Lambda_3] + [J] \quad [7]$$

where $[\Lambda_3]$ is diagonal and $[J]$ contains the off-diagonal component. The *Jordan* form for case (a) is discussed in *Hildebrand* (14) while for cases (b) and (c), the *Jordan* forms are given in *Martin* and *Mizel* (15). For (a), (b) and (c), e^{-sL} has the following forms:

$$\text{(a)}\ e^{-sL} = M_1 e^{-s\Lambda_1} M_1^{-1}; \quad [8]$$

$$\text{(b)}\ e^{-sL} = M_2 e^{-s\Lambda_2} Q(s) M_2^{-1}; \quad [9]$$

$$\text{(c)}\ e^{-sL} = M_3 e^{-s\Lambda_3} (I - sJ) M_3^{-1}. \quad [10]$$

In deriving the above forms, we have used the following well-known results:

(i) $e^{M\Lambda M^{-1}} = M e^{\Lambda} M^{-1}$

for every non-singular M.

(ii) Since Λ_2 commutes with W, i.e., $\Lambda_2 W = W \Lambda_2$,

$\exp(\Lambda_2 + W) = \exp(\Lambda_2)\exp(W)$.

Now $\exp(-sW)$, where W is skew-symmetric is orthogonal. This expression has been put as $Q(s)$ in [9] and $Q(s)$ is orthogonal.

(iii) Similarly, Λ_3 commutes with J and $J^2 = 0$, and hence

$$e^{-sL} = M_3 e^{-s\Lambda_3} e^{-sJ} M_3^{-1}$$

$$= M_3 e^{-s\Lambda_3}(1 - sJ) M_3^{-1}. \quad [11]$$

Note that $e^{-s\Lambda_i}$, $i = 1, 2, 3$ are all diagonal matrices with exponential terms such as $\exp(-s\lambda_1)$, $\exp(2\alpha s)$, etc.

In [8]–[10], M_i, $i = 1, 2, 3$, are constant tensors, and since L and L^T have the same eigenvalues, and since the strain history [1] is a product of e^{-sL^T} and e^{-sL}, it is clear that the strain history will have an exponentially growing component if one of the eigenvalues is positive, or has a positive real part. Thus a strong flow is characterized by at least one positive eigenvalue of L, while in a weak flow, no positive eigenvalues can occur. It is therefore immediate that the flow is weak if the eigenvalues of L are all zero or two are imaginary conjugates (then the third is automatically zero for it is real). Hence a weak flow falls under case (b) above with $\alpha = 0$, $\beta \neq 0$ or $\lambda_i \equiv 0$, $i = 1, 2, 3$. We shall now substantiate the claim in (6) that the flow in the *Maxwell* orthogonal rheometer is weak. The velocity field in this flow has the form

$$v_1 = -\Omega x_2 + \Omega\psi x_3,$$

$$v_2 = \Omega x_1,$$

$$v_3 = 0. \quad [12]$$

The velocity gradient L has the matrix form

$$[L] = \begin{bmatrix} 0 & -\Omega & \Omega\psi \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [13]$$

and its eigenvalues are $\{i\Omega, -i\Omega, 0\}$. Thus this flow is weak.

3. Conclusions

We have seen that for the case when the "restoring force" or time-constant terms are omitted we reduce the problem to determining whether or not the velocity gradient matrix L has any eigenvalues with a positive real part; if it does, then we have, ignoring the restoring forces in the material, a strong flow. The reference of *Giesekus* (1) should be consulted by the interested reader; it may easily be seen by comparison that only the "pure elliptic" and "pure parabolic" flows in his classification are "weak" flows – the rest are "strong". Thus, working from reference (1) or the present work, if we are given any particular flow field, we can now determine whether or not the flow behaves basically like a shear flow (weak flow) or an elongational flow (strong flow). Since this criterion depends only on standard computations, we feel that it will be useful in estimating how a real fluid may behave in a given flow field.

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Summary

There are some flows in which certain strain components grow exponentially in time, while there are other flows in which the components depend otherwise on the time. In this paper the former type are called strong flows and the latter weak. An examination of the *Jordan* form of the matrix of the velocity gradient of a steady, homogeneous, isochoric flow is made, along with the eigenvalues of such a matrix, to discover when such a flow is strong or weak. It is shown that if the eigenvalues are all zero or if one is zero and the other two purely imaginary, then the flow is weak, with the remaining cases leading to strong flows.

Zusammenfassung

Es gibt Strömungen, in denen gewisse Deformationskomponenten exponentiell mit der Zeit anwachsen, und es gibt solche, in denen die Zeitabhängigkeit der Deformationskomponenten eine andere mathematische Form aufweist. In dieser Abhandlung sind die ersteren als *starke* Strömungen und die letztgenannten als *schwache* Strömungen bezeichnet. Eine Untersuchung der *Jordan*schen Form der Matrix des Geschwindigkeitsgradienten einer stationären, homogenen Strömung, zusammen mit den Eigenwerten einer solchen Matrix, erlaubt zu bestimmen, ob die Strömung stark oder schwach ist. Es wird gezeigt, daß die Strömung schwach ist, wenn entweder alle Eigenwerte verschwinden oder aber wenn ein Eigenwert verschwindet und die beiden anderen rein imaginär sind. Alle übrigbleibenden Fälle entsprechen starken Strömungen.

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