Math. Ann. 266, 251–259 (1984)

Mathematische Annalen

The Square Sieve and Consecutive Square-Free Numbers

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1. Introduction

Let $\mathscr{A}$ be a sequence of integers. Suppose we have information about the distribution of $\mathscr{A}$ to certain moduli. How many squares can $\mathscr{A}$ contain? To formulate the problem precisely let $w(n) \geqq 0$ for each integer n (positive, negative or zero) and suppose $\sum w(n) < \infty$. We write $\mathscr{A}$ for the sequence $(w(n))$, and we define

$$S(\mathscr{A}) = \sum_{1}^{\infty} w(n^2)$$

We seek an upper bound for $S(\mathscr{A})$. Sieves which give answers to this problem have been provided by Gallagher [3] and Montgomery [7; Corollary 3.2]. These sieves concern much more general problems – and consequently are ill adapted for the finer questions discussed below. We shall prove the following simple bound.

Theorem 1. *Let $\mathscr{P}$ be a set of P primes. Suppose that $w(n) = 0$ for $n = 0$ or $|n| \geqq e^P$. Then*

$$S(\mathscr{A}) \ll P^{-1} \sum_n w(n) + P^{-2} \sum_{p \neq q \in \mathscr{P}} \left| \sum_n w(n) \left(\frac{n}{pq}\right) \right|, \tag{1}$$

where $\left(\frac{n}{pq}\right)$ is the Jacobi symbol.

This result is motivated by the method used by Hooley [5] in his proof that the number of representations $v(n)$ of n as a sum of 4 non-negative cubes, satisfies $v(n) \ll n^{11/18+\varepsilon}$. Hooley's method, when abstracted, is more complicated than Theorem 1, but not essentially weaker.

Both the side conditions of the theorem are necessary in some form, for if $p|n$ for some fixed $n > 0$ and every $p \in \mathscr{P}$, and $w(n^2) = 1$, $w(m) = 0$ for $m \neq n^2$, then $S(\mathscr{A}) = 1$, while the right hand side of (1) is $O(P^{-1})$. The theorem will in general be weak by $\log P$ factors at least. As an illustration let $w(n) = 1$ for $1 \leqq n \leqq x$ and $w(n) = 0$ otherwise, and take $\mathscr{P}$ to be the set of primes $p \leqq x^{1/2}$. Then the Pólya-Vinogradov inequality yields

$$\sum_n w(n) \left(\frac{n}{pq}\right) \ll x^{1/2} \log x,$$

so that the right hand side of (1) is $O(x^{1/2}\log x)$ rather than $O(x^{1/2})$. This log factor could be recovered by complicating the form of the result. However one is usually interested in saving powers of P, rather than $\log P$.

One may count cubes or higher powers similarly (indeed one may bound $\sum w(f(n))$ for any polynomial $f(.)$) using power residue symbols in place of the Jacobi symbol. However, it seems, for example, that one can prove nothing better than

$$\sum_{n^3\leqq x} 1 \ll x^{1/2}$$

in this way.

The form of the second term on the right of (1) is reminiscent of that occuring in the linear sieve (see Iwaniec [6]). Here we may give the coefficient of the inner sum explicitly; the term in question is merely

$$\sum_{p\neq q\in\mathscr{P}} \sum_n w(n)\left(\frac{n}{pq}\right).$$

However the averaging over p and q has not yet been used successfully to improve bounds for $S(\mathscr{A})$.

As an application of Theorem 1 we investigate the number of consecutive square-free numbers below x. (I should like to thank Dr. R. Hall for suggesting this problem to me.) We shall prove:

Theorem 2. *Let $E(n)=1$ if n is square-free and $E(n)=0$ otherwise. Then*

$$\sum_{n<x} E(n)E(n+1)=Cx+O(x^{7/11}(\log x)^7),$$

where

$$C=\prod_p (1-2p^{-2}).$$

Elementary methods yield an error term $O(x^{2/3+\varepsilon})$ only (see Carlitz [1] for example). Our improvement is rather small. However, even for the much simpler problem of estimating the difference

$$\sum_{n\leqq x} E(n)-\frac{6}{\pi^2}x,$$

the bound $O(x^{1/2})$ cannot be improved without invoking the prime number theorem. We have made no effort at economy with the $\log x$ factors occuring in Theorem 2.

