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On a Spherical Integral Transformation and Sections of Star Bodies

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Abstract. Let K be a d-dimensional star body (with respect to the origin o). It is known that the $(d-1)$-dimensional volume of the intersections of K with the hyperplanes through o does not uniquely determine K. Uniqueness can only be achieved under additional assumptions, such as central symmetry. Here it is pointed out that if one uses, instead of intersections by hyperplanes, intersections by half-planes that contain o on the boundary, then, without any additional assumptions, the volume of these intersections determines K uniquely. This assertion, and more general results of this kind, together with stability estimates, are obtained from uniqueness results and estimates concerning a particular spherical integral transformation.

1. Introduction

Let $\mathbf{E}^d$ denote the euclidean d-dimensional space. It is always assumed that $d \geqslant 2$. The origin of $\mathbf{E}^d$ will be denoted by o, and the volume (Lebesgue measure) in $\mathbf{E}^d$ by v_d. We let κ_d denote the volume, and ω_d the surface area of a d-dimensional unit ball. A nonempty compact subset K of $\mathbf{E}^d$ is called a *star body* if it contains for every $p \in K$ the half-open line segment $[o,p)$ in its interior. S^{d-1} denotes the unit sphere in $\mathbf{E}^d$ (centered at o), and for any $u \in S^{d-1}$ we let $u^\perp$ denote the $(d-1)$-dimensional linear subspace of $\mathbf{E}^d$ that is orthogonal to u. It is well-known that there are star bodies K, L such that for all $u \in S^{d-1}$

$$v_{d-1}(K \cap u^\perp) = v_{d-1}(L \cap u^\perp) \tag{1}$$

but $K \neq L$. For example, this happens already for $d = 2$ if K is a circular disc and L a non-circular equichordal set of appropriate size. On the other hand, it also is well-known that for centrally symmetric star bodies (1) does imply $K = L$. For references regarding these matters see the books [2] or [4].

In the present article we consider intersections of star bodies with half-planes (i.e. half-hyperplanes) of the form

$$H(u,w) = \{x : x \in u^\perp,\, x \cdot w \geqslant 0\},$$

where $u \in S^{d-1}$, $w \in S^{d-1} \cap u^\perp$, and the dot indicates the inner product. The intersections $K \cap H(u,w)$ will be called the *half-sections* of K. As a special case of

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our results it will follow that, without any symmetry assumptions on the given bodies, it can be concluded that $K = L$ if for all such half-sections $v_{d-1}(K \cap H(u,w)) = v_{d-1}(L \cap H(u,w))$.

The primary subject of our work, however, is a certain spherical integral transformation, particularly its injectivity, stability, and geometric significance. Such transformations have been considered for $d = 3$ by BACKUS [1] in connection with a problem of geophysics. Backus pointed out that although a sufficiently smooth function on the unit sphere is not uniquely determined by its integrals over great circles (i.e., by its spherical Radon transform), it is uniquely determined by integrals over great semicircles. Injectivity results for the spherical Radon transformation and the corresponding transformation, where integrals are extended over hemispheres, have been known for some time and various geometric consequences have been deduced (see GARDNER [2] for an exposition and references regarding this matter). However, the geometric relevance of half-sections has apparently been overlooked.

In the following section we consider, for all $d \geqslant 2$, the transformation suggested by the work of Backus and prove our stability results. In Section 3 several geometric implications of the results obtained in Section 2 are presented.

2. The Integral Transformation $\mathscr{B}$

For any $u \in S^{d-1}$ let $S(u)$ denote the $(d-2)$-dimensional unit sphere $S^{d-1} \cap u^{\perp}$. Furthermore if $w \perp u$, or equivalently, $w \in S(u)$, define $S(u,w)$ as the $(d-2)$-dimensional hemisphere $S^{d-1} \cap H(u,w)$. Hence,

$$S(u,w) = \{z : z \in S^{d-1}, z \cdot u = 0, z \cdot w \geqslant 0\}.$$

If F is a continuous function on S^{d-1}, and if u and w are as just described, we define a function $F^{\mathscr{B}}(u,w)$ by

