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# On a Conjecture of R. E. Miles About the Convex Hull of Random Points

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*Dedicated to Professor E. Hlawka on the occasion of his seventieth birthday*

**Abstract.** Denote by $p_{d+i}(B_d, d+m)$ the probability that the convex hull of $d+m$ points chosen independently and uniformly from a $d$-dimensional ball $B_d$ possesses $d+i (i = 1, \ldots, m)$ vertices. We prove Miles' conjecture that, given any integer $m$, $p_{d+m}(B_d, d+m) \to 1$ as $d \to \infty$. This is obvious for $m = 1$ and was shown by Kingman for $m = 2$ and by Miles for $m = 3$. Further, a related result by Miles is generalized, and several consequences are deduced.

## 1. Introduction

We shall consider a generalization of Sylvester's problem which, in its classical version, asks for the probability $p_3(C, 4)$ that the convex hull of 4 points chosen independently and uniformly from a given plane convex body $C$ possesses only 3 vertices. It is easy to see (cf., e. g., SANTALÓ [21, pp. 63—64]) that $p_3(C, 4)$ is four times the expected area of the convex hull of 3 points chosen independently and uniformly from $C$ if the area of $C$ is 1. BLASCHKE [3], [4, pp. 55—60] proved that $p_3(C, 4)$ attains its minimum if $C$ is an ellipse and its maximum if $C$ is a triangle:

$$\frac{35}{12\pi^2} = p_3\,(\text{ellipse}, 4) \leqslant p_3(C, 4) \leqslant p_3(\text{triangle}, 4) = \tfrac{1}{3}\,.$$

(Note that $p_3(C, 4)$ is invariant under nonsingular affine transformations of $C$.) A special case of a recent result [5] gives explicit values of $p_3(C, 4)$ for any convex polygon $C$; cf. [5] also for further references.

A three-dimensional version of Sylvester's problem was treated by HOSTINSKÝ [12, pp. 22—26], who determined the probability that the convex hull of 5 points chosen from a three-dimensional ball $B_3$ is a tetrahedron:

$$p_4(B_3, 5) = \tfrac{9}{143} .$$

Decades later, KINGMAN [14] calculated the probability that the convex hull of $d+2$ points chosen from a $d$-dimensional ball $B_d$ is a simplex:

$$p_{d+1}(B_d, d+2) = \frac{d+2}{2^d} \binom{d+1}{(d+1)/2}^{d+1} \binom{(d+1)^2}{(d+1)^2/2}^{-1} .$$

(Here as well as throughout the present paper, in the case of even $d$, the binomial coefficients are defined on replacing $d!$ by $\Gamma(d+1)$.)

GROEMER [9], [10] extended the left hand side of Blaschke's inequality by showing that, for any $d, p_{d+1}(C, d+2)$ attains its minimum among all $d$-dimensional convex bodies $C$ if $C$ is an ellipsoid. Thus the values obtained by Hostinský and Kingman provide lower bounds. For $d = 3$, SOLOMON [23, p. 125] conjectures that the maximum is attained if $C$ is a tetrahedron. (BLASCHKE [3, p. 452] asserted that the method used to establish his planar result works in all dimensions and that this method shows that the maximum is attained for the $d$-simplex. However, a proof has apparently never been published, and it is not obvious how to proceed.)

MILES [17, p. 354 and pp. 369—374] generalized Sylvester's problem in the sense of finding the probabilities $p_{d+i}(B_d, d+m)$ that the convex hull of $d+m$ points chosen from a $d$-dimensional ball $B_d$ possesses $d+i \, (i = 1, \ldots, m)$ vertices. If $m = 2$, Kingman's result provides a complete answer, the case $m = 3$ was solved by Miles. (For $d = 2$ and $d = 3$, Miles calculated explicit values, but refrained from deriving concise formulae for $d \geqslant 4$, although he had definitely solved the problem in all dimensions. However, LITTLE's survey [16, p. 105] only mentions the planar and the three-dimensional case, SANTALÓ [21, p. 65] even states that the 'general', i.e., the higher-dimensional, values are unknown.) For $m \geqslant 4$, it seems to be difficult to obtain the probabilities $p_{d+i}(B_d, d+m)$ explicitly if $i \neq 1$.

Kingman's result shows that

$$\lim_{d \to \infty} p_{d+2}(B_d, d+2) = 1 ,$$

and Miles' result gives

$$\lim_{d\to\infty} p_{d+3}(B_d, d+3) = 1 .$$

Miles therefore conjectured that, generally,

$$\lim_{d\to\infty} p_{d+m}(B_d, d+m) = 1 .$$

In Section 2 of the present article, Miles' conjecture is verified for any integer $m$. Further, the order of convergence is estimated (Theorem 1).

