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$L_p - L_{p'}$-Estimates for Fourier Integral Operators Related to Hyperbolic Equations

Philip Brenner

Department of Mathematics, Chalmers University of Technology and the University of Göteborg, Fack, S-40220 Göteborg 5, Sweden

1. Introduction

The purpose of this note is to prove local $L_p - L_{p'}$-estimates for certain Fourier integral operators, and to apply these estimates to obtain existence and uniqueness results in $L_{p'}$, $p' > 2$, for some semilinear hyperbolic problems. Let us here remark that estimates of this type were suggested, and to some extent proved, in a slightly different setting, by Littman [8]. See also [1]. In the case of semi-linear problems for the wave-equation the corresponding results are due to Strichartz [13, 14].

The solutions of a large class of hyperbolic initial value problems in $(x, t) \in \mathbf{R}^n \times \mathbf{R}$, with data on $t=0$ may be written as a finite sum of (properly supported) Fourier integral operators and of integral operators with C^∞-kernels (see [2–4]). Here the Fourier integral operators are given (for t fixed) by locally canonical graphs, and for $t=0$ they reduce to pseudo-differential operators (for these concepts, and other properties of Fourier integral operators, we refer to [3, 4] and [6]). This means that locally, in a neighborhood of $t=0$, say, the operators may be written

$$Q(t)\, u(x) = \iint e^{2\pi i \phi(t; x, y, \xi)} q(t; x, y, \xi)\, u(y)\, dy\, d\xi, \tag{1}$$

where ϕ is a real non-degenerate operator phase function, depending smoothly on the parameter t, and q is a symbol in the class $S^{-\nu}$, some $\nu \geqq 0$. In addition, if $\phi(0; x, y, \xi) = \psi(x, y, \xi)$, then $\operatorname{grad}_\xi \psi = 0$ for $\xi \neq 0$ exactly on the diagonal $x = y$, where also $\operatorname{grad}_x \psi = -\operatorname{grad}_y \psi = \xi$, so that

$$\psi(x, y, \xi) = \langle x - y, \xi \rangle + O(|x-y|^2\, |\xi|), \qquad x \to y. \tag{2}$$

In view of this we shall in the following assume that q for small values of t, $|t| < \varepsilon$ say, has support in some (sufficiently small) neighborhood of the diagonal $x = y$, adding if necessary an integral operator with C^∞-kernel to $Q(t)$. Thus $Q(t)$ is assumed proper (see [6]).

In Section 4 below we shall in particular use the phase-function $\phi(t; x, y, \xi) = \psi(x, y, \xi) \pm t\, p(y, \xi)$, with ψ satisfying (2) and $p(y, \xi) \neq 0$ for $\xi \neq 0$.

The existence of local $L_p - L_{p'}$-estimates, that is estimates $L_p^{\text{comp}} \to L_{p'}^{\text{loc}}$, for integral operators with C^∞-kernels are obvious. It therefore remains to obtain such estimates for (properly supported) operators with local representations (1) for $|t|$ small, using the oscillatory character of the integral (1).

As local $L_p - L_{p'}$-estimates (with $1 < p \leqq 2$, $1/p + 1/p' = 1$, as we shall assume from now on) imply local $L_{p'} - L_{p'}$-estimates, the nonexistence of the latter for sufficiently large values of p' for the wave-equation ([9]) shows that some restrictions on p will be necessary, say in terms of the size of $\delta = 1/p - 1/2 = 1/2 - 1/p'$.

Assume in addition to the above conditions that the phase functions ϕ used in (1) satisfy

(*) the hessian matrix $\frac{d}{dt} \phi''_{\xi\xi}(t; x, y, \xi)$ at $t=0$ has for $\xi \neq 0$ rank at least ρ on the diagonal $x = y$.

As an example, let $p(y, \xi) = \left(\sum_{k,l=1}^{n} a_{kl}(y) \xi_k \xi_l \right)^{1/2}$, where $(a_{kl}(y))_{k,l}$ is a real positive definit $n \times n$-matrix for $y \in \mathbf{R}^n$, and let as above $\phi = \psi \pm tp$. Then $\rho = n-1$. In general, the homogeneity of ϕ implies that $0 \leqq \rho \leqq n-1$.

Under the above assumptions on $Q(t)$, locally represented by (1), we shall prove that for each compact set $K \subset \mathbf{R}^n$ there are $\varepsilon > 0$ and a constant $C = C_K$ such that for $(2n - \rho)\delta \leqq \nu$,

$$\|Q(t)u\|_{L_{p'}(K)} \leqq C_K |t|^{\nu - 2n\delta} \|u\|_p, \quad 0 < |t| < \varepsilon, \; u \in C_0^\infty. \tag{3}$$

The proof of (3), and in fact a slightly more general inequality formulated in terms of certain Besov spaces, will be carried out in Sections 2 and 3 below (Theorems 1 and 2). The method of proof will depend strongly on that used for the "constant coefficient" case in [1]. As an application we scetch in Section 4 some existence and uniqueness results for a class of semi-linear hyperbolic problems. Using (3), these results are obtained by using the known (local) structure of the solution of the linear problem (in our case found in [2]) and a strightforward application of the ideas used by Strichartz [14] for the corresponding problems for the wave-operator.

2. Some Preliminary Results

Let $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, $j > 0$, and $\varphi_0 = 1 - \sum \varphi_j$, where $\varphi \in C_0^\infty(\mathbf{R}^n)$, $\varphi \geqq 0$, and $\operatorname{supp} \varphi \subseteq \{\xi; 1/2 < |\xi| < 2\}$ is such that

$$\sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

Define the Fourier transform $\hat{u}$ of an L_1-function u by

$$\mathscr{F}u(\xi) = \hat{u}(\xi) = \int_{R^n} e^{2\pi i \langle x, \xi \rangle} u(x)\, dx,$$

and then define $B_p^{s,q}$, s real, as the closure of C_0^∞ in the norm

$$\|v\|_{B_p^{s,q}} = \left(\sum_0^\infty (2^{js}\|\mathscr{F}^{-1}(\varphi_j \hat{v})\|_p)^q\right)^{1/q},$$

with the usual modification for $q=\infty$. Here $\|\cdot\|_p$ denotes the L_p-norm.

With $\omega_s(\xi) = (1+|\xi|^2)^{s/2}$, we also define L_p^s as the closure of C_0^∞ in the norm

$$\|v\|_{p,s} = \|v\|_{L_p^s} = \|\mathscr{F}^{-1}(\omega_s \hat{v})\|_p.$$

Notice that $L_p^0 = L_p$, $1 \leqq p < \infty$.

The following inclusion lemma will be useful.

Lemma 1. *Let $1 < p \leqq 2$, $1/p + 1/p' = 1$, and $s \geqq 0$. Then*

(i) $B_p^{s,p} \subseteq L_p^s \subseteq B_p^{s,2}$,

(ii) $B_{p'}^{s,p'} \supseteq L_{p'}^s \supseteq B_{p'}^{s,2}$.

Proof. See [15, Thm. 15] and [10, 12] and also [16, Thm. 5.2.3].

The following asymptotic estimate can be found e.g. in [7] (see also [3] and [6], p. 145).