$$F^{\mathscr{B}}(u,w) = \int_{S(u,w)} F(z) d\sigma_u(z),$$

where $\sigma_u(z)$ denotes the surface area differential on $S(u)$ at z. Thus, $F^{\mathscr{B}}$ is a function on the set

$$\mathbf{B}^d = \{(u,w) : u, w \in S^{d-1}, w \perp u\}.$$

Instead of writing $F^{\mathscr{B}}$ it will sometimes be more convenient to write $\mathscr{B}F$, and to consider $\mathscr{B}$ as a transformation that maps the set of continuous functions on S^{d-1} into the set of functions on $\mathbf{B}^d$. If $d = 3$ the function $F^{\mathscr{B}}(u,w)$ is the integral of F over the great semicircle determined by u and w. If $d = 2$ we let $F^{\mathscr{B}}(u,w) = F(w)$.

The transformation $\mathscr{B}$ is closely related to two other spherical integral transformations, namely the spherical Radon transformation $\mathscr{R}$, defined by

$$F^{\mathscr{R}}(u) = \int_{S(u)} F(z) d\sigma_u(z),$$

and the 'hemispherical transformation' $\mathscr{T}$, defined by

$$F^{\mathscr{T}}(u) = \int_{S^+(u)} d\sigma,$$

where $S^+(u)$ denotes the $(d-1)$-dimensional hemisphere $\{z : z \in S^{d-1}, z \cdot u \geqslant 0\}$, and $d\sigma$ indicates the surface area differential on $S^+(u)$. The transformation $\mathscr{B}$ can be expressed in terms of $\mathscr{T}$ by

$$F^{\mathscr{B}}(u,w) = (F \wedge S(u))^{\mathscr{T}}(w), \tag{2}$$

where $F \wedge S(u)$ signifies the restriction of F to $S(u)$ and $\mathscr{T}$ refers to the hemispherical transformation on the $(d-2)$-dimensional sphere $S(u)$. We also note that evidently for all $(u,w) \in \mathbf{B}^d$

$$F^{\mathscr{B}}(u,w) + F^{\mathscr{B}}(u,-w) = F^{\mathscr{R}}(u). \tag{3}$$

The problem is now to estimate the deviation between two functions F and G on S^{d-1} in terms of the deviation between the functions $\mathscr{B}F$ and $\mathscr{B}G$. As deviation measure for the relevant functions we use the pertinent L_2-norms. The following notation will be used:

If Θ is a function on S^{d-1}, let

$$\|\Theta\|_{S^{d-1}} = \|\Theta(u)\|_{u \in S^{d-1}} = \left(\int_{S^{d-1}} \Theta(u)^2 d\sigma(u)\right)^{1/2};$$

if Φ is a function on $S(u)$, let

$$\|\Phi\|_{S(u)} = \|\Phi(w)\|_{w \in S(u)} = \left(\int_{S(u)} \Phi(w)^2 d\sigma_u(w)\right)^{1/2},$$

and if Ψ is a function on $\mathbf{B}^d$, let

$$\|\Psi\|_{\mathbf{B}^d} = \|\Psi(u,w)\|_{(u,w) \in \mathbf{B}^d} = \left(\int_{S^{d-1}} \int_{S(u)} \Psi(u,w)^2 d\sigma_u(w) d\sigma(u)\right)^{1/2}.$$

We obviously have

$$\|\Psi\|_{\mathbf{B}^d} = \| \, \|\Psi(u,w)\|_{w \in S(u)} \|_{u \in S^{d-1}}.$$

It is clear that estimates of the desired kind will depend on the smoothness of the functions F and G. As a measure of smoothness of a function Θ on S^{d-1} we use the gradient $\nabla_o \Theta$ with S^{d-1} as underlying space. It can be defined by

$$(\nabla_o \Theta)(u) = (\nabla \Theta(x/|x|))_{x=u},$$

where $u \in S^{d-1}$ and ∇ is the gradient operator $(\partial/\partial x_1, \cdots, \partial/\partial x_d)$. Similarly one defines the gradient for functions on $S(u)$. The notation $\|\nabla_o \Theta\|$ is used to indicate the L_2-norm of the euclidean norm of $\nabla_o \Theta$.