2. Proof of Theorem 1

Consider the expression

$$\Sigma=\sum_n w(n)\left(\sum_{p\in\mathscr{P}}\left(\frac{n}{p}\right)\right)^2.$$

Each n is clearly counted with non-negative weight. Moreover if $n=m^2$, then

$$\sum_{p\in\mathscr{P}}\left(\frac{n}{p}\right)=\sum_{p\in\mathscr{P},\,p\nmid m} 1 \geqq P-\sum_{p\mid m} 1 \gg P,$$

since

$$\sum_{p|m} 1 \ll \frac{\log m}{\log\log m}.$$

Hence $\Sigma \gg P^2 S(\mathscr{A})$. However,

$$\Sigma = \sum_{p,q\in\mathscr{P}} \sum_n w(n)\left(\frac{n}{pq}\right) = \sum_{p\in\mathscr{P}} \sum_{n;\, p\nmid n} w(n) + \sum_{p\neq q\in\mathscr{P}} \sum_n w(n)\left(\frac{n}{pq}\right)$$
$$\leqq P\sum_n w(n) + \sum_{p\neq q\in\mathscr{P}} \left|\sum_n w(n)\left(\frac{n}{pq}\right)\right|,$$

and the theorem follows.

3. Theorem 2 – Preliminaries

Since

$$E(n) = \sum_{j^2|n} \mu(j),$$

we have

$$\sum_{n<x} E(n)E(n+1) = \sum_{j,k} \mu(j)\mu(k)N(x,j,k),$$

where

$$N(x,j,k) = \#\{n<x;\, j^2|n,\, k^2|n+1\}.$$

We observe that $N(x,j,k) = xj^{-2}k^{-2} + O(1)$ if $(j,k)=1$, and $N(x,j,k)=0$ otherwise. The terms with $jk \leqq y$ (where y will be specified later) therefore contribute

$$x \sum_{\substack{jk\leqq y\\ (j,k)=1}} \mu(j)\mu(k)(jk)^{-2} + O\Big(\sum_{jk\leqq y} 1\Big)$$
$$= x \sum_{(j,k)=1} \mu(jk)(jk)^{-2} + O(x \sum_{n>y} d(n)n^{-2}) + O\Big(\sum_{n\leqq y} d(n)\Big)$$
$$= Cx + O(xy^{-1}\log y) + O(y\log y).$$

(Here $d(n)$ is the divisor function.) The remaining values of j, k lie in $O((\log x)^2)$ ranges $J<j\leqq 2J$, $K<k\leqq 2K$, where

$$JK \gg y, \qquad J, K \ll x^{1/2}.$$

Hence there exist some such, J, K for which

$$\sum_{jk>y} \mu(j)\mu(k)N(x,j,k) \ll N(\log x)^2,$$

where

$$N = \#\{(j,k,u,v);\, J<j\leqq 2J,\, K<k\leqq 2K,\, j^2u+1=k^2v\leqq x\}.$$

We will choose $x^{1/2} \leqq y \leqq x$, whence

$$\sum_{n<x} E(n)E(n+1) = Cx + O(y\log x) + O(N(\log x)^2). \tag{2}$$

It remains to bound N. We first give an elementary auxilliary bound. We have

$$N \ll \sum_{K<k\leqq 2K} \sum_{u \leqq xJ^{-2}} \sum_{\substack{J<j\leqq 2J \\ j^2u \equiv -1 \pmod{k^2}}} 1 .$$

Since the above congruence condition has $\ll d(k)$ solutions $(\bmod k^2)$ the innermost sum is $\ll (1+JK^{-2})d(k)$. Thus

$$N \ll xJ^{-2}(1+JK^{-2}) \sum_k d(k) \ll \{xKJ^{-2}+x(JK)^{-1}\} \log x \tag{3}$$

Henceforth we shall assume $J \geqq K$, the alternative case being similar. Since $JK \gg y$, the bound (3) yields $N \ll xy^{-1/2} \log x$. On taking $y=x^{2/3}$ the estimate (2) would show that Theorem 2 is true with the weaker error term $O(x^{2/3}(\log x)^3)$. This is already better than the result of Carlitz [1], by an x^{ε} factor.