Lemma 2. *Let $\phi = \phi(\xi)$ be real and C^∞ in a neighborhood of the support of $v \in C_0^\infty$. Assume that the rank of $\phi''_{\xi\xi} = \left(\frac{\partial^2 \phi}{\partial \xi_k \partial \xi_l}\right)_{k,l}$ is at least ρ on the support of v. Then for some integer M,*

$$\|\mathscr{F}^{-1}(e^{it\phi} v)\|_\infty \leqq C(1+|t|)^{-\frac{1}{2}\rho} \sum_{|\alpha| \leqq M} \|D^\alpha v\|_1.$$

Here C depends on the bounds of the derivatives of ϕ on supp v, *on a lower bound for the maximum of the absolute values of the minors of order ρ of $\phi''_{\xi\xi}$ on* supp v, *and on* supp v.

For the general theory of Fourier integral operators, we refer to [3, 4] and [6] (for pseudo-differential operators, see also [5]). Here we will only in a very incomplete way present some of the main concepts that will be used below:

Let $\Omega \subset \mathbf{R}^n$ be open and let $1/2 < \sigma \leqq 1$. We say that $a \in C^\infty(\Omega \times \Omega \times \mathbf{R}^n)$ belongs to $S_\sigma^m(\Omega \times \Omega \times \mathbf{R}^n) = S_{\sigma,1-\sigma}^m(\Omega \times \Omega \times \mathbf{R}^n)$ if for any multi-indices α, β and γ and compact set $\tilde{K} \subset \Omega \times \Omega$,

$$|D_y^\gamma D_x^\beta D_\xi^\alpha a(x,y,\xi)| \leqq C_{\alpha,\beta,\gamma,\tilde{K}}(1+|\xi|)^{m-\sigma|\alpha|+(1-\sigma)(|\beta|+|\gamma|)}, \quad \text{for } (x,y) \in \tilde{K}.$$

We write S_σ^m for this space, whenever convenient, and S^m for S_1^m.

A Fourier integral operator can locally be represented by an oscillatory integral of the form (1) with a phase function ϕ and an amplitude $q \in S_\sigma^m$, some m and σ, $1/2 < \sigma \leqq 1$. If the Fourier integral operator is locally defined by a relation which is a canonical graph, then we may take ϕ as a non-degenerate phase function such that $D(\phi) = \det \begin{pmatrix} \phi''_{\xi\xi} & \phi''_{\xi x} \\ \phi''_{y\xi} & \phi''_{xy} \end{pmatrix} \neq 0$ on the set where $\operatorname{grad}_\xi \phi = 0$ (cf. [6], Sect. 4.1). That ϕ is an operator phase function means that ϕ is real, homogeneous of degree 1 in ξ and C^∞ for $\xi \neq 0$, and that $\operatorname{grad}_{x,\xi}\phi$ and $\operatorname{grad}_{y,\xi}\phi$ doesn't vanish for $\xi \neq 0$. That ϕ is nondegenerate means that $d\left(\frac{\partial \phi}{\partial \xi_j}\right)$, $j = 1, \ldots, n$ are linearly independent.

From now on we assume that ϕ and q are the phase function and amplitude, respectively, in the representation (1), having in particular the properties assumed in Section 1, so that $q \in S^m$ with $\sigma = 1$ and $m = -\nu$.

Since an amplitude in $S^{-\infty}$ gives an integral operator with C^∞-kernel, we may assume that $q = 0$ for $|\xi| < 1$, say. As $\operatorname{grad}_{x,\xi}\phi \neq 0$ for $\xi \neq 0$, we may assume that $|\operatorname{grad}_x \phi| \geqq c|\xi|$, $c = c(x,y) > 0$, on the support of q (uniformly for $|t| < \varepsilon$, $\varepsilon > 0$ small enough), the contribution from the set where $\operatorname{grad}_\xi \phi$ is bounded away from zero (uniformly in t) being an integral operator with C^∞-kernel (cf. [4]). In the same way we may also assume that $|\operatorname{grad}_\xi \psi| \geqq c|x-y|$ on supp q, where as above $\psi(x,y,\xi) = \phi(0; x, y, \xi)$.

Remember that the support of q may be choosen to be contained in any suitable neighborhood of the diagonal $x = y$ for $|t| < \varepsilon$, $\varepsilon > 0$ small enough. In addition, ϕ and q depend smoothly on t, and $q \in S^{-\nu}$.

Let $\chi \in C_0^\infty(R^n)$, $\chi \geqq 0$, and with support in the set $\{\xi; c < |\xi| < c^{-1}\}$, for some $c > 0$. Define $\chi_j(\xi) = \chi(2^{-j}\xi)$, $j \geqq 0$, and let $\chi_j(t; x, y, \xi) = \chi_j(\operatorname{grad}_x \phi(t; x, y, \xi))$. Notice that for any $\sigma \in (0, 1]$,

$$|D_\xi^\alpha \chi_j(\xi)| \leqq C 2^{-j(1-\sigma)|\alpha|}(1+|\xi|)^{-\sigma|\alpha|}, \tag{4}$$

and, since by our assumptions above $c|\xi| \leqq |\operatorname{grad}_x \phi| \leqq C|\xi|$ on supp q,

$$|D_y^\gamma D_x^\beta D_\xi^\alpha \chi_j(t; x, y, \xi)| \leqq C_{\alpha,\beta,\gamma} 2^{-j(1-\sigma)(|\alpha|+|\beta|+|\gamma|)}(1+|\xi|)^{-m}, \tag{5}$$

with $m = -\sigma|\alpha| + (1-\sigma)(|\beta|+|\gamma|)$, for (x, y) in compact subsets of $\Omega \times \Omega$, and $(x, y, \xi) \in \operatorname{supp} q$.

Define the Fourier integral operator $Q_j(t)$ by the local representation

$$Q_j(t)\, u(x) = \iint e^{2\pi i \phi(t; x, y, \xi)} \chi_j(\operatorname{grad}_x \phi(t; x, y, \xi))\, q_0(t; x, y, \xi)\, u(y)\, dy\, d\xi, \tag{6}$$

where ϕ is as above, and q_0 has the same properties as q, but now belongs to $S^{-\mu}$. Hence $Q_j(t)$ is given by a locally canonical graph, $2^{j\mu}\chi_j(\operatorname{grad}_x \phi)\, q_0 \in S_\sigma^0$, for $1/2 < \sigma \leqq 1$, uniformly for $j \geqq 0$, and q_0 is a properly supported symbol (i.e. has support in a neighborhood $|x-y| < \varepsilon'$, say, of the diagonal $x = y$). Then Theorem 4.3.1 in [6] applies, and proves the following result:

Lemma 3. *Under the above assumptions, for each compact set $K \subset \Omega$, there are constants C and $\varepsilon > 0$, such that if $f \in C_0^\infty$ and* $\operatorname{supp} f \subseteq K$, *then*

$$\|Q_j(t) f\|_2 \leqq C 2^{-j\mu} \|f\|_2, \quad 0 \leqq |t| < \varepsilon,\ j \geqq 0.$$