We now can prove the following theorem.

Theorem 1. *If F and G are two continuous functions on S^{d-1} and $\mathscr{B}F = \mathscr{B}G$, then $F = G$.*

Furthermore, let an $\epsilon \in [0,1]$ be given and assume that F and G are twice continuously differentiable and that C is a constant such that

$$\|\nabla_o F\|_{S^{d-1}} \leqslant C, \quad \|\nabla_o G\|_{S^{d-1}} \leqslant C, \tag{4}$$

and for all $u \in S^{d-1}$

$$\|\nabla_o(F \wedge S(u))\|_{S(u)} \leqslant C, \quad \|\nabla_o(G \wedge S(u))\|_{S(u)} \leqslant C. \tag{5}$$

Then the condition

$$\|\mathscr{B}F - \mathscr{B}G\|_{\mathbf{B}^d} \leqslant \epsilon \tag{6}$$

implies

$$\|F - G\|_{S^{d-1}} \leqslant \eta_d(C)\epsilon^{2/(d+1)}, \tag{7}$$

where $\eta_d(C)$ *depends on* C *and* d *only.*

It would be possible to explicitly determine a suitable $\eta_d(C)$. This will become evident from the proof, but will not be done here. The proof depends on the following lemma concerning the transformations $\mathscr{R}$ and $\mathscr{T}$. A proof of this lemma can be found [4, Propositions 3.4.11, 3.4.12, and Theorem 3.4.14]. (Regarding proofs of the first statement of this lemma see also the older work cited in [4]). If F is a function on S^{d-1} it will be convenient to define functions F_+ and F_- by

$$F_+(u) = \frac{1}{2}(F(u) + F(-u)), \quad F_-(u) = \frac{1}{2}(F(u) - F(-u)).$$

Lemma. *If* F *and* G *are two continuous functions on* S^{d-1}, *then the condition* $\mathscr{R}F = \mathscr{R}G$ *implies* $F_+ = G_+$, *and the condition* $\mathscr{T}F = \mathscr{T}G$ *implies* $F_- = G_-$.

Furthermore, if F *and* G *are twice continuously differentiable and* $\epsilon \geqslant 0$ *the following two statements are valid.*

(a) *If* $n \geqslant 3$ *and* $\|\mathscr{R}F - \mathscr{R}G\|_{S^{n-1}} \leqslant \epsilon$, *then*

$$\|F_+ - G_+\|_{S^{n-1}} \leqslant a_n(\|\nabla_o F\|^2_{S^{n-1}} + \|\nabla_o G\|^2_{S^{n-1}} + b_n\epsilon^2)^{(n-2)/2n}\epsilon^{2/n}.$$

(b) *If* $n \geqslant 2$ *and* $\|\mathscr{T}F - \mathscr{T}G\|_{S^{d-1}} \leqslant \epsilon$, *then*

$$\|F_- - G_-\|_{S^{n-1}} \leqslant c_n(\|\nabla_o F\|^2_{S^{n-1}} + \|\nabla_o G\|^2_{S^{n-1}})^{n/2(n+2)}\epsilon^{2/(n+2)}.$$

The coefficients a_n, b_n, *and* c_n *depend on* n *only (and can be explicitly determined).*

Proof of Theorem 1. The first part of Theorem 1 follows immediately from (2), (3), and the first statement in the above lemma.