For our principal bound for N, we write N as $\sum N_u$ according to the value of u, and divide the range for u into intervals $U<u\leqq 2U$. We write

$$N(U) = \sum_{U<u\leqq 2U} N_u ,$$

so that

$$N \ll N(U) \log x \tag{4}$$

for some U,

$$U \ll xJ^{-2} . \tag{5}$$

By examining the available range for v we see that

$$N_u \leqq \#\{(j,k,v);\, k^2v-1=j^2u,\, K<k\leqq 2K,\, L\leqq v\leqq M\},$$

where

$$L=\mathrm{Max}(1, \tfrac{1}{4}J^2UK^{-2}), \qquad M=K^{-2}(1+8J^2U) .$$

We define $w(n)$ to be zero unless $u|n$, in which case

$$w(mu) = \#\{(k,v);\, u|k^2v-1,\, m=k^2v-1,\, K<k\leqq 2K,\, L\leqq v\leqq M\} .$$

Thus $N_u \leqq S(\mathscr{A})$. We are now in a position to apply Theorem 1. We shall take $\mathscr{P}$ to be the set of primes $p \nmid u$, $Q<p\leqq 2Q$. Here Q, which will be independent of u, lies in the range

$$(\log x)^2 \leqq Q \leqq x , \tag{6}$$

and will be chosen optimally later. We have $P \sim Q(\log Q)^{-1}$ so that $w(n)=0$ for $|n| \geqq e^P$ (and for $n=0$). Hence

$$N_u \ll Q^{-1}(\log x) \sum w(n) + P^{-2} \sum_{p\neq q} \left| \sum_{k,v} \left(\frac{u(k^2v-1)}{pq} \right) \right| . \tag{7}$$

In the final sum the conditions on k, v are

$$K<k\leqq 2K, \qquad L\leqq v\leqq M, \qquad u|k^2v-1 , \tag{8}$$

In this section we deal with the first term on the right in (7), leaving the second term for Sect. 4. The first term contributes to $N(U)$ a total

$$\ll Q^{-1}(\log x)\sum_{k,v}\sum_{u|k^2v-1}1\ll Q^{-1}(\log x)\sum_{k}\sum_{v}d(k^2v-1). \tag{9}$$

To bound the sum over v we use the following lemma, for which see Shiu [8], for example.

Lemma. *Let $\delta>0$ be given. Then*

$$\sum_{\substack{n\leqq x\\ n\equiv a(\bmod q)}} d(n)\ll\phi(q)q^{-2}x\log x,$$

uniformly for $q\leqq x^{1-\delta}$, $(q,a)=1$.

Taking $\delta=1/8$, and noting, by (5), that $J^2U\ll x$, we have

$$\sum_{v} d(k^2v-1)\ll K^{-2}x\log x$$

for $K\leqq x^{1/2-1/16}$, whence (9) is $O((QK)^{-1}x(\log x)^2)$. In case $x^{1/2-1/16}\leqq K\leqq x^{1/2}$ we may use the trival bound $d(k^2v-1)\ll x^{1/16}$, which shows that (9) is

$$\ll Q^{-1}(\log x)K(xK^{-2})x^{1/16}\ll x^{5/8},$$

by (6). Thus the total contribution to $N(U)$ arising from the first term on the right of (7) is

$$\ll x^{5/8}+(QK)^{-1}x(\log x)^2. \tag{10}$$

Let us now see what will be necessary to obtain some form of Theorem 2 with an exponent $<2/3$. We will apply (2) with y slightly less than $x^{2/3}$. Since $J\geqq K$ and $JK\geqq y$ the estimate (3) will then be satisfactory unless J and K are both close to $x^{1/3}$. In this last case the contribution (10) will be sufficiently small if Q is any positive power of x. There remains the second term of (7). Here there are many terms k, v and these are very well distributed $(\bmod pq)$; this produces the key saving.