In Section 3 we first prove that the expected volume of the convex hull of $d+2$ points chosen from a $d$-dimensional convex body $C$ is $(d+2)/2$ times the expected volume of a random simplex in $C$ (Theorem 2). This observation is used to express the probabilities $p_{d+i}(C, d+3)$ $(i = 1, 2, 3)$ in terms of the first and the second moment of the volume of a random simplex in $C$ (Theorem 3).

Subsequently (Remarks 6—13), we have a look at some consequences of Theorem 3. If $C$ is a ball (Remark 6), in which case these moments were derived by Kingman and Miles, we rediscover Miles' results avoiding the consideration of '$m$-filled simplices' which are Miles' main tool. (Yet, Miles' idea is related to Theorem 2.) Apart from the ball, first *and* second moments of the volume of a random simplex are apparently only known for the triangle and for the parallelogram (Remark 7). However, using a theorem due to Blaschke, we deduce estimates for the probabilities $p_3(C, 5)$, $p_4(C, 5)$ and $p_5(C, 5)$, where $C$ is an arbitrary plane convex body (Remark 8). Further, we give estimates for $p_{d+i}(S_d, d+3)$ $(i = 1, 2, 3)$, $S_d$ denoting a $d$-dimensional simplex (Remarks 9—11). Finally, special attention is paid to the probability $p_{d+1}(C, d+m)$ (Remarks 12—13).

For related results see the monograph by KENDALL and MORAN [13] and the later surveys by MORAN [18], [19], LITTLE [16], and BADDELEY [2], as well as the monograph by SANTALÓ [21]. A list of references is also contained in [7].

## 2. Proof of Miles' Conjecture

**Theorem 1.** *For $i = 1, \ldots, m$, let $p_{d+i}(B_d, d+m)$ denote the probabilities that the convex hull of $d+m$ points chosen independently and uniformly from a d-dimensional ball $B_d$ possesses $d+i$ vertices. Then*

$$1 - p_{d+m}(B_d, d+m) = o\left(\left(\frac{1}{2\pi d}\right)^{(d-2m-2)/2}\right) \text{ as } d \to \infty ,$$

*and*

$$p_{d+i}(B_d, d+m) = o\left(\left(\frac{1}{2\pi d}\right)^{(d-2m-2)/2}\right) \text{ as } d \to \infty$$

*for* $i = 1, \ldots, m-1$.

*Remark* 1. In the cases $m = 2$ and $m = 3$, the exact order of convergence may be derived from Kingman's and Miles' results mentioned in the introduction:

$$1 - p_{d+2}(B_d, d+2) = O\left(\left(\frac{1}{2\pi d}\right)^{(d-3)/2}\right) \text{ as } d \to \infty ,$$

$$1 - p_{d+3}(B_d, d+3) = O\left(\left(\frac{1}{2\pi d}\right)^{(d-5)/2}\right) \text{ as } d \to \infty .$$

*Proof of Theorem* 1. We denote by $S(B_d)$ and $V(B_d)$ the surface area and the volume of the given ball $B_d$ and by $P_n$, $S_n$ and $V_n$ the number of vertices, the surface area and the volume of the convex hull of $n$ points chosen independently and uniformly from $B_d$. The expected number $E(P_{d+m})$ of vertices of the convex hull of $d+m$ points satisfies

$$E(P_{d+m}) = \sum_{i=1}^{m} (d+i) p_{d+i}(B_d, d+m) \leqslant$$

$$\leqslant (d+m-1)\,(1 - p_{d+m}(B_d, d+m)) + (d+m) p_{d+m}(B_d, d+m) =$$

$$= d + m - 1 + p_{d+m}(B_d, d+m) .$$

One of the $d+m$ points is a vertex of their convex hull if it is not contained in the convex hull of the remaining $d+m-1$ points. Obviously, this event occurs with probability $1 - E(V_{d+m-1})/V(B_d)$. As all points are independently and identically distributed, the expected number of vertices of the convex hull is given by

$$E(P_{d+m}) = (d+m)\left(1 - \frac{E(V_{d+m-1})}{V(B_d)}\right).$$

Thus we have shown that

$$1 - p_{d+m}(B_d, d+m) \leqslant (d+m)\frac{E(V_{d+m-1})}{V(B_d)} .$$

The isoperimetric inequality

$$\left(\frac{V_{d+m-1}}{V(B_d)}\right)^{d-1} \leqslant \left(\frac{S_{d+m-1}}{S(B_d)}\right)^{d}$$

