Z. Phys. B - Condensed Matter 54, 357-370 (1984)

Condensed Matter
Zeitschrift für Physik B

Eigenvalues for the Extremely Underdamped Brownian Motion in an Inclined Periodic Potential

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Received November 28, 1983

The eigenvalues for the Brownian motion in a periodic potential with an additive constant force are investigated in the low friction limit. First the Fokker-Planck equation for the distribution function in velocity and position space is transformed to energy and position coordinates. By a proper averaging process over the position coordinate a differential equation for the distribution function depending on the energy only is obtained. Next the eigenvalues and eigenfunctions are calculated from this equation by a Runge-Kutta method. Finally the problem is formulated in terms of an integral equation from which the lowest non-zero eigenvalue is obtained analytically in the bistability region in the zero temperature limit.

1. Introduction

The Brownian motion of particles in the periodic potential $f(x)=f(x+2\pi)$ with an additional constant force F i.e. with an additional potential $-Fx$ is described by the Langevin equation

$$\ddot{x}+\gamma\dot{x}+f'(x)=F+\Gamma(t) \tag{1.1}$$

$$\langle\Gamma(t)\rangle=0; \quad \langle\Gamma(t)\Gamma(t')\rangle=2\gamma\Theta\delta(t-t'). \tag{1.2}$$

In (1.1,2) γ is the damping constant and Θ the noise power (or normalized temperature) of the Langevin force $\Gamma(t)$. This Brownian motion problem arises in a number of fields in physics, chemical physics and electrical engineering [1-19]. Usually the simplest periodic potential, i.e. that of a cosine potential

$$f(x)=-d\cos x \tag{1.3}$$

appears in these applications. The equations (1.1, 2) are equivalent to the Fokker-Planck equation [20, 21]

$$\partial W/\partial t=\mathbf{L}_{FP}W \tag{1.4}$$

$$\mathbf{L}_{FP}=-\frac{\partial}{\partial x}v+\frac{\partial}{\partial v}(\gamma v+f'(x)-F)+\gamma\Theta\frac{\partial^2}{\partial v^2}. \tag{1.5}$$

Here $v=\dot{x}$ is the velocity and $W=W(x,v,t)$ is the distribution function in position - velocity space. To specify the solution, boundary conditions must be added. Here we require natural boundary conditions for v (i.e. W must go to zero for $|v|\to\infty$) and periodic boundary conditions in x

$$W(x,v,t)=W(x+2\pi,v,t). \tag{1.6}$$

If x is an angle variable and if we do not distinguish whether an additional full rotation has been made or not the boundary condition (1.6) is appropriate. For further discussion, see [22]. As discussed elsewhere [5, 22-27] there exist bistable solutions, a locked and a running one, for certain parameters γ, d, F if the noise $\Gamma(t)$ is neglected. With the addition of the noise term $\Gamma(t)$ transitions between these two solutions are possible [26]. The transition rates between these two solutions follow from the eigenvalues of the Fokker-Planck operator (1.5) and from the stationary solution of (1.4) as discussed recently [22]. In that paper the eigenvalues have been determined by the matrix continued fraction method. This method was first used to determine the stationary mobility [28] and the stationary distribution function [29] of the above Brownian motion problem and then was applied to the determination of the eigenvalues of the Fokker-Planck operator [22, 30-33].

The matrix continued fraction method, however, does not work in the limit of low friction γ. The low friction case is the main topic of the present paper. As we know from classical mechanics the energy is a constant of motion if the damping is absent. Therefore in the limit of small damping the energy will become a slow or relevant variable, which should then be used instead of x and v to describe the motion. The energy-variable was already introduced in [34] to describe the Brownian motion for low damping constants (for a general discussion of slow and fast variables see [35]). The transformation of the Fokker-Planck equation to energy and position variables will be made in Chap. 2. In this chapter we also perform a proper averaging process of the fast variable x leading finally to an ordinary second order differential equation for the energy dependent distribution function. From this equation the eigenvalues and eigenfunctions are calculated in Chap. 3 by a Runge-Kutta method. The main result for the eigenvalues is the following: For low noise strength Θ a sharp transition to nearly degenerate eigenvalues occurs for those forces, where bistable solutions are possible. This will be discussed in terms of a potential model in this chapter too. In Chap. 4 we apply a boundary layer theory, by which an x-dependence of the distribution function near the critical trajectory in phase space is used. By including the boundary layer the validity of the method can be extended to somewhat higher friction constants as it was shown [36] for the stationary problem. Finally in Chap. 5 the ordinary second order differential equation for the energy-dependent distribution function is transformed to an integral equation. In the bistability region the kernel degenerates to a product-kernel for low temperatures. For this reason we obtain an analytic expression for the lowest non-zero eigenvalue in the bistability region and in the zero-temperature limit.

Transition rates between the locked and the running solutions have been investigated recently. In [26] the transition rate out of the running solution was calculated for the zero-temperature limit whereas in [27] the transition rate out of the locked state was determined in the zero-temperature limit. Though the last reference has practically the same title as the present paper, some results are different as will be discussed in detail in Chap. 6.

2. Transformation to Energy- and Position-Variables

As already discussed in Chap. 1 the energy $\tilde{E}$ defined by

$$\tilde{E}=v^2/2+f(x)-F\cdot x \tag{2.1}$$

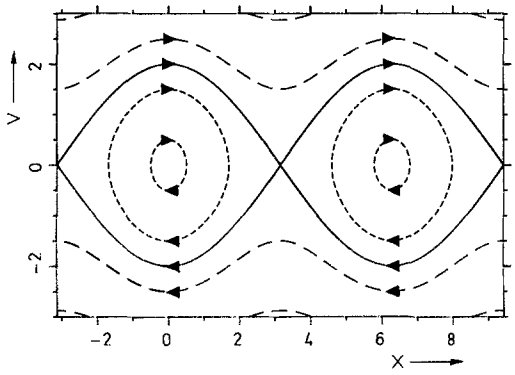

Fig. 1. The lines of constant energy for the cosine potential $f(x)=-\cos x$. The critical trajectory $E=E_0=1$ is shown by the solid line

will become a slow or relevant variable for small friction γ. If the friction is small the external force F must also be small, because otherwise the energy gain of the particles due to the external force F cannot be compensated by energy dissipation and no stationary solution would exist in the low-friction limit for final forces F. If

$$F=\gamma\cdot F_0 \tag{2.2}$$

it turns out that in the low-friction limit a stationary solution does exist for finite F_0. Therefore we may approximately neglect the potential $-F\cdot x$ of the additional force in (2.1) i.e. we use the energy variable [36]

$$E=v^2/2+f(x). \tag{2.3}$$

This energy variable has the advantage that a function of E strictly fulfills the periodicity condition (1.6). The lines of constant energy are shown in Fig. 1.

A transformation to an energy variable for particles moving in an arbitrary potential was already made in [34]. However, in order to retain the full information of the distribution function the method of [34] has to be modified so that two separate energy distribution functions, one for each sign of the velocity, have to be taken into account:

$$\begin{aligned} W_+(x,E,t)&=W(x,v(x,E),t),\\ W_-(x,E,t)&=W(x,-v(x,E),t), \end{aligned} \tag{2.4}$$

$$v(x,E)=+\sqrt{2(E-f(x))}. \tag{2.4a}$$

For further calculations the sum (S) and the difference (D) of W_+ and W_- are sometimes more suitable:

$$W_{\substack{S\\D}}(x,E,t)=W_+(x,E,t)\pm W_-(x,E,t). \tag{2.5}$$

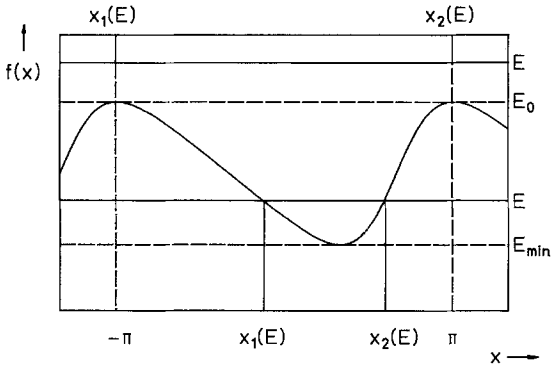

Fig. 2. The periodic potential $f(x)$, E_0, E_{min} and $x_1(E)$, $x_2(E)$ for $E \leqq E_0$. In the following $x_1(E)=\pi$, $x_2(E)=\pi$ is used for $E \geqq E_0$