4. Theorem 2 – Continuation of the Proof

We have now to estimate

$$\sum_{k,v}\left(\frac{k^2v-1}{pq}\right)=S,$$

say, subject to the conditions (8). We transform S as follows, (using the notation $e(x)=\exp(2\pi i x)$).

$$S=\sum_{\substack{\alpha,\beta=1\\ u|\alpha^2\beta-1}}^{upq}\left(\frac{\alpha^2\beta-1}{pq}\right)\left|\sum_{\substack{K<k\leqq 2K\\ k\equiv\alpha(\bmod upq)}}1\right|\left|\sum_{\substack{L\leqq v\leqq M\\ v\equiv\beta(\bmod upq)}}1\right|$$
$$=\sum_{\alpha,\beta}\left(\frac{\alpha^2\beta-1}{pq}\right)\left\{\frac{1}{upq}\sum_{\gamma=1}^{upq}\sum_{K<k\leqq 2K}e\left(\frac{\gamma(\alpha-k)}{upq}\right)\right\}$$
$$\cdot\left\{\frac{1}{upq}\sum_{\delta=1}^{upq}\sum_{L\leqq v\leqq M}e\left(\frac{\delta(\beta-v)}{upq}\right)\right\}$$
$$=(upq)^{-2}\sum_{\gamma,\delta=1}^{upq}S(u,pq;\gamma,\delta)\theta_\gamma\phi_\delta, \tag{11}$$

where

$$S(u,pq;\gamma,\delta)=\sum_{\substack{\alpha,\beta=1\\ u|\alpha^2\beta-1}}^{upq}\left(\frac{\alpha^2\beta-1}{pq}\right)e\left(\frac{\gamma\alpha+\delta\beta}{upq}\right), \tag{12}$$

$$\theta_\gamma=\sum_{K<k\leqq 2K}e\left(\frac{-\gamma k}{upq}\right)\ll\mathrm{Min}\left(K,\left\|\frac{\gamma}{upq}\right\|^{-1}\right), \tag{13}$$

$$\phi_\delta=\sum_{L\leqq v\leqq M}e\left(\frac{-\delta v}{upq}\right)\ll\mathrm{Min}\left(J^2UK^{-2},\left\|\frac{\delta}{upq}\right\|^{-1}\right). \tag{14}$$

Here $\|x\|$ denotes the distance from x to the nearest integer or integers.

We note that u, p and q are relatively prime in pairs. Moreover we suppose that $u=\prod r^f$ is the canonical decomposition of u into prime power factors. Then the sum (12) factorizes as

$$S_1(p;c,d)S_1(q;c,d)\prod S_2(r^f;c,d), \tag{15}$$

where

$$S_1(p;c,d)=\sum_{\alpha,\beta=1}^{p}\left(\frac{\alpha^2\beta-1}{p}\right)e\left(\frac{c\alpha+d\beta}{p}\right),$$
$$S_2(r^f;c,d)=\sum_{\substack{\alpha,\beta=1\\ r^f|\alpha^2\beta-1}}^{r^f}e\left(\frac{c\alpha+d\beta}{r^f}\right),$$

and c, d are integers such that

$$(c,upq)=(\gamma,upq)\,,\qquad (d,upq)=(\delta,upq)\,.$$

We shall see later that

$$S_1(p;c,d)\ll p\,, \tag{16}$$
$$|S_2(r;c,d)|\leqq 3r^{1/2}(r,c,d)^{1/2}\,, \tag{17}$$

and for $f\geqq 2$ we shall use the trivial bound

$$|S_2(r^f;c,d)|\leqq r^f\,. \tag{18}$$

(Note that, for S_2, each α corresponds to at most one β.) It follows that (15) is

$$\ll Q^2Ud_3(w)w^{-1/2}(w,c,d)^{1/2},$$

where w is the product of those prime factors $r|u$ for which $f=1$. Moreover $d_k(.)$ denotes the usual generalized divisor function. Note that it would be possible to improve upon the trivial bound (18), but fortunately this is unnecessary for our purposes.