Now, in addition to the above conditions, ϕ also satisfies condition (*) of Section 1. Choosing the support of q_0 sufficiently close to the diagonal $x = y$, we may assume that for $y \in K \subset \Omega$ compact and $0 < |t| < \varepsilon$, $\frac{d}{dt}\phi''_{\xi\xi}$ has rank at least ρ for $\xi \neq 0$ in the support of q_0. Then

$$|Q_j(t)\, u(x)| \leqq \sup_{y \in K} \left|\int e^{2\pi i \phi(t; x, y, \xi)} (\chi_j(\operatorname{grad}_x \phi)\, q_0)(t; x, y, \xi)\, d\xi\right| \|u\|_1 .$$

By a change of variables

$$\begin{aligned} I_{j,t} &= \int e^{2\pi i \phi(t; x, y, \xi)} (\chi_j(\operatorname{grad}_x \phi)\, q_0)(t; x, y, \xi)\, d\xi \\ &= 2^{j(n-\mu)} \int e^{2\pi i 2^j \phi(t; x, y, \xi)} \chi(\operatorname{grad}_x \phi)\, q_0(t; x, y, 2^j \xi)\, 2^{j\nu}\, d\xi. \end{aligned}$$

Since

$$\phi(t; x, y, \xi) = \psi(x, y, \xi) + t\tilde{\phi}(x, y, \xi) + O(t^2)$$

we obtain in supp $q_0 \cap$ supp $\chi(\operatorname{grad}_x \phi)$ for $|x-y| \geqq C|t|$, C a suitably large constant, that $|\operatorname{grad}_\xi \phi| \geqq c|t|$, $c>0$, and hence repeated partial integrations give that $I_{j,t}$ is a rapidly decreasing function of $2^j|t|$, uniformly for $y \in K$. On the other hand, if $|x-y| \leqq C|t|$, then by (2)

$$\phi(t; x, y, \xi) = \langle x-y, \xi\rangle + t\tilde{\phi}(x, y, \xi) + O(t^2),$$

and thus $\tilde{\phi}''_{\xi\xi}$ has rank at least ρ in supp $q_0 \cap$ supp $(\operatorname{grad}_x \phi)$ for $|x-y| \leqq C|t|$, $y \in K$, provided $|t| < \varepsilon$, $\varepsilon > 0$ small enough. We may then apply Lemma 2, with $u \in C_0^\infty$ and supp $u \subseteq K$,

$$\begin{aligned} |Q^j(t)\,u(x)| &\leqq C 2^{j(n-\mu)}(2^j|t|+1)^{-\frac{1}{2}\rho}\,\|u\|_1 \\ &\leqq C \begin{cases} 2^{j(\frac{1}{2}(2n-\rho)-\mu)}|t|^{-\frac{1}{2}\rho}\,\|u\|_1, & 2^j|t|>1, \\ 2^{j(n-\mu)}\,\|u\|_1, & 2^j|t| \leqq 1, \end{cases} \quad 0<|t|<\varepsilon. \end{aligned} \tag{7}$$

Notice that $\varepsilon = \varepsilon(K)$ in the above argument.

Interpolating between Lemma 3 and (7), we end this section with an estimate of $Q_j(t)$:

Proposition 1. *Let* $1 \leqq p \leqq 2$, $1/p + 1/p' = 1$, $\delta = 1/p - 1/2$. *Assume that* $Q_j(t)$ *has the local representation* (6) *in* $\Omega \subseteq \mathbf{R}^n$, *with* ϕ, q_0 *and* χ_j *as above. In particular* ϕ *satisfies* (∗) *of Section* 1, *and* $q_0 \in S^{-\mu}$. *Then for each compact set* $K \subset \Omega$, *there are* $\varepsilon > 0$ *and* $C = C_K$ *such that for* $u \in C_0^\infty$ *with* supp $u \subseteq K$,

$$\|Q_j(t)\,u\|_{p'} \leqq C_K \begin{cases} 2^{j(\delta(2n-\rho)-\mu)}|t|^{-\rho\delta}\,\|u\|_p, & 2^j|t|>1, \\ 2^{j(2n\delta-\mu)}\,\|u\|_p, & 2^j|t| \leqq 1, \end{cases} \quad 0<|t|<\varepsilon. \tag{8}$$

Remark. It was suggested by Anders Melin that the above method should apply also if $q_0 \in S_\sigma^{-\mu}$ for $1/2 < \sigma \leqq 1$: By [6], pp. 144–145 (cf. also [3], Sect. 1.2) a more precise version of Lemma 2 is expressed by the following inequality:

$$\|\mathscr{F}^{-1}(e^{it\phi}v)\|_\infty \leqq C|t|^{-\rho/2}\left\{\sum_0^{k-1} |t|^{-l} \sum_{|\alpha|=2l} \|D^\alpha v\|_\infty + |t|^{-k} \sum_{|\alpha| \leqq 2k+\rho+1} \|D^\alpha v\|\right\}.$$

If $q_0 \in S_\sigma^{-\mu}$, we shall apply this to $v = \chi(\operatorname{grad}_x \phi)\, q_0(t, x, y, 2^j\xi)\, 2^{j\mu}$, so that

$$|D_\xi^\alpha v| \leqq C 2^{j(1-\sigma)|\alpha|},$$

and replace t by $2^j t$. Thus, as above

$$|I_{j,t}| \leqq C 2^{j(n-\mu)}(2^j|t|)^{-\frac{1}{2}\rho}\left\{\sum_0^{k-1} (2^{j(2\sigma-1)}|t|)^{-l} + 2^{j(\rho+1)}(2^{j(2\sigma-1)}|t|)^{-k}\right\},$$

and so

$$|I_{j,t}| \leqq \begin{cases} 2^{j(n-\mu-\rho/2)}|t|^{-\rho/2}, & 2^{j(2\sigma-1)}|t|>1, \\ 2^{j(n-\mu)}, & 2^{j(2\sigma-1)}|t| \leqq 1, \end{cases}$$

in the case $1/2 < \sigma \leqq 1$, $0 < |t| < \varepsilon$.

By interpolation we obtain under the assumptions of Proposition 1, but now assuming that $q_0 \in S_\sigma^{-\mu}$ for some $\sigma \in (\frac{1}{2}, 1]$,

$$\|Q_j(t)\,u\|_{p'} \leqq C_K \begin{cases} 2^{j(\delta(2n-\rho)-\mu)}|t|^{-\rho\delta}\|u\|_p, & 2^{j(2\sigma-1)}|t|>1, \\ 2^{j(2n\delta-\mu)}\|u\|_p, & 2^{j(2\sigma-1)}|t|\leqq 1. \end{cases} \tag{8'}$$

3. Proof of the Main Results

In this section we shall prove a result which in particular implies the estimate (3). We keep the notations and assumptions of Section 2. To avoid duplication of some computations, we shall frequently refer to [4] and [5].

As follows from (4), $\varphi_j \in S_\sigma^0$, and

$$|D_\xi^\alpha \varphi_j(\xi)| \leqq C_\alpha 2^{-j(1-\sigma)|\alpha|}(1+|\xi|)^{-\sigma|\alpha|}, \qquad 0<\sigma\leqq 1. \tag{9}$$

Let $\Phi_j f(x) = \mathscr{F}^{-1}(\varphi_j \hat{f})(x)$. The following is a consequence of (the proof of) Theorem 2.16 in [5].