For simplicity we assume that the potential has only one maximum and therefore only one minimum in the period length 2π. Assuming further that the maxima E_0 of the periodic potential are located at $-\pi$ and π and that $x_1 = x_1(E)$ and $x_2 = x_2(E)$ are the minimum and maximum values for the space coordinate for $E < E_0$ in the region $-\pi \leqq x \leqq \pi$ (see Fig. 2) we require the following continuity conditions for the distribution functions

$$\begin{aligned} W_{\pm}(-\pi, E, t) &= W_{\pm}(\pi, E, t) \quad \text{for } E > E_0 \\ W_{+}(x_1, E, t) &= W_{-}(x_1, E, t) \quad \text{for } E < E_0 \end{aligned} \tag{2.6}$$

or

$$\begin{aligned} W_S(-\pi, E, t) &= W_S(\pi, E, t) \quad \text{for } E > E_0 \\ W_D(x_1, E, t) &= W_D(x_2, E, t) = 0 \quad \text{for } E < E_0. \end{aligned} \tag{2.7}$$

The Fokker-Planck equation (1.4, 5) now reads for the functions (2.4)

$$\frac{\partial}{\partial t} W_{\pm} = v(x,E)\left[\mp \frac{\partial}{\partial x} W_{\pm} + \gamma \frac{\partial}{\partial E}\left\{ v(x,E)\left[1 + \Theta \frac{\partial}{\partial E}\right] \mp F_0 \right\} W_{\pm} \right] \tag{2.8}$$

and for (2.6)

$$\frac{\partial}{\partial t} W_D = v(x,E)\left[-\frac{\partial}{\partial x} W_S + \gamma \frac{\partial}{\partial E}\left\{ v(x,E)\left[1 + \Theta \frac{\partial}{\partial E}\right] W_D - F_0 W_S \right\} \right], \tag{2.9}$$

where F_0 is defined by (2.2). It should be emphasized that $W_{\pm}$ are the distribution functions in (x, v)-space. The distribution function $\hat{W}$ in (x, E) - space is obtained by multiplying $W_{\pm}$ with the Jacobian

$$\frac{\partial(x,v)}{\partial(x,E)} = \frac{dv}{dE} = \frac{1}{v(x,E)} = \frac{1}{\sqrt{2(E - f(x))}}. \tag{2.10}$$

The distribution i.e.

$$\hat{W}_{\pm}(x,E,t) = \frac{1}{v(x,E)} W_{\pm}(x,E,t) \tag{2.11}$$

in (x, E)-space is not introduced because the equation for $W_{\pm}$ and W_S have a simpler form.

Averages over Trajectories

For small damping constants the particles stay a long time on the trajectories $E = \text{const}$, see Fig. 1. We therefore expect that the distribution function depends on E only i.e.

$$W_{\pm}(x,E,t) = \tilde{W}_{\pm}(E,t), \qquad W_D(x,E,t) = \tilde{W}_D(E,t) \tag{2.12}$$

(The ansatz (2.12) will be modified near the critical trajectory, see Chap. 4). The dependence of $W_{\pm}$ on energy is determined by the small change of energy. The Langevin equation (1.1) transforms for the energy variable (2.3) to

$$\dot{E} = -2\gamma(E - f(x)) \pm \gamma v(x,E) F_0 \pm v(x,E)\Gamma \quad \text{for} \quad \begin{matrix} v > 0 \\ v < 0 \end{matrix} \tag{2.13}$$

where $v(x, E)$ is defined by (2.4a). For the distribution function the energy change is described by the right-handside of (2.8). As best seen from (2.13), the small energy gain $\gamma v(x,E) F_0$ due to the external field for $v > 0$ cancels the small energy loss $-\gamma v(x,E) F_0$ for $v < 0$ if the motion is closed, i.e. for $E < E_0$. Therefore the distribution functions $W_+(E)$ and $W_-(E)$ are identical (and W_D is zero), in agreement with the second boundary condition in (2.6 or 2.7).

For $E > E_0$ the motion of the particles for $v > 0$ and $v < 0$ is separated. Therefore the particles may gain (lose) energy by the external field F_0 for $v > 0$ ($v < 0$) and the distribution functions $W_{\pm}$ will be different for $E > E_0$, compatible with the first boundary condition in (2.6).

We now insert the ansatz (2.12) into (2.8) and (2.9) respectively and average over the time T (this time should be small compared to the time $\sim 1/\gamma$ in which $\tilde{W}(E,t)$ changes its value appreciably). Because of

$$\int \ldots dt = \int \ldots \frac{dx}{v(x,E)} \tag{2.14}$$

we may as well divide (2.8) and (2.9) by $v(x,E)$ and take an x-average. We thus obtain

$$E<E_0$$

$$T(E)\frac{\partial \tilde{W}(E,t)}{\partial t}=\gamma\frac{\partial}{\partial E}I(E)\left(1+\Theta\frac{\partial}{\partial E}\right)\tilde{W}(E,t) \tag{2.14a}$$

$$\tilde{W}(E,t)=\tilde{W}_+(E,t)=\tilde{W}_-(E,t)$$

$$E>E_0$$

$$T(E)\frac{\partial \tilde{W}_\pm(E,t)}{\partial t}=\gamma\frac{\partial}{\partial E}\left\{I(E)\left(1+\Theta\frac{\partial}{\partial E}\right)\mp 2\pi F_0\right\}W_\pm \tag{2.14b}$$

$$E<E_0$$

$$T(E)\frac{\partial \tilde{W}_S(E,t)}{\partial t}=\gamma\frac{\partial}{\partial E}I(E)\left(1+\Theta\frac{\partial}{\partial E}\right)\tilde{W}_S(E,t) \tag{2.14c}$$

$$\tilde{W}_D(E,t)=0$$

$$E>E_0$$

$$T(E)\frac{\partial}{\partial t}\underset{S}{\tilde{W}_D}(E,t)$$

$$=\gamma\frac{\partial}{\partial E}\left[I(E)\left(1+\Theta\frac{\partial}{\partial E}\right)\underset{S}{\tilde{W}_D}(E,t)-2\pi F_0\underset{D}{\tilde{W}_S}(E,t)\right] \tag{2.14d}$$

where $I(E)$ and $T(E)$ are defined by

$$I(E)=\int_{x_1(E)}^{x_2(E)}v(x,E)\,dx,$$

$$I'(E)=T(E)=\int_{x_1(E)}^{x_2(E)}[1/v(x,E)]\,dx. \tag{2.15}$$

Here $x_1(E)$ and $x_2(E)$ are the minimal and maximal x-values in the potential, see Fig. 2. For $E>E_0$ we have $x_1=-\pi$, $x_2=+\pi$ and the averaged velocity used in [36] is then given by the first part of (2.15) divided by 2π i.e.

$$\bar{v}(E)=I(E)/(2\pi) \quad \text{for } E\geqq E_0. \tag{2.16}$$

It should be noted that the usual definition $\hat{I}$ of an action integral for $E<E_0$ is twice the value (2.15)

$$\hat{I}(E)=\oint v(x,E)\,dx=2I(E). \tag{2.17}$$

We do not use the definition (2.17) because the action integral would then be discontinuous at $E=E_0$. The time for one cycle for $E<E_0$ is twice the derivative of the action integral $I'(E)=T(E)$. For the potential $-d\cos x$, $I(E)$ and $T(E)$ are given by:

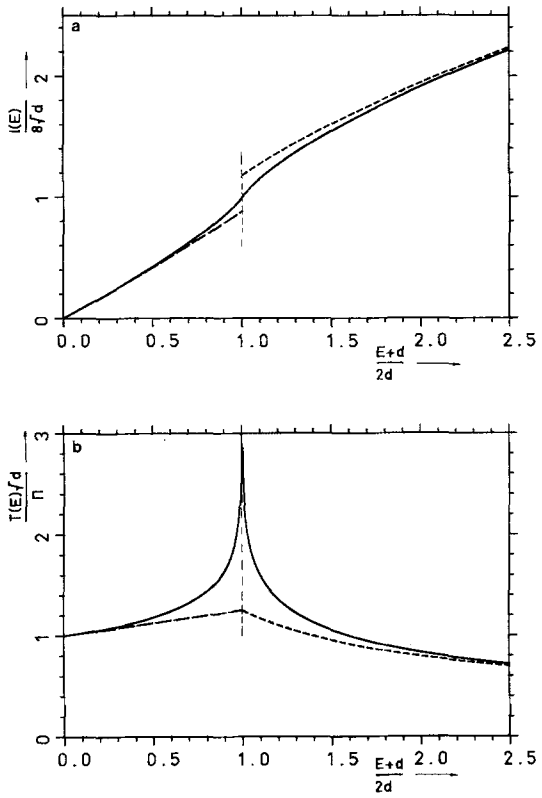

Fig. 3a and b. The action integral $I(E)$ **a** and its derivative $T(E)=I'(E)$ **b** for the potential $-d\cos x$ (solid line). The expressions (2.18) for $E+d\ll d$ and $E\gg d$ are shown by the broken line

$$E_{\min}=-d\leqq E\leqq E_0=d$$

$$I(E)=8\sqrt{d}\{\mathbf{E}[(E+d)/(2d)]-[1-(E+d)/(2d)]\,\mathbf{K}[(E+d)/(2d)]\}$$

$$\approx(\pi/\sqrt{d})(E+d)[1+(E+d)/(16d)] \quad \text{for } E\gtrsim E_{\min},$$

$$I'(E)=T(E)=(2\sqrt{d})\,\mathbf{K}[(E+d)/(2d)]$$

$$\approx(\pi/\sqrt{d})[1+(E+d)/(8d)] \quad \text{for } E\gtrsim E_{\min} \tag{2.18a}$$

$$E\geqq E_0=d$$

$$I(E)=4\sqrt{2(E+d)}\,\mathbf{E}[2d/(E+d)]$$

$$\approx 2\pi\sqrt{2(E+d)}-2\pi d/\sqrt{2(E+d)} \quad \text{for } E\gg d$$

$$I'(E)=T(E)=(4/\sqrt{2(E+d)})\,\mathbf{K}[2d/(E+d)]$$

$$\approx 2\pi/\sqrt{2(E+d)}+2\pi d/\sqrt{2(E+d)^3} \quad \text{for } E\gg d. \tag{2.18b}$$