Let $u \in S^{d-1}$. If part (b) of the preceding lemma is applied in the case $n = d - 1$ with $S(u)$ serving as S^{n-1}, we obtain, observing also (5),

$$\|F_- \wedge S(u) - G_- \wedge S(u)\|_{S(u)} \leqslant c_{d-1}(2C^2)^{(d-1)/2(d+1)}\epsilon^{2/(d+1)}. \tag{8}$$

Since for any continuous function Θ on S^{d-1} we have

$$\int_{S^{d-1}} \int_{S(u)} \Theta(w) d\sigma_u(w) d\sigma(u) = \omega_{d-1} \int_{S^{d-1}} \Theta(v) d\sigma(v)$$

(see, for example, [4, Lemma 1.3.3]) it follows from (8) that

$$\begin{aligned}\|F_- - G_-\|_{S^{d-1}}^2 &= \int_{S^{d-1}} (F_-(v) - G_-(v))^2 d\sigma(v)\\ &= \frac{1}{\omega_{d-1}} \int_{S^{d-1}} \int_{S(u)} (F_-(w) - G_-(w))^2 d\sigma_u(w) d\sigma(u)\\ &= \frac{1}{\omega_{d-1}} \int_{S^{d-1}} \|F_- \wedge S(u) - G_- \wedge S(u)\|_{S(u)}^2 d\sigma(u)\\ &\leqslant c_{d-1}^2 \frac{\omega_d}{\omega_{d-1}} (2C^2)^{(d-1)/(d+1)} \epsilon^{4/(d+1)}.\end{aligned}$$

Hence, there is a $g_d(C)$, depending on d and C only, such that

$$\|F_- - G_-\|_{S^{d-1}} \leqslant g_d(C) \epsilon^{2/(d+1)}. \tag{9}$$

To estimate $\|F_+ - G_+\|$ we note that (3) shows that for any choice of $w_u \in S(u)$

$$\|\mathscr{R}F - \mathscr{R}G\|_{S^{d-1}} = \|F^{\mathscr{B}}(u, w_u) + F^{\mathscr{B}}(u, -w_u) - G^{\mathscr{B}}(u, w_u) - G^{\mathscr{B}}(u, -w_u)\|_{u \in S^{d-1}}.$$

Noting that on the right hand side the expression within the norm sign does not depend on w_u and using (6) we can infer that

$$\begin{aligned}&\|\mathscr{R}F - \mathscr{R}G\|_{S^{d-1}} =\\ &= \frac{1}{\sqrt{\omega_{d-1}}} \| \, \|F^{\mathscr{B}}(u, w) - G^{\mathscr{B}}(u, w) + F^{\mathscr{B}}(u, -w) - G^{\mathscr{B}}(u, -w)\|_{w \in S(u)} \|_{u \in S^{d-1}}\\ &\leqslant \frac{1}{\sqrt{\omega_{d-1}}} (\| \, \|F^{\mathscr{B}}(u, w) - G^{\mathscr{B}}(u, w)\|_{w \in S(u)} \|_{u \in S^{d-1}}\\ &\qquad + \| \, \|F^{\mathscr{B}}(u, -w) - G^{\mathscr{B}}(u, -w)\|_{w \in S(u)} \|_{u \in S^{d-1}})\\ &\leqslant \frac{2\epsilon}{\sqrt{\omega_{d-1}}}.\end{aligned}$$

If this is combined with (4) and part (a) of the above lemma, letting $n = d$, we obtain

$$\|F_+ - G_+\|_{S^{d-1}} \leqslant a_n \left(2C^2 + b_n \frac{4\epsilon^2}{\omega_{d-1}}\right)^{(d-2)/2d} \left(\frac{2\epsilon}{\sqrt{\omega_{d-1}}}\right)^{2/d}.$$

Since $\epsilon \in [0, 1]$ this shows that

$$\|F_+ - G_+\|_{S^{d-1}} \leqslant h_d(C) \epsilon^{2/d} \leqslant h_d(C) \epsilon^{2/(d+1)}, \tag{10}$$

where $h_d(C)$ depends on d and C only. Combining (9) and (10) we find

$$\|F - G\|_{S^{d-1}} \leqslant \|F_+ - G_+\|_{S^{d-1}} + \|F_- - G_-\|_{S^{d-1}} \leqslant (g_d(C) + h_d(C)) \epsilon^{2/d+1},$$

which obviously implies (7).