Proposition 1. *Let $q \in S^{-\nu}$, $\nu \geqq 0$, and ϕ be as above. Then for $1/2<\sigma\leqq 1$,*

$$e^{-i\phi}\Phi_j(e^{i\phi}q) = \sum_{|\alpha|<N} \frac{1}{\alpha!}\varphi_j^{(\alpha)}(\operatorname{grad}_x\phi)\, D_z(q\, e^{i\phi_x''(x,y,\xi)})|_{z=x} + 2^{-j(1-\sigma)N} r_{-\nu-(\sigma-\frac{1}{2})N}, \tag{10}$$

where

$$\phi_x''(z,y,\xi) = \phi(z,y,\xi) - \phi(x,y,\xi) - \langle z-x, \phi_x'(x,y,\xi)\rangle,$$

and $r_{-M} \in S_\sigma^{-M}$ uniformly for $j\geqq 0$, $|t|<\varepsilon$.

Proof. By (9) and formula (2.19) in [5], the difference between the left hand side and the sum over $|\alpha|<N$ on the right hand side of (10) is of the order $2^{-j(1-\sigma)N}|\xi|^{n-(\frac{1}{2}-\sigma)N}$. By an obvious modification of Theorem 2.9 in [5], it follows that the left hand side of (10) belongs to S_σ^0 uniformly for $j\geqq 0$, and that the above difference belongs to $S_\sigma^{-\nu-(\sigma-\frac{1}{2})N}$ with a bound $O(2^{-j(\sigma-\frac{1}{2})N})$ for $j\geqq 0$ (this is where a modification of Theorem 2.9 in [5] is used).

Compare also with the derivation of Lemmas 2.11 and 2.12 in [4], where the case $\sigma=1$ is considered.

By operating under the sign of integration, we find by Proposition 2 that

$$\Phi_j Q(t)\, f = \sum_{|\alpha|<N} \frac{1}{\alpha!} Q_j^{(\alpha)}(t)\, f + 2^{-j(1-\sigma)N} R_N f, \qquad f\in C_0^\infty, \tag{11}$$

where the operators on the right hand side are all locally defined by oscillatory integrals with phase functions ϕ.

The amplitudes for the oscillatory integrals defining $Q_j^{(\alpha)}$ are by Proposition 2

$$q_j^{(\alpha)}(t;x,y,\xi) = \varphi_j^{(\alpha)}(\operatorname{grad}_x\phi)\, D_z^\alpha(q(t;z,y,\xi)\, e^{i\phi_x''(z,y,\xi)})|_{z=x}, \tag{12}$$

where ϕ_x'' is defined as in Proposition 2. Notice that $2^{j(1-\alpha)|\alpha|}q_j^{(\alpha)}$ belongs uniformly to $S_1^{-\nu-(\sigma-\frac{1}{2})|\alpha|}$ for $j\geqq 0$, $|t|<\varepsilon$, and has support in $c2^j\leqq|\xi|\leqq C2^j$, for some $C, c>0$. Since we assume that $\operatorname{supp} f\subseteq K$, a compact subset of Ω, we may as well assume that $q_j^{(\alpha)}(t;x,y,\xi)=0$ if $(x,y)\notin K'\times K$, for some compact set K'.

With $f = \sum f_l$, $f_l = \mathscr{F}^{-1}(\varphi_l \hat{f})$, we have

$$Q_j^{(\alpha)}(t) f = \sum_{l=0}^{\infty} Q_j^{(\alpha)}(t) f_l$$

whenever the sum is convergent. The estimate

$$\| Q_j^{(\alpha)}(t) f_l \|_{p'} \leqq C_K \begin{cases} 2^{j((2n-\rho)\delta - \nu - \frac{1}{2}|\alpha|)} |t|^{-\rho\delta} \| f_l \|_p, & 2^j|t| > 1, \\ 2^{j(2n\delta - \nu - \frac{1}{2}|\alpha|)} \| f_l \|_p, & 2^j|t| \leqq 1 \end{cases} \tag{13}$$

now follows from Proposition 1. Merely use that the support of $q_j^{(\alpha)}$ is contained in $K' \times K \times \mathbf{R}^n$. If for some constant c, $|l-j| \leqq c$, the estimate (13) will be enough. In case $|l-j| \geqq c$, the following lemma shows that (13) may be improved.

Lemma 4. *For each $\kappa > 0$ there is a constant $c > 0$ such that*

$$\| Q_j^{(\alpha)}(t) f_l \|_{p'} \leqq c 2^{-j\left(\nu + \frac{|\alpha|}{2}\right)} \begin{cases} 2^{-j\kappa} \| f_l \|_p, & l \leqq j - c, \\ 2^{-l\kappa} \| f_l \|_p, & l \geqq j + c. \end{cases} \tag{14}$$

Proof. Write, with $\tilde{\varphi}_l = \varphi_{l-1} + \varphi_l + \varphi_{l+1}$, $\varphi_{-1} = 0$,

$$\begin{aligned} Q_j^{(\alpha)}(t) f_l(x) &= \iint e^{2\pi i \phi(t; x, y, \xi)} q_j^{(\alpha)}(t; x, y, \xi) \int e^{-2\pi i \langle \eta, y \rangle} \tilde{\varphi}_1(\eta) \hat{f}_l(\eta)\, d\eta\, dy\, d\xi \\ &= \int e^{-2\pi i \langle x, \eta \rangle} \tilde{\varphi}_l(\eta) g_j(x, \eta) \hat{f}_l(\eta)\, d\eta, \end{aligned}$$

where (remember that $\operatorname{supp} q_j^{(\alpha)} \subseteq K' \times K \times \mathbf{R}^n$)

$$g_j(x, \eta) = \iint e^{2\pi i (\phi(t; x, y, \xi) + \langle x - y, \eta \rangle)} q_j^{(\alpha)}(t; x, y, \xi)\, dy\, d\xi. \tag{15}$$

We claim that there is a constant c such that for each integer A,

$$\sup_{x, \eta} |g_j(x, \eta) \tilde{\varphi}_l(\eta)| \leqq C_A 2^{jn} 2^{-j\left(\nu + \frac{|\alpha|}{2}\right)} |2^l - 2^j|^{-\sigma A}, \qquad |l - j| \geqq c. \tag{16}$$

To see this, notice that $\operatorname{grad}_y(\phi + \langle x - y, \eta \rangle = \operatorname{grad}_y \phi - \eta$, and so for the x, y, ξ in consideration and for $\eta \in \operatorname{supp} \tilde{\varphi}_l$, there is a constant c such that

$$|\operatorname{grad}_y(\phi + \langle x - y, \eta \rangle)| \geqq c |2^j - 2^l|.$$

Repeated partial integrations in (15), and observing that $q_j^{(\alpha)} \in S_\sigma^{-\nu - \sigma|\alpha| + \frac{1}{2}|\alpha|}$, then proves (16).