Here $\mathbf{K}(m)$ and $\mathbf{E}(m)$ are the complete elliptic integrals of first and second kind [37]. The action integral (2.18) and its derivative are plotted in Fig. 3. At $E=E_0$, $T(E)$ has a weak logarithmic singularity of the form

$$I'(E)=T(E)=(2/\sqrt{d})\ln\{32d/|E-d|\} \quad \text{for } E\approx d.$$

Eigenvalue Equation

The separation ansatz

$$\tilde{W}_{\substack{D\\S}}(E,t)=\Phi^{(n)}_{\substack{D\\S}}(E)e^{-\lambda_n t} \tag{2.19}$$

leads to the following equation for the eigenvalues

$$A_n=\lambda_n/\gamma \tag{2.20}$$

and the eigenfunctions $\Phi^{(n)}_{\substack{D\\S}}(E)$:

$E<E_0$

$$\frac{d}{dE}I(E)\left(1+\Theta\frac{d}{dE}\right)\Phi^{(n)}_S+A_nI'(E)\Phi^{(n)}_S=0 \tag{2.21a}$$

$$\Phi_D=0$$

$E>E_0$

$$\frac{d}{dE}\left[I(E)\left(1+\Theta\frac{d}{dE}\right)\Phi^{(n)}_{\substack{D\\S}}-2\pi F_0\Phi_{\substack{S\\D}}\right]+A_nI'(E)\Phi^{(n)}_{\substack{D\\S}}=0. \tag{2.21b}$$

Boundary Condition at $E=E_{\min}$

Because $I(E)$ vanishes at $E=E_{\min}$, we obtain from (2.21a) for $E=E_{\min}$

$$I'(E_{\min})\left(1+\Theta\frac{d}{dE}\right)\Phi^{(n)}_S(E)\Big|_{E=E_{\min}}+A_nI'(E_{\min})\Phi^{(n)}_S(E_{\min})=0$$

i.e.

$$\Theta\,d\Phi^{(n)}_S(E)/dE|_{E=E_{\min}}+(A_n+1)\Phi^{(n)}_S(E_{\min})=0. \tag{2.22}$$

Thus the derivative of $\Phi^{(n)}_S$ can be expressed in terms of $\Phi^{(n)}_S$ and A_n at $E=E_{\min}$.

Continuity Conditions at $E=E_0$

Without the boundary layer we can require the continuity of $\Phi^{(n)}_S$ and its derivative and the continuity of $\Phi^{(n)}_D$, but not the continuity of the derivative of $\Phi^{(n)}_D$ as explained in Chap. 4, i.e. we have

$$\Phi^{(n)}_S(E_0-0)=\Phi^{(n)}_S(E_0+0) \tag{2.23}$$

$$d\Phi^{(n)}_S(E)/dE|_{E=E_0-0}=d\Phi^{(n)}_S(E)/dE|_{E=E_0+0} \tag{2.24}$$

$$\Phi^{(n)}_D(E_0+0)=0. \tag{2.25}$$

The eigenvalues $A_n=\lambda_n/\gamma$ are thus determined by (2.21-25) and the requirement that $\Phi^{(n)}_D$ and $\Phi^{(n)}_S$ vanish for $E\to\infty$. The weak logarithmic singularity of $I'(E)$ at $E=E_0$ does not lead to serious difficulties.

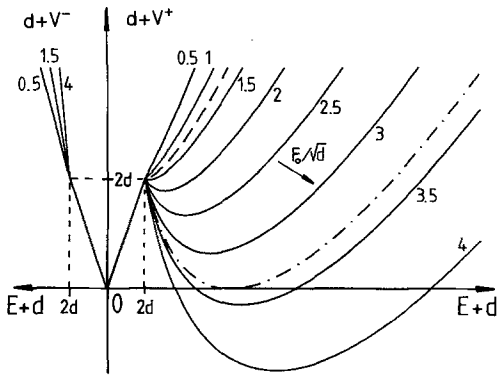

Fig. 4. The effective potential (2.30) as a function of $E+d$ for various external forces $F_0/\sqrt{d}$ and for the cosine potential $f(x)=-d\cos x$. The effective potential V^+ (V^-) for positive (negative) velocities is plotted to the right (left). The effective potential for the critical force $F_{01}/\sqrt{d}=4\pi$ is shown by a broken line and for the critical force $F_{02}/\sqrt{d}=3.3576$ by a broken line with dots

Stationary Solution

Because the probability current in E - direction is zero the stationary solution ($A_0=\lambda_0/\gamma=0$) follows immediately from (2.21) [36]

$$\begin{aligned}\tilde{W}_\pm&=N\exp(-E/\Theta) &&\text{for } E<E_0\\ \tilde{W}_\pm&=N\exp[-(E\mp F_0 g(E))/\Theta] &&\text{for } E>E_0\end{aligned} \tag{2.26}$$

$$N^{-1}=\sqrt{2\pi\Theta}\int_{-\pi}^{\pi}e^{-f(x)/\Theta}dx+8\pi^2\cdot\int_{E_0}^{\infty}T(E)e^{-E/\Theta}[\cosh(F_0g(E)/\Theta)-1]\,dE. \tag{2.27}$$

where $g(E)$ is defined by

$$g(E)=2\pi\int_{E_0}^{E}dE'/I(E'). \tag{2.28}$$

The function (2.26) may be written in the form

$$\tilde{W}_\pm(E)=N\exp(-V^\pm(E)/\Theta) \tag{2.29}$$

where the effective low friction potential $V^\pm(E)$ is given by

$$V^\pm(E)=\begin{cases}E\\ E\mp F_0g(E)\end{cases}\quad\text{for}\quad\begin{matrix}E\leqq E_0\\ E\geqq E_0.\end{matrix} \tag{2.30}$$

This potential is needed in Chap. 3 for the interpretation of the eigenvalues. A plot of the potential is shown in Fig. 4. As explained in [25] we may define a critical force $F_{01}/\sqrt{d}=4/\pi$ above which a running solution of the equation (1.1) without noise is possible in the zero-friction limit. As further explained in [25] a second critical force F_{02} exists, above which

the running solution becomes globally stable. As seen in Fig. 4 for forces F_0 above the first critical force F_{01}, a second minimum corresponding to the running solution occurs. For forces F_0 in the range $F_{01}<F_0<F_{02}$, this minimum is higher than the minimum at $E+d=0$ corresponding to the locked solution and therefore the running solution is not globally stable in this region. For forces $F_0>F_{02}$ the minimum corresponding to the running solution is the lowest one and the running solution then becomes globally stable. The energy $\bar{E}$ at which a minimum of V^+ for $E>E_0$ occurs is given by

$$dV^+(E)/dE|_{E=\bar{E}}=1-F_0/\bar{v}(\bar{E})=1-2\pi F_0/I(\bar{E})=0 \quad (2.31)$$

3. Calculation and Results of the Eigenvalues and Eigenfunctions

The eigenvalues and eigenfunctions are obtained in [38] by the following numerical integration method. First the equation (2.21a) is integrated by a Runge-Kutta method from $E_{\min}$ to E_0. The derivative of Φ_S at $E_{\min}$ is determined by (2.22). Thus we have $\Phi_S(E_{\min})=a$ as the only integration constant. Next (2.21b) is integrated by the Runge-Kutta method from E_0 to E_∞. The starting values of Φ_S and of its derivative at $E=E_0$ are given by (2.23, 24). For the difference equation however only $\Phi_D(E_0)$ is given, see (2.25) whereas the value $\Phi'_D(E_0)=b$ must still be determined. To be more precise the ratio b/a must be determined, the constant a respectively b follows from the normalization. Finally the ratio b/a and the eigenvalue $A_n=\lambda_n/\gamma$ follow from the requirement that both functions Φ_S and Φ_D vanish for sufficiently large E_∞ i.e.