From (11), (13) and (14) we now have

$$S\ll U^{-1}Q^{-2}d_3(w)w^{-1/2}\sum_{\gamma,\delta}|\theta_\gamma\phi_\delta|(w,\gamma,\delta)^{1/2}$$

and

$$\sum_{\gamma,\delta}|\theta_\gamma\phi_\delta|(w,\gamma,\delta)^{1/2}$$
$$\ll J^2UK^{-1}w^{1/2}+J^2U^2Q^2K^{-2}\sum_{1\leqq\gamma\leqq\frac{1}{2}Upq}\gamma^{-1}(w,\gamma)^{1/2}$$
$$+KUQ^2\sum_{1\leqq\delta\leqq\frac{1}{2}Upq}\delta^{-1}(w,\delta)^{1/2}+U^2Q^4\sum_{1\leqq\gamma,\delta\leqq\frac{1}{2}Upq}(\gamma\delta)^{-1}(w,\gamma,\delta)^{1/2}.$$

We observe here that, for example,

$$\sum_{1\leqq\gamma\leqq\frac{1}{2}Upq}\gamma^{-1}(w,\gamma)^{1/2}\leqq\sum_{d|w}d^{1/2}\sum_{d|\gamma}\gamma^{-1}\ll\sum_{d|w}d^{-1/2}\log x\ll d(w)\log x.$$

Moreover $d_3(w)d(w)\leqq d_6(w)$, whence

$$S\ll J^2K^{-1}Q^{-2}d_3(w)+\{J^2K^{-2}U+K+UQ^2\}w^{-1/2}d_6(w)(\log x)^2.$$

In order to compute the contribution of S to $N(U)$ we must sum over u. Firstly we have

$$\sum_{U<u\leqq 2U}d_3(w)\leqq\sum_{U<u\leqq 2U}d_3(u)\ll U(\log U)^2.$$

Moreover u decomposes uniquely as $u=ws$, where s is 'square-full' (that is, $p|s$ implies $p^2|s$) and the number of such $s\leqq z$ is $O(z^{1/2})$, (see Erdös and Szerkeres [2] for example). Hence

$$\sum_{U<u\leqq 2U}d_6(w)w^{-1/2}\leqq\sum_{w\leqq 2U}d_6(w)w^{-1/2}\sum_{U<ws\leqq 2U}1$$
$$\ll\sum_{w\leqq 2U}d_6(w)w^{-1/2}(U/w)^{1/2}$$
$$\ll U^{1/2}(\log U)^6.$$

We conclude that the total contribution to $N(U)$ arising from the second term on the right of (7) is

$$\ll\{J^2K^{-1}UQ^{-2}+J^2K^{-2}U^{3/2}+KU^{1/2}+U^{3/2}Q^2\}(\log x)^8$$
$$\ll\{xK^{-1}Q^{-2}+x^{3/2}J^{-1}K^{-2}+x^{1/2}J^{-1}K+x^{3/2}J^{-3}Q^2\}(\log x)^8,$$

by (5). Comparison with (10) yields

$$N(U)\ll x^{5/8}+\{Q^2x^{3/2}J^{-3}+Q^{-1}xK^{-1}+x^{3/2}J^{-1}K^{-2}+x^{1/2}J^{-1}K\}(\log x)^8.$$

We therefore take

$$Q = x^{-1/6} J K^{-1/3} + (\log x)^2;$$

since $J \leqq x^{1/2}$ the condition (6) is satisfied. Then, since $J \geqq K$, $JK \gg y$, we have

$$\begin{aligned} N(U) &\ll x^{5/8} + x^{3/2} J^{-3} (\log x)^{12} \\ &\quad + \{x^{7/6} J^{-1} K^{-2/3} + x^{3/2} J^{-1} K^{-2}\} (\log x)^8 \\ &\ll x^{5/8} + x^{3/2} y^{-3/2} (\log x)^{12} \\ &\quad + x^{7/6} y^{-5/6} (\log x)^8 + x^{3/2} J^{-1} K^{-2} (\log x)^8 . \end{aligned}$$