3. Geometric Consequences

If M is a star body in $\mathbf{E}^d$ let ρ_M denote the radial function of M. This means that $\rho_M(u)$ signifies for any $u \in S^{d-1}$ the length of the line segment $M \cap \{\tau u : \tau \geqslant 0\}$. Our geometric applications concern the *radial power integrals* I_λ^{d-1} of the half-sections $M \cap H(u, w)$. If λ is a real number and $(u, w) \in \mathbf{B}^d$, then these functionals are defined by

$$I_\lambda^{d-1}(M \cap H(u, w)) = \int_{S(u,w)} r_M^\lambda(u) d\sigma_u(z).$$

In particular, if $\lambda = d - 1$, then

$$\frac{1}{d-1} I_{d-1}^{d-1}(M \cap H(u, w)) = v_{d-1}(M \cap H(u, w)),$$

where v_{d-1} denotes again the volume in $\mathbf{E}^{d-1}$. In the case when $\lambda = 1$ then $\frac{2}{\omega_{d-1}} I_1^{d-1}(M \cap H(u, w))$ is the *average radial length* of $M \cap H(u, w)$.

Theorem 1, together with results proved in [3] and [4], can now be used to obtain the following statement. Here, convex bodies in $\mathbf{E}^d$ are defined as nonempty compact convex subsets of $\mathbf{E}^d$. The closed unit ball in $\mathbf{E}^d$ centered at o will be denoted by Q^d.

Theorem 2. *If K and L are two star bodies in $\mathbf{E}^d$ and if for some $\lambda \neq 0$ and all $(u, w) \in \mathbf{B}^d$*

$$I_\lambda^{d-1}(K \cap H(u, w)) = I_\lambda^{d-1}(L \cap H(u, w)),$$

then

$$K = L.$$

Furthermore, let K and L be two convex bodies in $\mathbf{E}^d$, and assume that there are numbers r and R such that $0 < r < R$, $rQ^d \subset K \subset RQ^d$, and $rQ^d \subset L \subset RQ^d$. If for some $\epsilon \in [0, 1]$ and $\lambda \neq 0$

$$\|I_\lambda^{d-1}(K \cap H(u, w)) - I_\lambda^{d-1}(L \cap H(u, w))\|_{\mathbf{B}^d} \leqslant \epsilon,$$

then

$$\|\rho_K - \rho_L\|_{S^{d-1}} \leqslant \beta_d(\lambda, R, r)\epsilon^{2/(d+1)} \tag{11}$$

and

$$\delta(K, L) \leqslant \gamma_d(\lambda, R, r)\epsilon^{4/(d+1)^2}, \tag{12}$$

where δ denotes the Hausdorff distance, and both $\beta_d(\lambda, R, r)$ and $\gamma_d(\lambda, R, r)$ depend solely on d, λ, R and r.

Proof. The first part of this theorem is obtained from the first part of Theorem 1. Indeed, letting $F(u) = \rho_K^\lambda(u)$, $G(u) = \rho_K^\lambda(u)$ we have

$$F^{\mathscr{B}}(u, w) = I_\lambda^{d-1}(K \cap H(u, w)), \quad G^{\mathscr{B}}(u, w) = I_\lambda^{d-1}(L \cap H(u, w)), \tag{13}$$

and it follows from Theorem 1 that $\rho_K^\lambda = \rho_L^\lambda$. Since $\lambda \neq 0$ this implies $\rho_K = \rho_L$ and therefore $K = L$. Concerning the second part we apply the second part of

Theorem 1 again with $F = \rho_K$ and $G = \rho_L$ and assume that these functions are twice continuously differentiable. Observing (13) we find that under the assumptions (4) and (5)

$$\|\rho_K^\lambda - \rho_L^\lambda\| \leqslant \eta_d(C)\epsilon^{2/(d+1)}. \tag{14}$$

It is shown in [3, formula (52)] and in [4, formula (5.6.9)] that for any convex body M in $\mathbf{E}^n$ with $rQ^n \subset M \subset RQ^n$ and twice continuously differentiable radial function ρ_M we have

$$\|\nabla_o \rho_M^\lambda\| \leqslant \sqrt{(n-1)\omega_d}\frac{|\lambda|}{r}\max\{r^{\lambda+1}, R^{\lambda+1}\}.$$