$$\Phi_S(E_\infty, b/a, A_n)=0, \quad \Phi_D(E_\infty, b/a, A_n)=0. \quad (3.1)$$

To find the roots b/a and A_n of (3.1) a two dimensional regula falsi method was used in [38]. The cutoff energy E_∞ was chosen in such a way, that an increase of E_∞ did not alter the results beyond a given accuracy. The results for the eigenvalues as a function of the applied force $F=F_0\gamma$ are shown in Fig. 5 for two d/Θ values. As seen the eigenvalues for small and large F_0 are essentially given by $\gamma_n/\gamma\approx n (n=0, 1, 2 \ldots)$ for low noise powers Θ; they are not degenerate for low F_0 but nearly twofold degenerate for large F_0. This can be interpreted as follows. For low F_0 the effective potential in Fig. 4 has only one well whereas for large F_0 two wells occur. For low noise powers Θ the wells are separated by a high barrier. The eigenvalues in the left and right well nearly coincide and are given by $\lambda_n/\gamma\approx n$ for small Θ as may be derived by expanding

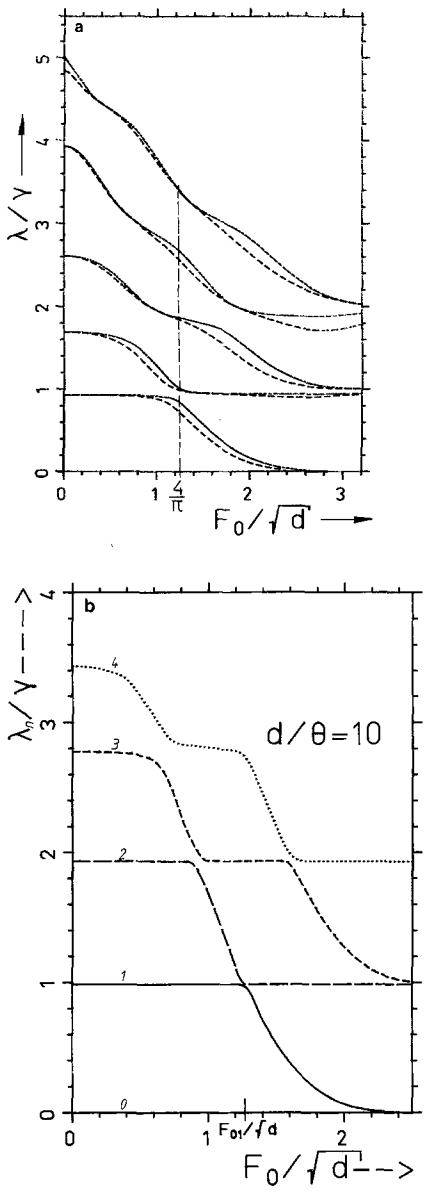

Fig. 5a and b. The eigenvalue divided by the friction constant in the low friction limit for the cosine potential (1.3) as a function of $F_0/\sqrt{d}$ for $d/\Theta=5$ **a** and $d/\Theta=10$ **b**. The critical force $F_{01}/\sqrt{d}=4/\pi$ is also indicated. In Fig. 5a we have also plotted the values of λ/γ according to the boundary layer theory of Chap. 4 with $\gamma=0.01$ (broken line)

$I(E)$ around the minima. Thus in the bistability region $F_0>F_{01}$ a degeneracy occurs. In the plot the transition of the nondegenerate eigenvalues to the degenerate ones by changing F_0 from low to large values is clearly seen. This transition at F_{01} is similar to the transition at F_1 in Fig. 3 of [22]. Notice the interesting intermediate plateau of λ_3/γ and λ_4/γ near the critical force F_{01}. In Chap. 5 an analytical value for the lowest non-zero eigenvalue in the bistability region $F_0>F_{01}$ is derived. In Fig. 6 the sta-

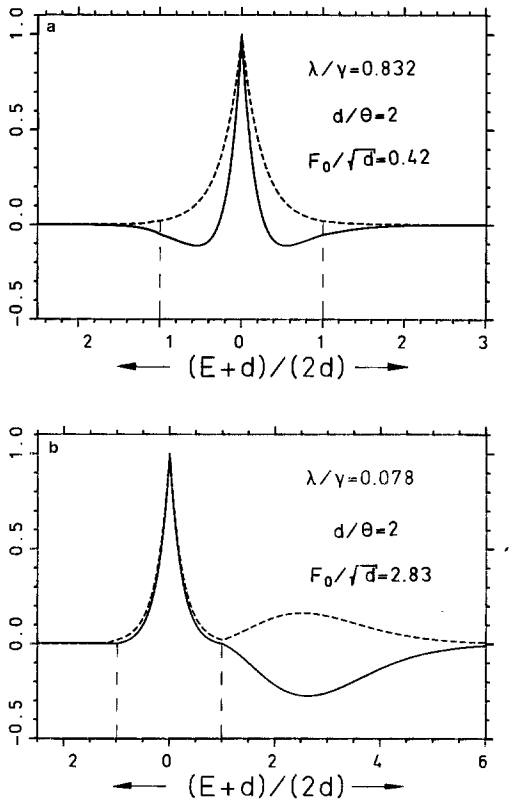

Fig. 6a and b. Eigenfunctions $\Phi_+ = (\Phi_S^{(n)} + \Phi_D^{(n)})/2$ plotted to the right and $\Phi_- = (\Phi_S^{(n)} - \Phi_D^{(n)})/2$ plotted to the left as a function of E for $d/\Theta = 2$ and $F_0/\sqrt{d} = 0.42$ **a** and 2.83 **b** (arbitrary units). The stationary solution is shown by the broken line

tionary solution and the eigenfunction belonging to the lowest non-zero eigenvalue are shown for some typical forces. Whereas for $F_0 < F_{01}$ values the stationary solution and the eigenfunction belonging to the first non-zero eigenvalue are different, they agree approximately for $F_0 > F_{01}$ for small energies and have opposite signs for larger energies in accordance with (2.13, 14) of [22].

Zero External Force

For the special case of zero external force $F_0 = 0$ the equations (2.21b) for Φ_D and Φ_S are no longer coupled. The eigenvalues for the sum function Φ_S agree quite well with the eigenvalues obtained in [33] with the matrix continued fraction method for small γ. The first non-zero eigenvalue for the cosine potential with $d/\Theta = 2$ according to the present method is given by $\lambda_1^{(S)}/\gamma = 0.848$ whereas the matrix continued fraction for $\gamma = 0.2$ leads to $\lambda_1^{(S)}/\gamma = 0.868$. The eigenvalues for the difference function Φ_D do not agree so well as those for the sum functions. The first eigenvalue for $d/\Theta = 2$ according to the present theory is $\lambda_1^{(D)}/\gamma = 3.19$ whereas the matrix continued fraction leads to $\lambda_1^{(D)} = 2.64$ for $\gamma = 0.2$. If however a boundary layer is considered (Chap. 4) the agreement is again quite good.

4. Boundary Layer Theory

By the ansatz (2.12) we have assumed that the distribution function depends on energy only. Near the critical trajectory $E = E_0$ (see Fig. 1) this assumption cannot be valid for finite damping constants as seen as follows: Particles moving along closed trajectories near $E \lessapprox E_0$ are confronted at each turn with two different groups of particles; in one group the particles move to the right ($v > 0$) along trajectories near $E \gtrapprox E_0$ and in the other they move to the left ($v < 0$) along trajectories near $E \gtrapprox E_0$. If the probability of these two groups is different (which is always the case for $F_0 \neq 0$) one therefore expects for finite γ a strong diffusion perpendicular to the $E = E_0$ trajectory leading to an x-dependence of the distribution function. The process takes place in a boundary layer around the $E = E_0$ trajectory. To describe the x-dependence in this boundary layer we make the following ansatz for $W_S(x, E, t)$ and $W_D(x, E, t)$:

$$\underset{D}{W_S}(x, E, t) = \underset{D}{\tilde{W}_S}(E, t) + \underset{D}{w_S}(x, E, t). \tag{4.1}$$