Now,

$$Q_j^{(\alpha)}(t) f_l(x) = \mathscr{F}_{\eta \to z}^{-1}(e^{-2\pi i \langle x, \eta \rangle} g_j(x, \eta) \tilde{\varphi}_l(\eta) \hat{f}_l(\eta))|_{z = x} = (\mathscr{F}^{-1}(g_j(x, \cdot) \tilde{\varphi}_l) * f_l)(x).$$

Since $g_j(x, \eta) \tilde{\varphi}_l(\eta)$ satisfies (16) and has support in $c 2^l \leqq |\eta| \leqq C 2^l$, we find for $|l - j| \geqq c$,

$$\| Q_j^{(\alpha)}(t) f_l \|_\infty \leqq C_A 2^{(j+1)n} 2^{-j(\nu + \frac{1}{2}|\alpha|)} |2^l - 2^j|^{-\sigma A} \| f_l \|_1. \tag{17}$$

An L_2-estimate is obtained e.g. in the following way:

$$\begin{aligned} \| Q_j^{(\alpha)}(t) f_l \|_2^2 &= \int dx \left| \int e^{-2\pi i \langle x, \eta \rangle} \tilde{\varphi}_l(\eta) g_j(x, \eta) \hat{f}_l(\eta)\, d\eta \right|^2 \\ &\leqq \iint |\tilde{\varphi}_l(\eta) g_j(x, \eta)|^2\, d\eta\, dx \int |\hat{f}_l(\eta)|^2\, d\eta \\ &= \iint |\tilde{\varphi}_l(\eta) g_j(x, \eta)|^2\, d\eta\, dx\, \| f_l \|_2^2. \end{aligned}$$

As above,

$$\left(\iint|\tilde{\varphi}_l(\eta)\, g_j(x,\eta)|^2\, d\eta\, dx\right)^{\frac{1}{2}} \leqq C 2^{(j+\frac{1}{2}l)n} 2^{-j(\nu+\frac{1}{2}|\alpha|)} |2^l-2^j|^{-\sigma A},$$

and hence

$$\|Q_j^{(\alpha)}(t)\, f_l\|_2 \leqq C 2^{(j+l)n} 2^{-j(\nu+\frac{1}{2}|\alpha|)} |2^l-2^j|^{-\sigma A} \|f_l\|_2. \tag{18}$$

Interpolation between (17) and (18), taking A large enough,then proves the lemma.

Let

$$\tilde{Q}_j(t) = \sum_{|\alpha|<N} \frac{1}{\alpha!} Q_j^{(\alpha)}(t).$$

Adding the estimates of Lemma 4 over $|\alpha|<N$ and over $|l-j|\geqq c$, we obtain

$$\|\tilde{Q}_j(t)\, f\|_{p'} \leqq C 2^{j(n-\nu-\kappa)} \|f\|_{B_p^{0,2}}$$

which with (13) implies that if $\kappa>n-\nu$ and if $(2n-\rho)\,\delta\leqq\nu$, then

$$\sum_{j=0}^{\infty} \|\tilde{Q}_j(t)\, f\|_{p'}^2 \leqq C|t|^{\nu-2n\delta} \|f\|_{B_p^{0,2}}^2. \tag{19}$$

We still have to consider the error term $R_N f$:

Lemma 5. *Let* $1\leqq p\leqq 2$, $1/p+1/p'=1$ *and let* R_N *as above be defined by* (11). *Then there is an* N_0 *such that for* $f\in C_0^\infty$ *with* $\operatorname{supp} f\subseteq K$,

$$\|R_N f\|_{p'} \leqq C\|f\|_{B_p^{0,2}}, \qquad N\geqq N_0, \tag{20}$$

where $C=C_K$ *is independent of* $j\geqq 0$ *and of* $|t|<\varepsilon$.

Proof. Let first N be so large that $\nu+(\sigma-1/2)N>n$. Then, here and in the remainder of the proof suppressing the t-dependence,

$$R_N f(x) = \int r(x,y)\, f(y)\, dy,$$

with

$$r(x,y) = \int e^{2\pi i\phi(x,y,\xi)} r_N(x,y,\xi)\, d\xi.$$

As assumed above, there is a compact set K' such that $x\notin K'$ implies that $q(x,y,\xi)=0$ and $q_j^{(\alpha)}(x,y,\xi)=0$, that is

$$e^{-2\pi i\phi(x,y,\xi)}\, \Phi_j(e^{2\pi i\phi(\cdot,y,\xi)} q(\cdot,y,\xi))(x) = r_N(x,y,\xi), \qquad x\notin K'.$$

Since we only are going to consider functions with support in K, we may (as for q and $q_j^{(\alpha)}$) assume that $r_N=0$ for $y\notin K$. By our assumptions on q, we also have $r_N=0$ for $|\xi|<1$. With $\check{\varphi}_j=\mathscr{F}^{-1}(\varphi_j)$,

$$\Phi_j(g) = \int \check{\varphi}_j(x-z)\, g(z)\, dz.$$

We may as well assume that $|x|\geqq 1$ on $\mathbf{R}^n\setminus K'$. Using that $\check{\varphi}$ is rapidly decreasing, we have for any M, $x\notin 2K'$,

$$|D_y^\alpha(e^{-2\pi i\phi}\Phi_j(e^{2\pi i\phi} q)| \leqq C|\xi|^{-\nu+|\alpha|} |x|^{-M} 2^{-jM}. \tag{21}$$

Let $\tilde{\varphi}_l = \varphi_{l-1} + \varphi_l + \varphi_{l+1}$, with $\varphi_{-1} = 0$. Then

$$|R_N f_l(x)| = |\iint f_l(z)\, \check{\tilde{\varphi}}_l(y-z)\, r(x,y)\, dy\, dz| \leqq |\int f_l(z)\, dz \int r(x,y)\, \check{\tilde{\varphi}}_l(y-z)\, dy|$$
$$\leqq \| f_l \|_1 \sup_z |\int r(x,y)\, \check{\tilde{\varphi}}_l(y-z)\, dy|.$$

By Parseval's formula,

$$\int r(x,y)\, \check{\tilde{\varphi}}_l(y-z)\, dy = \int \hat{r}(x,\eta)\, \tilde{\varphi}_l(\eta)\, e^{2\pi i \langle z,\eta\rangle}\, d\eta.$$

Since

$$\eta^\alpha \hat{r}(x,\eta) = (D_y^\alpha r(x,\cdot))\hat{\ }(\eta),$$

the estimate (21) implies for $M \geqq (1-\sigma) N + \nu$ that

$$|\eta^\alpha \hat{r}(x,\eta)| \leqq \int_{|\xi| \leqq |x|} |(D_y^\alpha r_N)(x,\eta,\xi)|\, d\xi + \int_{|\xi| \geqq |x|}$$
$$\leqq C|x|^{-M}(1+|x|^{-\nu+|\alpha|+n}) + C|x|^{+n+|\alpha|-M-\nu}, \qquad x \notin 2K',$$

and so, for $|\alpha| = n+1$, $M+\nu > 2n+1$, $M \geqq (1-\sigma) N + \nu$,

$$\sup_z |\int r(x,y)\, \check{\tilde{\varphi}}_l(y-z)\, dy| \leqq C 2^{-l} \{(1+|x|)^{2n+1-M-\nu} + (1+|x|)^{-M}\}, \tag{22}$$

for $x \notin 2K'$. Clearly (22) also holds for $x \in 2K'$.
Thus,

$$\sup_x |R_N f_l(x)| \leqq C 2^{-l} \| f_l \|_1, \qquad N \geqq N_{01},$$

from which, for $N \geqq N_{01}$,

$$\| R_N f \|_\infty \leqq \sum_{l=0}^{\infty} \| R_N f_l \|_\infty \leqq C \sum_0^\infty 2^{-l} \| f_l \|_1 \leqq C \| f \|_{B_1^{0,2}}. \tag{23}$$