and the relation $S_{d+m-1} < S(B)$ (which follows from the monotony of the surface area of convex bodies) yield

$$\frac{V_{d+m-1}}{V(B_d)} \leqslant \left(\frac{S_{d+m-1}}{S(B_d)}\right)^{d/(d-1)} < \frac{S_{d+m-1}}{S(B_d)},$$

and consequently

$$1 - p_{d+m}(B_d, d+m) < (d+m)\frac{E(S_{d+m-1})}{S(B_d)}.$$

A by-product of a recent paper [8, p. 758] is the integral representation

$$\frac{E(S_{d+m-1})}{S(B_d)} = \binom{d+m-1}{d} \frac{d}{(d+1)^{d-1}} \left(\frac{\varrho_{d-1}}{\varrho_d}\right)^{d} \cdot \int_{-1}^{1} \left(\frac{\varrho_{d-1}}{\varrho_d} \int_{p}^{1} (1-q^2)^{(d-1)/2}\, dq\right)^{m-1} (1-p^2)^{(d-1)(d+2)/2}\, dp,$$

where $\varrho_d$ is the volume of the unit ball. As

$$\int_{-1}^{1} (1-q^2)^{(d-1)/2}\, dq = \frac{\varrho_d}{\varrho_{d-1}},$$

it follows that

$$\frac{\varrho_{d-1}}{\varrho_d} \int_{p}^{1} (1-q^2)^{(d-1)/2}\, dq \leqslant 1 .$$

This yields

$$1 - p_{d+m}(B_d, d+m) < $$
$$< (d+m)\binom{d+m-1}{d} \frac{d}{(d+1)^{d-1}} \left(\frac{\varrho_{d-1}}{\varrho_d}\right)^{d} \frac{\varrho_{d^2+d-1}}{\varrho_{d^2+d-2}}.$$

From Wallis' formula (cf., e.g., Abramowitz and Stegun [1, p. 258]) we know that

$$\frac{\varrho_{d-1}}{\varrho_d} = \left(\frac{d}{2\pi}\right)^{1/2}\left(1 + O\left(\frac{1}{d}\right)\right) \text{ as } d \to \infty$$

and obtain

$$1 - p_{d+m}(B_d, d+m) = o\left(\left(\frac{1}{2\pi d}\right)^{(d-2m-2)/2}\right) \text{ as } d \to \infty .$$

For $i = 1, \ldots, m-1$, the corresponding expressions follow trivially. □

## 3. Some Further Results

**Theorem 2.** *The expected volume $E(V_n)$ of the convex hull of n points chosen independently and uniformly from a d-dimensional convex body satisfies*

$$E(V_{d+2}) = \frac{d+2}{2} E(V_{d+1}) .$$

*Remark* 2. In spaces of dimensions 2 and 3, this result was established earlier [6] by completely different arguments.

*Remark* 3. Even in the planar case, the ratio of $E(V_{n+1})$ to $E(V_n)$ depends on the convex body from which the points are chosen if $n \geqslant 4$. A table of numerical examples is contained in [6].

*Proof of Theorem* 2. Clearly, $d+2$ points in $d$-space determine $d+2$ simplices. The expected volume of each such simplex is $E(V_{d+1})$, the expected sum of all volumes thus is $(d+2)E(V_{d+1})$. On the other hand, the convex hull of $d+2$ points in $d$-space may be considered as a degenerate $(d+1)$-simplex being contained in a $d$-dimensional hyperplane. Its $d$-dimensional volume is half of its $d$-dimensional surface area, which is the sum of the volumes of its facets. These facets are exactly those $d+2$ different $d$-simplices which are defined by the considered $d+2$ points. □

*Remark* 4. Note that the proof of Theorem 2 does not make use of the fact that the points are uniformly distributed.

**Theorem 3.** *Denote by $M_r$ the r-th moment of the volume of the simplex spanned by $d+1$ points chosen independently and uniformly from a d-dimensional convex body C of volume* 1. *Then, the probabilities $p_{d+i}(C, d+3)$ $(i = 1, 2, 3)$ are given by*

$$p_{d+1}(C, d+3) = \binom{d+3}{2} M_2,$$

$$p_{d+2}(C, d+3) = \binom{d+3}{2} (M_1 - 2M_2),$$

$$p_{d+3}(C, d+3) = 1 - \binom{d+3}{2} (M_1 - M_2).$$

*Remark* 5. The first relation may be generalized:

$$p_{d+1}(C, d+m) = \binom{d+m}{m-1} M_{m-1}.$$

*Proof of Theorem* 3. The convex hull of $d + m$ points is a simplex if there exists any collection of $m - 1$ points which are all contained in the simplex spanned by the remaining $d + 1$ points. For any $m - 1$ points, the event that they lie in the simplex spanned by the other $d + 1$ points occurs with probability $M_{m-1}$. As there are $\binom{d+m}{m-1}$ possibilities of choosing $m - 1$ points out of $d + m$ and as, with probability 1, at most one collection has the desired property, it follows that

$$p_{d+1}(C, d+m) = \binom{d+m}{m-1} M_{m-1}. \tag{1}$$

Analogously to the proof of Theorem 1, we may express the expected number of vertices of the convex hull of $d + 3$ points by the expected volume of the convex hull of $d + 2$ points which we may again, by Theorem 2, express by the expected volume of the convex hull of $d + 1$ points:

$$\sum_{i=1}^{3} (d+i) p_{d+i}(C, d+3) = (d+3)\left(1 - \frac{d+2}{2} M_1\right). \tag{2}$$

Clearly,

$$\sum_{i=1}^{3} p_{d+i}(C, d+3) = 1. \tag{3}$$

Equations (1), (2) and (3) immediately lead to the claimed result. □

*Remark* 6. Theorem 3 yields explicit values in the case that $C$ is a ball, where, according to KINGMAN [14, p. 671] and MILES [17, p. 363],

$$M_1 = \frac{1}{2^d}\binom{d+1}{(d+1)/2}^{d+1}\binom{(d+1)^2}{(d+1)^2/2}^{-1},$$

$$M_2 = \frac{(d+1)^2}{2^{2d+2}(d+2)^d}\binom{d+1}{(d+1)/2}\frac{1}{\pi^{d-1}}.$$

Especially, in the planar case $M_1 = 35/48\pi^2$, $M_2 = 3/32\pi^2$, and in the three-dimensional case $M_1 = 9/715$, $M_2 = 3/1000\pi^2$, whence the values of $p_{d+i}(B_d, d+3)$ $(d = 2, 3; i = 1, 2, 3)$ calculated by MILES [17, p. 354 and p. 373] follow immediately. (These values are also referred to by SANTALÓ [21, p. 65].)

*Remark* 7. Apart from the ball, $M_1$ *and* $M_2$ are apparently only known for the triangle ($M_1 = 1/12$, $M_2 = 1/72$) and for the parallelogram ($M_1 = 11/144$, $M_2 = 1/96$). The values of $M_1$ already were known last century, the values of $M_2$ are due to REED [20, p. 197], but cf. HENZE [11, p. 123]. Thus we obtain for the triangle

$$p_3 = \tfrac{5}{36}, \quad p_4 = \tfrac{20}{36}, \quad p_5 = \tfrac{11}{36},$$

and for the parallelogram

$$p_3 = \tfrac{15}{144}, \quad p_4 = \tfrac{80}{144}, \quad p_5 = \tfrac{49}{144}.$$

*Remark* 8. For any plane convex body $C$, the probabilities $p_3(C,5)$, $p_4(C,5)$ and $p_5(C,5)$ satisfy

$$0.094\ldots = \frac{15}{16\pi^2} \leqslant p_3(C,5) \leqslant \tfrac{5}{36} = 0.138\ldots,$$

$$0.482\ldots = -\tfrac{5}{24} + \frac{655}{96\pi^2} < p_4(C,5) < \tfrac{55}{72} - \frac{45}{32\pi^2} = 0.621\ldots,$$

$$0.283\ldots = \tfrac{17}{72} + \frac{15}{32\pi^2} < p_5(C,5) < \tfrac{77}{72} - \frac{655}{96\pi^2} = 0.378\ldots.$$

These relations follow from a result due to BLASCHKE [3, p. 453], [4, p. 60] stating that, among all plane convex bodies $C$ of area 1, the expected value of any continuous, positive and nondecreasing function of the area of a random triangle in $C$ attains its minimum if $C$

is an ellipse and its maximum if $C$ is a triangle. The bounds for $p_4(C, 5)$ and $p_5(C, 5)$ are obtained on considering $(M_1 - \frac{1}{2} M_2) - \frac{3}{2} M_2$ and $(M_1 - \frac{1}{2} M_2) - \frac{1}{2} M_2$. Obviously, these bounds are not attained for any convex body $C$, and it seems to be an open problem to find the best possible bounds.