Here $\underset{D}{\tilde{W}_S}(E, t)$ are slowly varying functions in E not depending on x. The $\underset{D}{w_S}(x, E, t)$ are rapidly varying functions in E and slowly varying functions in x that contribute only in a thin boundary layer (skin) around $E = E_0$. As shown below, the thickness of the skin is of the order of magnitude of $\sqrt{\gamma}$. As it turns out the amplitudes of w_S are also of the order $\sqrt{\gamma}$. For $\underset{D}{w_S}$ only terms in lowest order will be taken into account. In accordance with this approximation we put $x_1(E) = \mp\pi$ in $w_\pm(x_1, E, t)$. Thus the continuity conditions (2.7) simplify for small γ values to

$$\begin{aligned} &\underset{D}{w_S}(\pi, E, t) = \underset{D}{w_S}(-\pi, E, t) \quad \text{for } E > E_0, \\ &w_D(\pm\pi, E, t) = \tilde{W}_D(E, t) = 0 \quad \text{for } E < E_0. \end{aligned} \tag{4.2}$$

Because of the assumptions made for $\underset{D}{w_S}$, the first derivatives in E are neglected compared with the second derivatives in E. Furthermore the velocity need to be considered for $E = E_0$ only. If we consider a time scale of the distribution functions of order $1/\gamma$, this time-dependence for the functions $\underset{D}{w_S}(x, E, t)$ need not be taken into account inside the very thin

boundary layer of thickness $\sqrt{\gamma}$, where a quasistationary distribution is rapidly established. Therefore the equations (2.9) for x-dependent solutions reduce to

$$\partial \underset{D}{w_S}/\partial x = \gamma \Theta v(x, E_0) \partial^2 \underset{S}{w_D}/\partial E^2. \tag{4.3}$$

Instead of the space coordinate x we introduce the variable u defined by

$$u = u(x) = -\pi + 2\pi \int_{-\pi}^{x} v(\xi, E_0)\, d\xi / I(E_0), \tag{4.4}$$

For the cosine potential (1.3) we have ($E_0 = d$)

$$u = \pi \sin(x/2), \tag{4.5}$$

$$I(E_0) = 8\sqrt{d}. \tag{4.6}$$

Using (2.16, 4.4) (4.3) simplifies to

$$\partial \underset{D}{w_S}/\partial u = \gamma \Theta \bar{v}(E_0) \partial^2 \underset{S}{w_D}/\partial E^2. \tag{4.7}$$

Solutions of (4.7) consistent with (4.2) and different from zero only in the vicinity of E_0 are ($-\pi \leqq u \leqq \pi$)

$E > E_0$:

$$w_D = w_0(t) \sum_{n=1}^{\infty} \mathrm{Im}\{a_n \exp[-(1+i)\alpha\sqrt{n}(E - E_0)/\Theta]\} \cos nu, \tag{4.8}$$

$$w_S = w_0(t) \sum_{n=1}^{\infty} \mathrm{Re}\{a_n \exp[-(1+i)\alpha\sqrt{n}(E - E_0)/\Theta]\} \sin nu \tag{4.9}$$

$E < E_0$:

$$w_D = w_0(t) \sum_{n=0}^{\infty} \mathrm{Im}\{b_n \exp[(1+i)\alpha\sqrt{n+1/2}(E - E_0)/\Theta]\} \cos(n+\tfrac{1}{2})u, \tag{4.10}$$

$$w_S = w_0(t) \sum_{n=0}^{\infty} \mathrm{Re}\{b_n \exp[(1+i)\alpha\sqrt{n+1/2}(E - E_0)/\Theta]\} \cdot \sin(n+\tfrac{1}{2})u. \tag{4.11}$$

Here α is given by

$$\alpha = \sqrt{\Theta/(2\gamma \bar{v}(E_0))} \tag{4.12}$$

and the imaginary and real parts are indicated by Im and Re. The prefactor

$$w_0(t) = \Theta \tilde{W}_D'(E_0 + 0, t)/\alpha \tag{4.13}$$

is chosen so that the complex amplitudes a_n and b_n are of the order of magnitude one. In (4.8, 9) the terms with $n = 0$ must be excluded, because they would give x-independent functions, which are already included in $\underset{D}{\tilde{W}_S}(E, t)$.

Continuity Conditions at $E = E_0$

We now require that at $E = E_0$, $\underset{D}{W_S}(x, E, t)$ and their derivatives with respect to E are continuous

$$\underset{D}{W_S}(x, E_0 - 0, t) = \underset{D}{W_S}(x, E_0 + 0, t) \tag{4.14}$$

$$\partial \underset{D}{W_S}(x, E, t)/\partial E|_{E=E_0-0} = \partial \underset{D}{W_S}(x, E, t)/\partial E|_{E=E_0+0}. \tag{4.15}$$

Because $w_S(x, E, t)$ and $\partial w_S(x, E, t)/\partial E$ is antisymmetric in $u(x)$ we obtain by inserting (4.1) and using $\tilde{W}_D(E, t) = 0$ for $E < E_0$

$$w_S(x, E_0 - 0, t) = w_S(x, E_0 + 0, t) \tag{4.14a}$$

$$w_D(x, E_0 - 0, t) = \tilde{W}_D(E_0 + 0, t) + w_D(x, E_0 + 0, t) \tag{4.14b}$$

$$\partial w_S(x, E, t)/\partial E|_{E=E_0-0} = \partial w_S(x, E, t)/\partial E|_{E=E_0+0} \tag{4.15a}$$

$$\partial w_D(x, E, t)/\partial E|_{E=E_0-0} = \tilde{W}_D'(E_0 + 0, t) + \partial w_D(x, E, t)/\partial E|_{E=E_0+0}. \tag{4.15b}$$

The derivative of $\tilde{W}_D(E, t)$ with respect to E is denoted by a prime. The x-independent part of (4.14) for $W_S(x, E, t)$ leads to

$$\tilde{W}_S(E_0 - 0, t) = \tilde{W}_S(E_0 + 0, t) = \tilde{W}_S(E_0, t) \tag{4.16}$$

whereas the x-independent part of (4.15) for $W_S(x, E, t)$ gives

$$\tilde{W}_S'(E_0 - 0, t) = \tilde{W}_S'(E_0 + 0, t) + O(\sqrt{\gamma}). \tag{4.17}$$

Because $\tilde{W}_S$ is slowly varying in E and w_S is rapidly varying in E, (4.15, 15a, b, 17) are only correct in the lowest order term $\sqrt{\gamma}^0$. Therefore the correction term in (4.17) is of the order $\sqrt{\gamma}$ [(4.16) is correct up to terms of the order $\sqrt{\gamma}$]. To find the correction term to (4.17) we use the continuity of the probability current for $\tilde{W}_S$ [see (2.14c, d)] at $E = E_0$. It follows from this condition and (4.16) that the jump condition

$$\tilde{W}_S'(E_0 - 0, t) = \tilde{W}_S'(E_0 + 0, t) - \tilde{W}_D(E_0 + 0, t) F_0/[\bar{v}(E_0)\Theta] \tag{4.18}$$

holds even in terms of the order $\sqrt{\gamma}$. Introducing the constant κ by

$$\tilde{W}_D(E_0 + 0, t) = \kappa w_0(t) = \kappa \Theta \tilde{W}_D'(E_0 + 0, t)/\alpha \tag{4.19}$$

and inserting (4.8-11) into the continuity conditions (4.14a-15b) and using (4.13) we have

$$\sum_{n=0}^{\infty} b_n^{(r)} \sin(n+\tfrac{1}{2})u = \sum_{n=1}^{\infty} a_n^{(r)} \sin nu \tag{4.20a}$$

$$\sum_{n=0}^{\infty} b_n^{(i)} \cos(n+\tfrac{1}{2})u = \kappa + \sum_{n=1}^{\infty} a_n^{(i)} \cos nu \tag{4.20b}$$

$$\sum_{n=0}^{\infty} (b_n^{(r)} - b_n^{(i)})\sqrt{n+1/2} \sin(n+\tfrac{1}{2})u = \sum_{n=1}^{\infty} (-a_n^{(r)} + a_n^{(i)})\sqrt{n} \sin nu \tag{4.21a}$$

$$\sum_{n=0}^{\infty} (b_n^{(r)} + b_n^{(i)})\sqrt{n+1/2} \cos(n+\tfrac{1}{2})u = 1 + \sum_{n=1}^{\infty} (-a_n^{(r)} - a_n^{(i)})\sqrt{n} \cos nu. \tag{4.21b}$$

Here the amplitudes with an upper index $r(i)$ are the real (imaginary) parts of the corresponding amplitudes. From (4.20a-21b) the amplitudes $a_n^{(r)}$, $a_n^{(i)}$, $b_n^{(r)}$, $b_n^{(i)}$ and the constant κ have to be determined. For the determination of the slowly varying functions $W_{\substack{S \\ D}}(E,t)$ we only need the constant κ. In [36] $\kappa = 0.859$ was obtained. More accurate recent calculations [39] lead to the value

$$\kappa = 0.855(4)\ldots . \tag{4.22}$$

By inserting (2.19) into (4.16, 18, 19) we obtain the following continuity conditions for the eigenfunctions $\Phi_{\substack{S \\ D}}$:

$$\Phi_S^{(n)}(E_0 - 0) = \Phi_S^{(n)}(E_0 + 0) \tag{4.23}$$

$$d\Phi_S^{(n)}(E)/dE|_{E=E_0-0} = d\Phi_S^{(n)}(E)/dE|_{E=E_0+0} - \Phi_D(E_0+0)2\pi F_0/[I(E_0)\Theta] \tag{4.24}$$

$$\Phi_D^{(n)}(E_0+0) = \kappa\, d\Phi_D^{(n)}(E)/dE|_{E=E_0+0} \sqrt{\gamma\Theta I(E_0)/\pi} \tag{4.25}$$