On comparing this with (2) and (4) we see that

$$\begin{aligned} \sum_{n<x} E(n)E(n+1) - Cx &\ll y \log x + x^{5/8} (\log x)^3 + x^{3/2} y^{-3/2} (\log x)^{15} \\ &\quad + x^{7/6} y^{-5/6} (\log x)^{11} + x^{3/2} J^{-1} K^{-2} (\log x)^{11} . \end{aligned}$$

We choose $y = x^{7/11} (\log x)^6$; this makes the first and fourth terms approximately equal. The expression above is then

$$\ll x^{7/11} (\log x)^7 + x^{3/2} J^{-1} K^{-2} (\log x)^{11} .$$

If we use the auxilliary bound (3) for N we see that

$$\begin{aligned} \sum_{n<x} E(n)E(n+1) - Cx &\ll y \log x + xKJ^{-2} (\log x)^3 + x(JK)^{-1} (\log x)^3 \\ &\ll x^{7/11} (\log x)^7 + xKJ^{-2} (\log x)^3 . \end{aligned}$$

Finally we observe that

$$\begin{aligned} \operatorname{Min}(x^{3/2} J^{-1} K^{-2}, xKJ^{-2}) &\leqq (x^{3/2} J^{-1} K^{-2})^{3/4} (xKJ^{-2})^{1/4} \\ &= x^{11/8} (JK)^{-5/4} \\ &\ll x^{11/8} y^{-5/4} \\ &\ll x^{51/88} , \end{aligned}$$

and that $\frac{51}{88} < \frac{7}{11}$. Thus Theorem 2 follows.

5. Exponential Sums

It remains to deal with the estimates (16) and (17). We first consider S_1. The terms with $\alpha = p$ contribute $O(p)$ trivially. If $\alpha \neq p$ let $\alpha\bar{\alpha} \equiv 1 (\bmod p)$. Then, on replacing β by $\beta + \bar{\alpha}^2$ we find

$$\begin{aligned} S_1(p; c, d) &= O(p) + \sum_{\alpha=1}^{p-1} \sum_{\beta=1}^{p} \left(\frac{\beta}{p}\right) e\left(\frac{c\alpha + d\bar{\alpha}^2 + d\beta}{p}\right) \\ &= O(p) + S_2(p; c, d) \sum_{\beta=1}^{p} \left(\frac{\beta}{p}\right) e\left(\frac{d\beta}{p}\right). \end{aligned}$$

Hence (16) is a consequence of (17) together with the well known bound for the quadratic Gauss sum, except possibly when $p|d$. However in the latter case

$$\sum_{\beta}\left(\frac{\beta}{p}\right)e\left(\frac{d\beta}{p}\right)=\sum_{\beta}\left(\frac{\beta}{p}\right)=0,$$

and (16) again follows.

Now consider (17). This is trivial if $r|c$, $r|d$. Suppose $r|c$, $r\nmid d$. Then

$$\begin{aligned}S_2(r;c,d)&=\sum_{\beta=1}^{r}e\left(\frac{d\beta}{r}\right)\sum_{\substack{\alpha=1\\ r|\alpha^2\beta-1}}^{r}1\\&=\sum_{\beta=1}^{r-1}e\left(\frac{d\beta}{r}\right)\left\{1+\left(\frac{\beta}{r}\right)\right\}\\&=\sum_{\beta=1}^{r}\left(\frac{\beta}{r}\right)e\left(\frac{d\beta}{r}\right)-1,\end{aligned}$$

and (17) follows. In case $r\nmid c$, $r|d$ we have

$$S_2(r;c,d)=\sum_{\alpha=1}^{r-1}e\left(\frac{c\alpha}{r}\right)=-1,$$

which again yields (17). Finally, if $r\nmid c$, $r\nmid d$, then the required estimate follows from Hayes [4; eq. (4)]. This last result is a consequence of Weil's "Riemann Hypothesis" for curves over finite fields.

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Reiceived August 24, 1981