*Remark* 9. For a $d$-dimensional simplex $S_d$, $M_2$ was determined by REED [20, p. 186]:

$$M_2 = \frac{d!}{(d+1)^d (d+2)^d},$$

but $M_1$ is not known (cf. KLEE [15]). The Cauchy-Schwarz inequality implies $M_1 < M_2^{1/2}$. Further, a higher-dimensional analogue to Blaschke's theorem (cf. Remark 8) due to SCHÖPF [22] states that, among all $d$-dimensional convex bodies $C$ of volume 1, the expected value of any continuous, positive and nondecreasing function of the volume of a random simplex in $C$ attains its minimum if $C$ is an ellipsoid. Thus the corresponding value for the ball (given in Remark 6) provides a lower bound for $M_1 - \frac{1}{2} M_2$. (It is easy to see that estimating $M_1 - \frac{1}{2} M_2$ is more profitable than considering merely $M_1$.) To summarize, the probabilities $p_{d+i}(S_d, d+3)$ $(i = 1, 2, 3)$ satisfy

$$p_{d+1}(S_d, d+3) = \binom{d+3}{2} \gamma_d,$$

$$\binom{d+3}{2} (\alpha_d - \tfrac{1}{2} \beta_d - \tfrac{3}{2} \gamma_d) < p_{d+2}(S_d, d+3) < \binom{d+3}{2} (\gamma_d^{1/2} - 2\gamma_d),$$

$$1 - \binom{d+3}{2} (\gamma_d^{1/2} - \gamma_d) < p_{d+3}(S_d, d+3) <$$

$$< 1 - \binom{d+3}{2} (\alpha_d - \tfrac{1}{2} \beta_d - \tfrac{1}{2} \gamma_d),$$

where

$$\alpha_d = \frac{1}{2^d} \binom{d+1}{(d+1)/2}^{d+1} \binom{(d+1)^2}{(d+1)^2/2}^{-1},$$

$$\beta_d = \frac{(d+1)^2}{2^{2d+2} (d+2)^d} \binom{d+1}{(d+1)/2} \frac{1}{\pi^{d-1}},$$

$$\gamma_d = \frac{d!}{(d+1)^d (d+2)^d}.$$

*Remark* 10. Especially, if $d = 3$,

$$p_4(S_3, 6) = 0.011\ldots,$$
$$0.169\ldots < p_5(S_3, 6) < 0.388\ldots,$$
$$0.600\ldots < p_6(S_3, 6) < 0.819\ldots.$$

By Monte Carlo experiments, REED [20, p. 197] obtained $M_1 \approx 0.01763$; similar results are due to Baker and, independently, to Marsaglia (cf. KLEE [15, p. 287]). Hence, approximately,

$$p_5(S_3, 6) = 0.241\ldots,$$
$$p_6(S_3, 6) = 0.746\ldots.$$

*Remark* 11. As $d$ tends to infinity,

$$\alpha_d = O\left(\left(\frac{1}{2\pi d}\right)^{(d-1)/2}\right),$$
$$\beta_d = O\left(\left(\frac{1}{2\pi d}\right)^{(2d-3)/2}\right),$$
$$\gamma_d = O\left(\left(\frac{1}{e d}\right)^{(2d-1)/2}\right).$$

Correspondingly, for large $d$,

$$p_{d+1}(S_d, d+3) = O\left(\left(\frac{1}{e d}\right)^{(2d-5)/2}\right),$$
$$c_1\left(\frac{1}{2\pi d}\right)^{(d-5)/2} \leqslant p_{d+2}(S_d, d+3) \leqslant c_2\left(\frac{1}{e d}\right)^{(2d-9)/4},$$
$$c_1\left(\frac{1}{2\pi d}\right)^{(d-5)/2} \leqslant 1 - p_{d+3}(S_d, d+3) \leqslant c_2\left(\frac{1}{e d}\right)^{(2d-9)/4},$$

where the constants $c_1$ and $c_2$ do not depend on $d$.

*Remark* 12. Given any integer $m$, the probability $p_{d+1}(C, d+m)$ attains its minimum among all $d$-dimensional convex bodies $C$ if $C$ is an ellipsoid. This is an obvious consequence of Remark 5 and of Groemer's theorem [9], [10] that $M_r$ (defined in Theorem 3) attains its minimum if $C$ is an ellipsoid.

*Remark* 13. Given any integer $m$, the probability $p_3(C, 2+m)$ attains its maximum among all plane convex bodies $C$ if $C$ is a triangle. This follows immediately from Remark 5 and from Blaschke's result stated in Remark 8. It would be very interesting to prove a higher-dimensional version.

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