Because of the closed motion $\Phi_D(E)$ must be zero for $E < E_0$ (see (4.2))

$$\phi_D(E) = 0 \quad \text{for } E < E_0. \tag{4.26}$$

Without the boundary layer, i.e. for $\gamma \to 0$ (4.23-25) simplify to (2.23-25). With the boundary layer the results of eigenvalues and eigenfunctions depend on $\sqrt{\gamma/\Theta}$ because $\Phi_D^{(n)}(E_0+0)$ depends on it. For small values of $\sqrt{\gamma/\Theta}$ one expects that the eigenvalues are a linear function of $\sqrt{\gamma/\Theta}$

$$A_n(\gamma) = A_n(0) + B_n\sqrt{\gamma/\Theta}. \tag{4.27}$$

Zero External Force

Without the force F_0 the equation for $\Phi_S^{(n)}$ and $\Phi_D^{(n)}$ are decoupled. The equation for the sum function Φ_S then does not contain any $\sqrt{\gamma/\Theta}$ term and the eigenvalues $\lambda_n^{(S)}$ should therefore not depend very much on γ, in agreement with the results mentioned at the end of Chap. 3. Because for the difference function $\Phi_D^{(n)}$ $\sqrt{\gamma/\Theta}$ enters in the boundary conditions, the eigenvalues $\lambda_n^{(D)}$ for the difference functions should be of the form (4.27)

$$\lambda_n^{(D)}(\gamma)/\gamma = \lambda_n^{(D)}(0)/\gamma + \tilde{B}_n\sqrt{\gamma}$$

for small γ. For the cosine potential with $d/\Theta = 2$ the numerical integration of (2.21a,b) with (4.23-25) leads to $\lambda_1^{(D)}(\gamma = 0.2)/\gamma = 2.56$ for the first eigenvalue. The last expression agrees again fairly well with the matrix continued fraction result $\lambda_1^{(D)}(0.2)/\gamma = 2.64$ found in [33].

Arbitrary External Force

For arbitrary external force the stationary solutions as well as the drift velocity and mobility have been obtained in [36] for the general continuity conditions (4.23-25). As shown in that paper the results agree very well with the results of the matrix continued fraction method. Some non-zero eigenvalues with the boundary condition (4.23-25) are shown in Fig. 5a.

5. Lowest Non-Zero Eigenvalue in the Bistability Region for the Zero-Temperature Limit

Transformation to an Integral Equation

The differential equations (2.21a,b) for the sum and difference functions (we now suppress the index n)

$$\Phi_{\pm} = \tfrac{1}{2}(\Phi_S^{(n)} \pm \Phi_D^{(n)}) \tag{5.1}$$

take the form

$E < E_0$

$$\frac{d}{dE} I(E)\left(1 + \Theta\frac{d}{dE}\right)\Phi + A\,T(E)\,\Phi = 0 \tag{5.2a}$$

$$\Phi = \Phi_+ = \Phi_-$$

$E > E_0$

$$\frac{d}{dE}\left[I(E)\left(1+\Theta\frac{d}{dE}\right)\Phi_{\pm} \mp 2\pi F_0 \Phi_{\pm}\right] + A\,T(E)\,\Phi_{\pm} = 0 \tag{5.2b}$$

At $E=E_{\min}$ we have to require (2.22), i.e.

$$\Theta \frac{d\Phi}{dE}\bigg|_{E=E_{\min}} + (A+1)\,\Phi(E_{\min}) = 0. \tag{5.3}$$

The continuity conditions (2.23-25) at $E=E_0$ take the form

$$2\,\Phi(E_0-0) = \Phi_+(E_0+0) + \Phi_-(E_0+0) \tag{5.4}$$

$$2\frac{d}{dE}\Phi(E)\bigg|_{E=E_0-0} = \frac{d}{dE}[\Phi_+(E)+\Phi_-(E)]\bigg|_{E=E_0+0}. \tag{5.5}$$

At $E\to\infty$ we have to require

$$\Phi_\pm(E)\to 0 \quad \text{for } E\to\infty. \tag{5.6}$$

Formal integration of (5.2a,b) subject to the boundary conditions (5.3-5.6) leads to

$E<E_0$

$$\Phi(E) = C e^{-\frac{E}{\Theta}} + \frac{A}{\Theta} e^{-\frac{E}{\Theta}} \int_E^{E_0} \frac{e^{E'/\Theta}}{I(E')} \left[\int_{E_{\min}}^{E'} T(E'')\,\Phi(E'')\,dE'' \right] dE' \tag{5.7a}$$

$E>E_0$

$$\Phi_\pm(E) = C e^{-\frac{V^\pm(E)}{\Theta}} + \frac{A}{\Theta} e^{-\frac{V^\pm(E)}{\Theta}} \int_{E_0}^{E} \frac{e^{V^\pm(E')/\Theta}}{I(E')} \cdot \left[\int_{E'}^{\infty} T(E'')\,\Phi_\pm(E'')\,dE'' \right] dE'. \tag{5.7b}$$

In (5.7a,b) C is an integration constant and $V^\pm(E)$ the effective low friction potential (2.30). By inserting (5.7a,b) into the boundary condition (5.5) we obtain

$$2\int_{E_{\min}}^{E_0} T(E)\,\Phi(E)\,dE + \int_{E_0}^{\infty} T(E)[\Phi_+(E)+\Phi_-(E)]\,dE = 0. \tag{5.8}$$

This condition says that the function $\Phi(E)=\Phi_+(E)=\Phi_-(E)$ for $E<E_0$ and $\Phi_\pm(E)$ for $E>E_0$ is orthogonal to the stationary ($A=0$) solution $\Phi=1$ of the adjoint equation to (5.2a,b) with the weight factor $T(E)$. By using proper partial integrations (5.7a,b) can be cast into the following standard integral equations

$E<E_0$

$$\Phi(E) = C e^{-\frac{E}{\Theta}} + A \int_{E_{\min}}^{E_0} K(E,E')\,\Phi(E')\,dE' \tag{5.9a}$$

$E>E_0$

$$\Phi_\pm(E) = C e^{-\frac{V^\pm(E)}{\Theta}} + A \int_{E_0}^{\infty} K_\pm(E,E')\,\Phi_\pm(E')\,dE' \tag{5.9b}$$

where the kernels K and $K_\pm$ are given by

$$K(E,E') = \frac{T(E')}{\Theta} e^{-\frac{E}{\Theta}} \begin{cases} y(E) \\ y(E') \end{cases} \text{for} \begin{matrix} E'<E<E_0 \\ E<E'<E_0 \end{matrix} \tag{5.10a}$$

$$K_\pm(E,E') = \frac{T(E')}{\Theta} e^{-\frac{V^\pm(E)}{\Theta}} \begin{cases} y_\pm(E') \\ y_\pm(E) \end{cases} \text{for} \begin{matrix} E_0<E'<E \\ E_0<E<E'. \end{matrix} \tag{5.10b}$$

Here $y(E)$ and $y_\pm(E)$ are defined by

$$y(E) = \int_E^{E_0} \frac{e^{E'/\Theta}}{I(E')}\,dE' \quad \text{for } E<E_0 \tag{5.11b}$$

$$y_\pm(E) = \int_{E_0}^{E} \frac{e^{V^\pm(E')/\Theta}}{I(E')}\,dE' \quad \text{for } E>E_0. \tag{5.11b}$$

From the integral equations (5.9a,b) together with the orthogonality condition (5.8) the eigenvalues A and the eigenfunctions $\Phi(E)$, $\Phi_\pm(E)$ have to be determined. If we use the functions

$$\begin{aligned} \psi(E) &= \sqrt{T(E)}\, e^{-E/(2\Theta)}\,\Phi(E); \\ \psi_\pm(E) &= \sqrt{T(E)}\, e^{-V^\pm(E)/(2\Theta)}\,\Phi_\pm(E) \end{aligned} \tag{5.12}$$

instead of $\Phi(E)$ and $\Phi_\pm(E)$ the integral equations for $\psi(E)$ and $\psi_\pm(E)$ will have symmetric kernels. Though symmetric kernels are usually much more convenient than nonsymmetric ones, we do not need the transformation (5.12) for the following considerations.