Next, we will prove a corresponding L_2-estimate. We have

$$\| R_N f \|_2^2 \leqq \sum_{l=0}^{\infty} \| R_N f_l \|_2^2,$$

and (as in the proof of Lemma 4),

$$\| R_N f_l \|_2^2 \leqq \int |\iint f_l(z)\, \check{\tilde{\varphi}}_l(y-z)\, r(x,y)\, dy\, dz|^2\, dx$$
$$\leqq \int |\int \check{\tilde{\varphi}}_l(y-z)\, r(x,y)\, dy|^2\, dz\, dx \int |f_l(z)|^2\, dz.$$

Now,

$$z^\alpha \int \hat{r}(x,\eta)\, \tilde{\varphi}_l(\eta)\, e^{2\pi i \langle z,\eta\rangle}\, d\eta = \int (D_\eta^\alpha(\hat{r}(x,\eta)\, \tilde{\varphi}_l(\eta))\, e^{2\pi i \langle z,\eta\rangle}\, d\eta,$$

and

$$D_\eta^\alpha \hat{r}(x,\eta) = \mathscr{F}_{y\to\eta}((-y)^\alpha r(x,y)).$$

Since $r=0$ for y outside the compact set K, we have the same type of estimate for $(\partial/\partial\eta)^\alpha \hat{r}(x,\eta)$ as for $\hat{r}(x,\eta)$, and hence for each β,

$$|\int r(x,y)\, \check{\tilde{\varphi}}_l(y-z)\, dy| \leqq C_\beta 2^{-l}(1+|z|)^{-|\beta|} \{(|x|+1)^{-M} + (|x|+1)^{-M-\nu+2n+1}\}$$

and so, for $|\beta| > n/2$, $M > 2n+1+n/2$, $M \geqq (1-\sigma) N + \nu$,

$$\| R_N f_l \|_2^2 \leqq C 2^{-2l} \| f_l \|_2^2, \qquad N \geqq N_{02},$$

and hence for $N \geqq N_{02}$,

$$\| R_N f \|_2^2 \leqq C \sum_{l=0}^{\infty} 2^{-2l} \| f_l \|_2^2 \leqq C \| f \|_{B_2^{0,2}}^2. \tag{24}$$

Interpolation between (23) and (24) completes the proof of the lemma.

Adding the estimates (19) and (20) we obtain for $f \in C_0^\infty$ with $\operatorname{supp} f \subseteq K$, assuming that $\frac{1}{2} < \sigma < 1$ and that $(2n-\rho)\,\delta \leqq \nu$,

$$\| Q(t) f \|_{B_{p'}^{0,2}} = \left(\sum_0^\infty \| \Phi_j Q(t) f \|_{p'}^2 \right)^{\frac{1}{2}} \leqq C |t|^{\nu - 2n\delta} \| f \|_{B_p^{0,2}}, \qquad |t| < \varepsilon.$$

We have proved the following theorem.

Theorem 1. *Let $1 \leqq p \leqq 2$, $1/p + 1/p' = 1$, $\delta = 1/p - 1/2$. Let $Q(t)$ be a properly supported Fourier integral operator defined by a relation which is locally a canonical graph. Assume that locally on $\Omega \subset \mathbf{R}^n$, in a neighborhood of $t=0$, $Q(t)$ is given by* (1), *with phase functions ϕ which satisfy* (*) *and, for $t=0$,* (2) *of Section* 1, *and with amplitudes $q \in S^{-\nu}$, $\nu \geqq 0$. Then for each compact set $K \subset \Omega$, there are $\varepsilon > 0$ and a constant C_K such that for $f \in C_0^\infty$ with* $\operatorname{supp} f \subseteq K$, *and for $(2n-\rho)\,\delta \leqq \nu$,*

$$\| Q(t) f \|_{B_{p'}^{0,2}} \leqq C_K |t|^{\nu - 2n\delta} \| f \|_{B_p^{0,2}}, \qquad |t| < \varepsilon. \tag{25}$$

Remark 1. If $1 < p \leqq 2$, Lemma 1 and Theorem 1 imply that

$$\| Q(t) f \|_{p'} \leqq C_K |t|^{\nu - 2n\delta} \| f \|_p, \qquad |t| < \varepsilon,\ f \in C_0^\infty,\ \operatorname{supp} f \subseteq K,$$

or, since $Q(t)$ is properly supported, for each compact set $K' \subset \Omega$,

$$\| Q(t) f \|_{L_{p'}(K')} \leqq C_{K'} |t|^{\nu - 2n\delta} \| f \|_p, \qquad |t| < \varepsilon,\ f \in C_0^\infty,$$

which is inequality (3) of Section 1.

Remark 2. As the proof shows, we may replace the $B_p^{0,2}$- and $B_{p'}^{0,2}$-norms by $B_p^{s,2}$- and $B_{p'}^{s,2}$-norms, respectively, in (25).

Remark 3. Under the assumptions of Theorem 1, however only assuming that $q \in S_\sigma^{-\nu}$ for $1/2 < \sigma \leqq 1$, we find with obvious modifications, using (8)′ instead of (8) in the proofs of the inequalities corresponding to (19) and (13), that provided $(2n-\rho)\,\delta \leqq \nu$ and $1/2 < \sigma \leqq 1$,

$$\| Q(t) f \|_{B_{p'}^{0,2}} \leqq C_K |t|^{\gamma(\sigma)} \| f \|_{B_p^{0,2}}, \qquad |t| < \varepsilon, \tag{25'}$$

where $\gamma(\sigma) = \min\{(\nu - 2n\delta - 2\rho\,\delta(\sigma-1))/(2\sigma-1), \quad (\nu - 2n\delta)/(2\sigma-1)\}$. Since $(2n-\rho)\delta \leqq \nu$, we have for $1 \geqq \sigma > 1/2$ that $\gamma(\sigma) \geqq -\rho\,\delta/(2\sigma-1)$, with equality if $(2n-\rho)\,\delta = \nu$.

In the "constant coefficient" situation we may improve slightly on (25). Let

$$Q_0(t) f(x) = \mathscr{F}_{\xi \to x}^{-1}(e^{itp(\xi)}\, q(t;\xi)\, \hat{f}(\xi)), \tag{26}$$

where $p(\xi)$ is a phase function satisfying (*) and $q \in S^{-\nu}$. From [1] (or under slightly different assumptions on p, also from [8]), we then have the following estimate.