If this inequality is applied to the radial functions of K and L it follows that there is a $k_d(\lambda, r, R)$ such that

$$\|\nabla_o \rho_K^\lambda\|_{S^{d-1}} \leqslant k_d(\lambda, r, R), \quad \|\nabla_o \rho_L^\lambda\|_{S^{d-1}} \leqslant k_d(\lambda, r, R).$$

Also, it is obvious that for any $u \in S^{d-1}$ we have $(rQ^d) \cap u^\perp \subset K \cap u^\perp \subset$ $\subset (RQ^d) \cap u^\perp$ and $(rQ^d) \cap u^\perp \subset L \cap u^\perp \subset (RQ^d) \cap u^\perp$. Thus, letting $n = d - 1$, and $M = K \cap u^\perp$ or $M = L \cap u^\perp$ we find

$$\|\nabla_o(\rho_K^\lambda \wedge S(u))\|_{S(u)} \leqslant k_{d-1}(\lambda, r, R), \quad \|\nabla_o(\rho_L^\lambda \wedge S(u))\|_{S(u)} \leqslant k_{d-1}(\lambda, r, R).$$

Hence, (4) and (5) are satisfied with $C = \max\{k_{d-1}(\lambda, r, R), k_d(\lambda, r, R)\}$, and it follows from (14) that $\|\rho_K^\lambda - \rho_L^\lambda\| \leqslant p_d(\lambda, r, R)\epsilon^{2/(d+1)}$. Since it is easy to show (see [3, Lemma 3], or [4, Lemma 2.3.2]) that

$$\|\rho_K - \rho_L\|_{S^{d-1}} \leqslant \frac{1}{|\lambda|}\max\{r^{1-\lambda}, R^{1-\lambda}\}\|\rho_K^\lambda - \rho_L^\lambda\|_{S^{d-1}}$$

(11) follows. Finally, (12) is a consequence of (11) since it can been shown (see the preceding references) that

$$\delta(K, L) \leqslant \mu_d R^2 r^{-(d+3)/(d+1)}\|\rho_K - \rho_L\|^{2/(d+1)},$$

with μ_d depending on d only.

If the differentiability assumptions on ρ_K andρ_L are not satisfied one can use well known approximation theorems to deduce the same inequalities in full generality (see, for example, the proof of Theorem 3 in [3], or of Theorem 5.6.3 in [4]).

In the special cases $\lambda = d - 1$ and $\lambda = 1$ one immediately obtains from the first part of this theorem the following result.

Corollary 1. *Let K and L be two star bodies in $\mathbf{E}^d$. If for all $(u, w) \in \mathbf{B}^d$*

$$v_{d-1}(K \cap H(u, w)) = v_{d-1}(L \cap H(u, w)),$$

then, $K = L$. The same conclusion is valid if the respective average radial lengths of the corresponding half-sections of K and L are equal.

Letting L be a ball in $\mathbf{E}^d$ one obtains the following result.

Corollary 2. *Let K be a star body in* $\mathbf{E}^d$. *If for some* $c > 0$, $\lambda \neq 0$, *and all* $(u, w) \in \mathbf{B}^d$

$$I_\lambda^{d-1}(K \cap H(u, w)) = c,$$

then K is ball. In particular, K is a ball if the volume or the average radial length of all half-sections is constant.

If obviously would be possible to use the second part of Theorem 2 to prove stability versions of these corollaries in the case when the star bodies are convex.

We finally note that the characterization of balls in terms of the volume of the half-sections can also be deduced from two well-known geometric results (cf. [4, Section 5.6]) as follows: If for fixed u the half-sections $K \cap H(u, w)$ have constant $(d-1)$-dimensional volume, then $K \cap u^\perp$, and therefore K itself, are known to be centrally symmetric. Moreover, since the half-sections $K \cap H(u, v)$ have constant volume, the same is true for the total sections $K \cap u^\perp$. But it is also known that symmetric bodies with this property are balls.

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