Approximate Solution of the Integral Equation

We now want to calculate the eigenvalue $A=\lambda/\gamma$ in the zero temperature limit $\Theta\to 0$ for forces in the bistability region $F_0>F_{01}=\bar{v}(E_0)$. Because of the large values of $V^-(E)$, see Fig. 4, we do not need to consider $\Phi_-(E)$. Next we observe that $\Phi(E)$ will be concentrated near $E=E_{\min}$ and $\Phi_+(E)$ near $\bar{E}$, where $\bar{E}$ is the energy corresponding to the minimum of $V^+(E)$. Therefore we only need to consider $y(E)$ and $y_+(E)$ for small $E-E_{\min}$ and small $E-\bar{E}$ respectively. In this case $y(E)$ and $y_+(E)$ may be approximated by constants in the limit $\Theta\to 0$

$$y(E) = \int_E^{E_0} \frac{e^{E'/\Theta}}{I(E')}\,dE' \approx \frac{\Theta\, e^{E_0/\Theta}}{I(E_0)} = y^0 \tag{5.13a}$$

$$\begin{aligned} y_+(E) &= \int_{E_0}^{E} \frac{e^{V^+(E')/\Theta}}{I(E')}\,dE' \approx \frac{e^{V^+(E_0)/\Theta}}{I(E_0)} \int_{E_0}^{E} e^{-\beta(E'-E_0)/\Theta}\,dE' \\ &\approx \frac{\Theta\, e^{E_0/\Theta}}{\beta I(E_0)} = y_+^0. \end{aligned} \tag{5.13b}$$

In (5.13b) β is the negative derivative of $V^+(E)$ at $E=E_0$ i.e.

$$\beta=-dV^+/dE|_{E=E_0}=F_0 g'(E_0)-1=F_0/\bar{v}(E_0)-1$$
$$=2\pi F_0/I(E_0)-1. \tag{5.14}$$

The constant β is positive in the bistability region

$$F_0>F_{01}=\bar{v}(E_0)=I(E_0)/(2\pi). \tag{5.15}$$

Because $y(E)\approx y^0$ and $y_+(E)\approx y_+^0$ is constant, the kernels K and K_+ are degenerate. Multiplying (5.9a,b) by $T(E)$ and integrating (5.9a) from $E_{\min}$ to E_0 and the + equation of (5.9b) from E_0 to infinity we obtain

$$x=NC+[Ay^0N/\Theta]x \tag{5.16a}$$

$$x_+=N_+C+[Ay_+^0N_+/\Theta]x_+ \tag{5.16b}$$

where we have used the abbreviations

$$\int_{E_{\min}}^{E_0} T(E)\Phi(E)dE=x \tag{5.17a}$$

$$\int_{E_0}^{\infty} T(E)\Phi_+(E)dE=x_+ \tag{5.17b}$$

$$\int_{E_{\min}}^{E_0} T(E)e^{-E/\Theta}dE=N \tag{5.18a}$$

$$\int_{E_0}^{\infty} T(E)e^{-V^+(E)/\Theta}dE=N_+. \tag{5.18b}$$

Because we have neglected Φ_-, (5.8) takes the form

$$2x+x_+=0. \tag{5.19}$$

Equations (5.16a,b, 5.19) are 3 linear homogeneous equations for the 3 unknowns x, x_+, C. Therefore the corresponding determinant must vanish, i.e.

$$\begin{vmatrix} 1-Ay^0N/\Theta & 0 & -N \\ 0 & 1-Ay_+^0N_+/\Theta & -N_+ \\ 2 & 1 & 0 \end{vmatrix}$$
$$=N_+(1-Ay^0N/\Theta)+2N(1-Ay_+^0N_+/\Theta)=0. \tag{5.20}$$

From this equation we obtain the eigenvalue

$$\frac{\lambda}{\gamma}=A=\frac{\Theta}{y^0+2y_+^0}\left[\frac{2}{N_+}+\frac{1}{N}\right]. \tag{5.21}$$

In the low temperature limit $\Theta\to 0$ we only need to consider energies near $E_{\min}$ and $\bar{E}$. Then the integrals (5.18a,b) can be approximated by

$$N\approx\Theta T(E_{\min})e^{-E_{\min}/\Theta} \tag{5.22a}$$

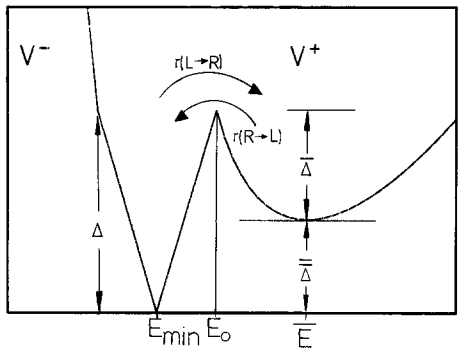

Fig. 7. The energies Δ, $\bar{\Delta}$, $\bar{\bar{\Delta}}$ and the transition rates $r(R\to L)$ and $r(L\to R)$ in the effective potential.

$$N_+\approx T(\bar{E})e^{-V^+(\bar{E})/\Theta}\int_{-\infty}^{\infty}\exp\left[-\frac{\delta}{2\Theta}(E-\bar{E})^2\right]dE$$
$$=\sqrt{\frac{2\pi\Theta}{\delta}}\,T(\bar{E})e^{-V^+(\bar{E})/\Theta}. \tag{5.22b}$$

The second derivative of $V^+(E)$ at $E=\bar{E}$ is denoted by δ. It can be connected to the mobility $(\gamma\mu)^R$ of the running solution without noise

$$\delta=d^2V^+/dE^2|_{E=\bar{E}}=-F_0g''(\bar{E})=F_0\bar{v}'(\bar{E})/\bar{v}(\bar{E})^2$$
$$=\bar{v}'(\bar{E})/F_0=I'(\bar{E})/(2\pi F_0)=1/[(\gamma\mu)^RF_0^2]. \tag{5.23}$$

In (5.23) we have used (2.31) and (3.9) of [25]. By inserting (5.13a,b) and (5.22a,b) in (5.21) and by using (2.16, 30, 5.14, 23) we finally obtain ($F_0=F/\gamma$)

$$\lambda=\gamma A=r(L\to R)+r(R\to L) \tag{5.24}$$

$$r(L\to R)=\gamma\frac{[2\pi F_0-I(E_0)]\,I(E_0)}{[2\pi F_0+I(E_0)]\,\Theta I'(E_{\min})}e^{-\frac{\Delta}{\Theta}} \tag{5.25a}$$

$$r(R\to L)=\gamma\frac{[2\pi F_0-I(E_0)]\sqrt{(\gamma\mu)^R}\,I(E_0)}{[2\pi F_0+I(E_0)]\sqrt{2\pi\Theta\pi}}e^{-\frac{\bar{\Delta}}{\Theta}}. \tag{5.25b}$$

Here Δ and $\bar{\Delta}$ are the energies between the maximum and the two minima of the potential $V(E)$ and $V^+(E)$

$$\Delta=E_0-E_{\min},\qquad \bar{\Delta}=E_0-V^+(\bar{E})=E_0-\bar{E}+F_0g(\bar{E}). \tag{5.26}$$

In (5.24) $r(L\to R)$ can be interpreted as the transition rate from the locked to the running state and vice versa for $r(R\to L)$ (see Fig. 7).

Special Values for the Cosine Potential

For the cosine potential (1.3) we have

$$E_{\min}=-d,\quad E_0=d,\quad \Delta=2d$$
$$I'(E_{\min})=\pi/\sqrt{d},\quad I(E_0)=8\sqrt{d} \tag{5.27}$$

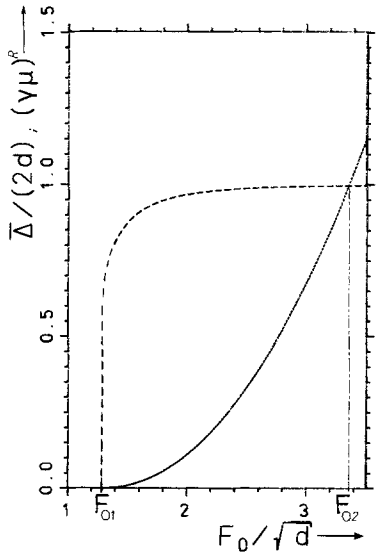

Fig. 8. The values $(\gamma\mu)^R$ (broken line) and $\bar{\Delta}/(2d)$ for the cosine potential (1.3) as a function of $F_0/\sqrt{d}$

and we thus obtain $(F_0 = F/\gamma)$

$$r(L\to R)=\gamma\frac{8}{\pi}\frac{d}{\Theta}\frac{F_0/\sqrt{d}-4/\pi}{F_0/\sqrt{d}+4/\pi}e^{-\frac{2d}{\Theta}} \tag{5.28a}$$

$$r(R\to L)=\gamma\frac{8}{\pi}\sqrt{\frac{(\gamma\mu)^R d}{2\pi\Theta}}\frac{F_0/\sqrt{d}-4/\pi}{F_0/\sqrt{d}+4/\pi}e^{-\frac{2d}{\Theta}\frac{\bar{\Delta}}{2d}}. \tag{5.28b}$$