Theorem 2. *Let $Q_0(t)$ be defined by* (26). *Assume that $1 \leqq p \leqq 2$, $1/p + 1/p' = 1$, $\delta = 1/p - 1/2$, and that $(2n-\rho)\,\delta \leqq \nu$. Then*

$$\| Q_0(t) f \|_{B_{p'}^{0,2}} \leqq C |t|^{\nu - 2n\delta} \| f \|_{B_p^{0,2}}, \qquad f \in C_0^\infty. \tag{27}$$

We omit the proof, which is essentially carried out in [1]. Again, the remarks to Theorem 1 also applies to Theorem 2, substituting global results for local ones.

4. Applications to Semi-Linear Hyperbolic Equations

Let $P = P(x, D_t, D_x)$ be a differential operator of order m on $\mathbf{R}^n \times \mathbf{R}$ (in the variables (x, t)), with C^∞-coefficients depending on x only and which are constant outside some compact set in $\mathbf{R}^n$. We assume that P has a principal part p with real coefficients, and that the coefficient for D_t^m is 1.

To be more specific, we assume, following Chazarain [2], that P is hyperbolic in the sense that

(H) the hyperplanes $\mathbf{R}^n \times \{t\}$, $t \in \mathbf{R}$, are non-characteristic for P, and the solutions $\tau = \lambda_k(x, \xi)$ of $p(x, \tau, \xi) = 0$ are real and have constant multiplicities r_k, for $\xi \neq 0$, $k = 1, \ldots, K$.

In addition we assume that P satisfies the Levi-condition (again see Chazarain [2]),

(L) if ϕ is real and satisfies $(\partial/\partial t)\,\phi - \lambda_k(x, \operatorname{grad}_x \phi) = 0$, then for $a \in C_0^\infty$ with $\operatorname{grad}_{x,t} \phi \neq 0$ on supp a,

$$e^{-i\lambda\phi} P(e^{i\lambda\phi} a) = O(\lambda^{m - r_k}), \qquad \lambda \to +\infty, \; k = 1, \ldots, K.$$

Then by [2] the Cauchy problem

$$(\mathrm{C}) \begin{cases} Pu = f \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}), \\ D_t^j u|_{t=0} = g_j \in C_0^\infty(\mathbf{R}^n), \qquad j = 0, \ldots, m-1, \end{cases}$$

has a unique solution, which can be written

$$u(x, t) = \sum_0^{m-1} (E_j(t) + R_j(t))\, g_j(x) + E(t)\,(Wf)\,(x) \tag{28}$$

which is a reformulation of (3.8) in [2]. The operators which appear in (28) have the following properties:

$E_j(t)$: This operator is a sum of properly supported Fourier integral operators $E_{jk}(t)$, $k = 1, \ldots, K$, of the type discussed in Section 1 and Theorem 1 (cf. [2], Remarque 2.4). The phase functions of $E_{jk}(t)$ may be choosen as $\phi_{jk} = \psi_{jk} + t\lambda_k$, where ψ_{jk} satisfies (2), since P is translation invariant in the t-variable (cf. [2], Lemma 2.1 and Hörmander [4]). The amplitude of $E_{jk}(t)$ belongs to $S^{-j+(r_k-1)}$.

$E(t)$: The operator $E(t)$ is given by

$$E(t)\, f = \int_0^t E_{m-1}(t-\tau)\, f(\cdot, \tau)\, d\tau. \tag{29}$$

W: By [2], $W = (1 - V)^{-1}$, where

$$Vf(\cdot, t) = \int_0^t V(t-\tau)\, f(\cdot, \tau)\, d\tau$$

with $V(t)$ an integral operator with C^∞-kernel; since P has constant coefficients for large $|x|$, and since the hyperbolic problem (C) has a finite speed of propagation of supports (cf. [2], Remarque 3.10), the support of the kernel of $V(t)$ is contained in a compact set of the form $\{(x, y);\, |x-y| \leqq C|t|,\, x \in K'' \text{ compact}\}$. In particular, $V(t)$ is a bounded operator on $B_p^{s,2}$ and hence $W = \sum_{n \geqq 0} V^n$ is a bounded operator on $L_p(I; B_p^{s,2})$ for each compact interval $I \subset \mathbf{R}$ and each $p \geqq 1$, $s \geqq 0$. Compare with Lemma 3.2 in [2]. Further, by [2], p. 193, $W\colon C^\infty(\mathbf{R}^{n+1}) \to C^\infty(\mathbf{R}^{n+1})$.

$R_j(t)$: $R_j(t) = E(t)\, W R_j'(t)$, where $R_j'(t)$ is an integral operator of the same type as $V(t)$ above. Since $W = I$ outside some compact set, and $E(t)$ is properly supported, $R_j(t)$ is also an integral operator of the same type as $V(t)$.

In the present situation, condition (*) takes the somewhat more easily verified form

(*)′ the Hessian $\lambda''_{k\,\xi\xi}$ of λ_k has rank at least ρ for $\xi \neq 0$, $k = 1, \ldots, K$.

By Theorems 1 and 2, and Remark 2, it now follows from (28) and (29) with $g = (g_0, \ldots, g_{m-1}) = 0$, that for some $\varepsilon > 0$ and $(2n-\rho)\,\delta \leqq m - r$, where $r = \max_k r_k$,

$$\|u(\cdot, t)\|_{B_{p'}^{s,2}} \leqq C \int_0^t |t-\tau|^{m-r-2n\delta}\, \|Pu(\cdot, \tau)\|_{B_p^{s,2}}\, d\tau, \qquad |t| < \varepsilon.$$

If $0 < 1 + m - r - 2n\delta = 2\delta' < 1$, then with $1/q + 1/q' = 1$ and $\delta' = 1/q - 1/2$, $1 < q \leqq 2$,

$$\|u\|_{L_{q'}((0,t);\, B_{p'}^{s,2})} \leqq C\, \|Pu\|_{L_q((0,t);\, B_p^{s,2})}, \qquad |t| < \varepsilon,$$

from well known estimates for Riesz' potentials. But, as mentioned above, P is translation invariant in t, and by translating and adding we obtain for each interval $I \subset \mathbf{R}$ that there is a constant C such that for $u \in C_0^\infty(\mathbf{R}^{n+1})$,

$$\|u\|_{L_{q'}(I;\, B_{p'}^{s,2})} \leqq C\, \|Pu\|_{L_q(I;\, B_p^{s,2})}. \tag{30}$$

In order to simplify the exposition below, we take $1 < p \leqq 2$ in (30), and then invoke Lemma 1:

$$\|u\|_{L_{q'}(I;\, L_{p'}^s)} \leqq C\, \|Pu\|_{L_q(I;\, L_p^s)}. \tag{30$'$}$$

In particular, if $s = 0$ and $q = p$, then

$$\|u\|_{p'} \leqq C\, \|Pu\|_p. \tag{30$''$}$$

Convention. From now on we assume that $1 < p \leqq 2$, $1/p + 1/p' = 1$, $\delta = 1/p - 1/2$ and that with ρ defined by (*)′, $(2n-\rho)\,\delta \leqq m - r$. We also assume that $1 < q \leqq 2$, $1/q + 1/q' = 1$ and $\delta' = 1/q - 1/2$ and $2\delta' = 1 + m - r - 2n\delta$.

Following Strichartz [14], we say that the (vector-valued) tempered distribution g belongs to $\mathscr{C}_{pq}^s(I)$ if the solution of $Pu = 0$ with data $g = (g_0, \ldots, g_{m-1})$, that is $D_t^j u|_{t=0} = g_j$, $j = 0, \ldots, m-1$, belongs to $L_{q'}(I; L_{p'}^s(\mathbf{R}^n))$. Some properties of $\mathscr{C}_{pq}^s(I)$ is collected in the following lemma. (Cf. Lemmas 2.3 and 4 of [14]).

Lemma 6. *Let I be an interval in* $\mathbf{R}$.

(a) *If $u \in L_{q'}(I; L_{p'}^s)$ and $Pu = 0$, then there exist $g \in \mathscr{C}_{pq}^s(I)$ such that*

$$u(x, t) = \sum_{j=0}^{m-1} (E_j(t) + R_j(t))\, g_j(x), \tag{31}$$

where $E_j(t)$ and $R_j(t)$ are the operators defined in (28).

(b) *If $u \in L_{q'}(I; L^s_{p'})$ and $Pu = f \in L_q(I; L^s_p)$, then there exists $g \in \mathscr{C}^s_{p,q}(I)$ such that (28) holds. Conversely, if $f \in L_q(I; L^s_p)$ and $g \in \mathscr{C}^s_{pq}(I)$, then the solution u of (C) belongs to $L_{q'}(I; L^s_{p'})$.*

(c) *Let $\mu = \max(m, 2(m-r))$. Then $\prod_{j=0}^{m-1} B_p^{s+\mu-j-1/p,\,p} \subseteq \mathscr{C}^s_{pp}(\mathbf{R})$.*

Proof. (a) By assumption $\mathbf{R}^n \times \{t\}$, $t \in \mathbf{R}$, are non-characteristic for P, and since $Pu = 0$, it follows that $t \to u(\cdot, t) \in \mathscr{S}'$ is a smooth function of t. In particular, $g_j = D_t^j u|_{t=0}$ are well defined, $j = 0, \ldots, m-1$. The uniqueness of the solution of (C) completes the proof of (a) (cf. [2], Prop. 3.2).

(b) Write u_0 for the solution of $Pu = f$ with zero initial data. Then by (30)$'$, $u_0 \in L_{q'}(I; L^s_{p'})$, and hence also $u_1 = u - u_0 \in L_{q'}(I; L^s_{p'})$. Since u_1 satisfies the assumptions of (a) above, by what we have already proved, u_1 and so u has data $g \in \mathscr{C}^s_{pq}(I)$. The converse is proved similarly.

(c) By the trace theorem, there is a $u \in B_p^{s+\mu,\,p}$ such that $D_t^j u|_{t=0} = g_j$, $j = 0, \ldots, m-1$, if $g_j \in B_p^{s+\mu-j-1/p,\,p}$. Then $Pu \in B_p^{s+\mu-m,\,p} \subseteq B_p^{s,\,p} \subseteq L_p(\mathbf{R}^{n+1})$, and by Sobolev's embedding theorem, $u \in B_p^{s+\mu,\,p} \subseteq B_p^{s+\mu-2(m-r),\,p} \subseteq B_{p'}^{s,\,p} \subseteq L_{p'}(\mathbf{R}^{n+1})$. An application of (b) above then proves that $g_j \in \mathscr{C}^s_{pp}(\mathbf{R})$. In the above inclusions, we have also applied Lemma 1.

Let $f = f(x, t, u) \in L^\sigma_q(I; L^{s-\sigma}_p)$ for each $u \in L^\sigma_{q'}(I; L^{s-\sigma}_{p'})$, $0 \leqq \sigma \leqq s$, $0 \leqq s \leqq s_0$, and each interval $I \subseteq \mathbf{R}$. Assume that for each $s \leqq s_0$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that if

$$\|u\|_{L_{q'}(I; L^s_{p'})} < \delta, \qquad \|v\|_{L_{q'}(I; L^s_{p'})} < \delta,$$

then

$$\|f(u) - f(v)\|_{L_q(I; L^s_p)} \leqq \varepsilon \|u - v\|_{L_{q'}(I; L^s_{p'})}. \tag{32}$$

Example 1. If $a = a(x, t) \in L_\infty(\mathbf{R}^{n+1})$ is continuous and $f(x, t, u) = a(x, t)\,|u|^M$, then f satisfies the above assumptions for $M > 1$ and with $Mp = p'$, $s = 0$.

If we notice that, since by assumption $q' < \infty$, if $u \in L_{q'}(I; L^s_{p'})$ then for each $\delta > 0$ there is some interval $I_\delta \subseteq I$ such that the norm of u in $L_{q'}(I_\delta; L^s_{p'})$ is at most δ. Also, notice that the norm of $g \in \mathscr{C}^s_{pq}(I)$ is naturally defined as the $L_{q'}(I; L^s_{p'})$-norm of the corresponding solution of $Pu = 0$. With this observations and inequality (30)$'$, the following result is proved in the same way as Theorem 3 in [14]. We omit (the, modulo [14], obvious) details of the proof.

Theorem 3. *With the above conventions and assumptions, assume that $g \in \mathscr{C}^{s_0}_{pq}(R)$ and that P as above satisfies* (H) *and* (L) *and that f satisfies* (32). *Then there is an interval $I_0 \subset R$, I_0 open and nonempty, such that the Cauchy problem*

$$(\mathrm{C})' \begin{cases} Pu = f(\cdot, u), \\ D_t^j u|_{t=0} = g_j, \qquad j = 0, \ldots, m-1, \end{cases}$$

has a solution $u \in L^\sigma_{q'}(I; L^{s_0-\sigma}_{p'})$ on each open interval I with $\bar{I} \subset I_0$, $0 \leqq \sigma \leqq s_0$. The solution is unique as long as it exists, and if it does not exist globally, then the $L^\sigma_{p'}(I; L^{s_0-\sigma}_{p'})$-norm of u tends to infinity as I tends to the maximal interval of existence.

Example 2. Let $a_{kl}(x)\in C^\infty$ be constant outside some compact set in $\mathbf{R}^n$, and assume that $(a_{kl}(x))_{k,l}$ is positive definite on $\mathbf{R}^n$. Let $P=\partial^2/\partial t^2-\sum_{k,l=1}^{n} a_{kl}(x)\,\partial^2/\partial x_k\partial x_l$, and let f be as in Example 1 above. Then $\rho=n-1$, $m=2$ and $r=1$. With $M=3$, the conditions of Theorem 3 are then satisfied for $n=3$, with $p=q=4/3$ and $p'=4$ and $q=p$.

Remark. If $s_0>1/q'$ we may take $\sigma>1/q'$ and by Sobolev's theorem obtain uniform bounds in the t-variable. In order to obtain uniform estimates also in the x-variables, we have to require essentially the same amount of smoothness of the initial data as that suggested by the use of L_2-methods (cf. Löfström and Thomée [11]). However, the smoothness assumptions on f will still in a sense be minimal by the use of the methods of this paper.

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