The values of $(\gamma\mu)^R$ and of $\bar{\Delta}/(2d)$ are shown in Fig. 8.
For $E>E_0$ we may approximate $I(E)$ by

$$I(E)=2\pi\sqrt{2E-E_0} \tag{5.29}$$

leading to

$$\bar{\Delta}/(2d)=(1/4)(F_0/\sqrt{d}-\sqrt{2})^2. \tag{5.30}$$

Figure 9 shows the exact expression (5.26), the analytical expression (5.30) and the analytic form of Ben Jacob et al. [26] in our notation

$$\bar{\Delta}/(2d)=(1/4)(F_0/\sqrt{d}-4/\pi)^2. \tag{5.31}$$

In Fig. 10 we compare the analytic expression for the eigenvalue λ/γ (5.24, 28a, b) with those from the numerical integration procedure. As seen the agreement is quite good for $F_0/\sqrt{d}\gtrapprox 1.8$. The deviations for smaller $F_0/\sqrt{d}$ occur, because the energy difference $\bar{\Delta}$ is not large compared to Θ.

Connection to the Stationary Mobility

As it was explained in [22] the stationary mobility $(\gamma\mu)_{St}=\gamma\langle v\rangle/F=\langle v\rangle/F_0$, the mobility of the running

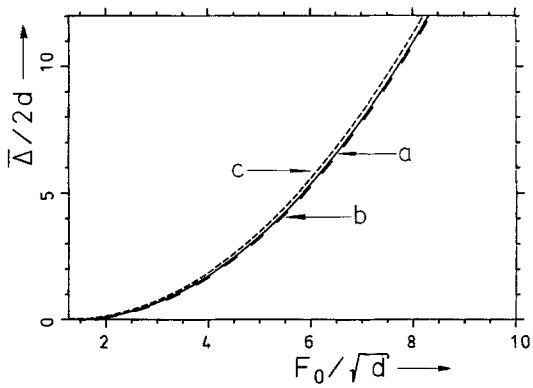

Fig. 9. The exact expression (5.26)(a) and the approximations (5.30)(b) and (5.31)(c)

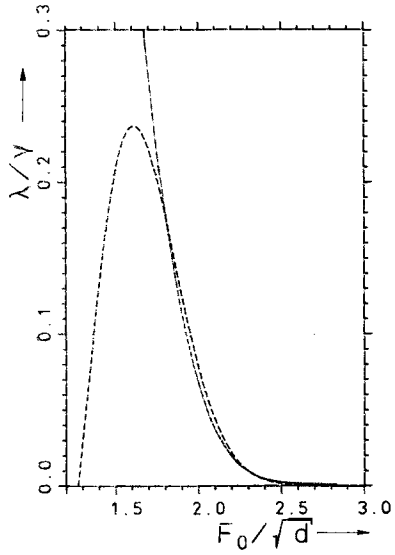

Fig. 10. Comparison of the lowest non-zero eigenvalue obtained by numerical integration as explained in Chap. 3 and the asymptotic expression (5.25a, b) (curve with the maximum)

solution without noise $(\gamma\mu)^R$, the eigenvalue λ and the transition rates (5.28a, b) are connected by (see (2.17a, b) of [22])

$$r(R\to L)=\lambda[1-(\gamma\mu)_{St}/(\gamma\mu)^R]$$
$$r(L\to R)=\lambda(\gamma\mu)_{St}/(\gamma\mu)^R. \tag{5.32}$$

Thus we may express $(\gamma\mu)_{St}$ by

$$(\gamma\mu)_{St}=(\gamma\mu)^R/\left[1+\frac{r(R\to L)}{r(L\to R)}\right]. \tag{5.33}$$

Insertion of (5.25a, b) yields

$$(\gamma\mu)_{St}=(\gamma\mu)^R/\{1+\sqrt{(\gamma\mu)^R\Theta/(2\pi)}[I'(E_{min})/\pi]\exp(\bar{\bar{\Delta}}/\Theta)\} \tag{5.34}$$

where $\bar{\bar{\Delta}}$ is the energy difference between the minima at $E=E_{min}$ and $E=\bar{E}$

$$\bar{\Delta} = \Delta - \bar{\Delta} = \bar{E} - E_{\min} - F_0 g(\bar{E}). \tag{5.35}$$

For the cosine potential (5.34, 35) specialize to

$$(\gamma\mu)_{\mathrm{St}} = (\gamma\mu)^R / [1 + \sqrt{(\gamma\mu)^R \Theta/(2\pi d)} \exp(\bar{\Delta}/\Theta)]$$
$$\bar{\Delta} = d + \bar{E} - F_0 g(\bar{E}). \tag{5.36}$$

This result was already obtained in [25], Eq. (4.11).

6. Discussion and Conclusion

In the present paper we have shown how to calculate the real eigenvalues and eigenfunctions for the Brownian motion in an inclined periodic potential in the low friction limit. The numerical method works very well for not too low temperatures Θ. These results agree with the results obtained by the matrix continued fraction method for low friction constants. As shown in Chap. 3 for the cosine potential (1.3), this numerical method works fine even for $d/\Theta = 1/10$. In the limit $\Theta \to 0$, the numerical method cannot be applied. In this low temperature limit we have derived an analytic expression for the lowest non-zero eigenvalue in the bistability region. We want to stress that this expression is valid for $\gamma \to 0$, $\Theta \to 0$ and $\sqrt{\gamma/\Theta} \to 0$, i.e. γ must be much smaller than the temperature Θ. Otherwise the additional terms to due to the boundary layer of Chap. 4 cannot be neglected or, for $\sqrt{\gamma/\Theta} \gtrsim O(1)$ cannot be used at all. (The expression (25) of [40] is not valid for $\sqrt{\gamma/\Theta} \to 0$.) The analytic expression (5.25a, b) should be compared with the results of Ben-Jacob et al. [26] and Büttiker et al. [27]. In [26], the lifetime $\tau = 1/r(R \to L)$ of the running solution was calculated in the low temperature limit, see (4.5) of [26].

The term ΔW in the exponential factor in (4.5) of [26] approximately agrees with our $\bar{\Delta}$ (compare Fig. 9) and the dependence of the prefactor on γ, Θ and ΔW also agrees. The other prefactor however, is different. Putting $(\gamma\mu)^R = 1$ and $\bar{\Delta} = \Delta W$ the ratio of the rate $r_{\mathrm{B.J.}} = 1/\bar{\tau}_S$ of Ben-Jacob et al. [26] and our rate out of the running state is given in our notation by

$$r_{\mathrm{B.J.}}/r(R \to L) = 1 + (\pi/8)(F_0/\sqrt{d} - 4/\pi). \tag{6.1}$$

This ratio is only 1 at the beginning of the bistability region, for forces F/γ larger than $(4/\pi)\sqrt{d}$ it is larger than one. An expression for the rate out of the locked state was obtained in [27] (Eq. (3.11)) in the zero temperature limit. The energy E_b in the exponential factor in (3.11) of [27] is the same as our energy Δ in (5.25a) for finite F/γ, i.e. for $F \to 0$. The last two prefactors in (3.11) of [27] also appear in our prefactors the other prefactor, however, is different. Though no distinction between the sum and the difference equations was made in [27], their eigenvalue equation (3.3) resembles our eigenvalue equation (2.21a, b). As it was remarked in [27], their parameter α was not obtained by the eigenvalue equation but was treated as an adjustable parameter (adjusted to the results of a numerical similation). Probably, because the boundary conditions are not exactly specified in [27], they could not obtain α by only using the eigenvalue equation (our eigenfunctions $\Phi_S(E)$ and $\Phi_D(E)$ are strictly periodic in x because we have used the energy (2.2) without the additional potential due to the external force). In [26] the method was also applied to finite friction constants in the zero temperature limit $\Theta \to 0$. In [26] the WKB approximation

$$W(x, v) = A(x, v) \exp[-S(x, v)/\Theta] \tag{6.2}$$

was used. Though this ansatz is valid in the zero friction limit (without any boundary layer region) it is not valid for small friction on the boundary layer region, compare (4.1, 8-12).

Finally we would like to remark that the eigenvalues for the Brownian motion problem in a double well potential as well as in other potentials in the low friction limit can be obtained similarly. The eigenvalues for the double well potential will be treated elsewhere [39].

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