Monatshefte für Mathematik 80, 325—330 (1975)

Completeness and Fixed-Points

By

P. V. Subrahmanyam, Madras

(Received 17 January 1974)

Abstract

In this note the converses of recent fixed-point theorems due to KANNAN and CHATTERJEA are obtained. An example is constructed to show that a metric space having the fixed-point property for homeomorphisms need not be metrically topologically complete. An example of CONNELL is formulated in a more general perspective.

§ 1. Introduction

HU [5] showed that a metric space is complete if and only if any contraction on closed subsets thereof has a fixed-point. In this context, it is easily seen from an example due to CONNELL ([3], p. 978, Example 3) that it is not however possible to conclude that a metric space is complete if any contraction on it has a fixed-point. In fact, the fixed-point property for even continuous maps does not insure the completeness of the metric space. Besides HU's [5], there are results converse to the contraction mapping principle. But mostly these assert the existence of complete metric topologies such that a class of mappings of an abstract set into itself with fixed-points consists of contractions (see e. g. [1]). Theorems 1 and 2 of Section 2 of this note on the other hand have in their conclusions the completeness of the metric space under a hypothesis that each member of a class of mappings with constraints on transformation of distance has fixed or periodic points. Incidentally it subsumes converses of recent results ([2], [6]) in the form of fixed-point theorems. Theorem 1 is independent of HU's result, mentioned above.

The author thanks the referee for drawing his attention to the note of HU [5].

The connection between metric-topological completeness of a metric space (i. e. the existence of a metric whose topology is the same as that of the given metric but under which the space is complete) and fixed-point property does not seem to have been studied. We observe in Section 3, that fixed-point property for homeomorphisms neither implies nor is implied by topological completeness.

Theorem 3, in Section 4, is an attempt to place CONNELL's example mentioned at the outset in a general perspective.

§ 2. Completeness and Fixed-Points

Theorem 1. *A metric space (X,d) in which every mapping T of X into itself, satisfying the conditions:*

(i) $d(T(x), T(y)) \leqslant \lambda \max\{d(x, T(x)), d(y, T(y))\}$, $x, y \in X$, *for a fixed* $\lambda > 0$;

(ii) $T(X)$ *is countable;*

has a fixed-point, is complete.

Proof: If possible, let $A = \{x_n\}$ be a non-convergent Cauchy sequence where x_n are distinct. For any $x \notin A$, $d(x,A) > 0$. $\{x_n\}$ being Cauchy, there exists a least positive integer $N(x)$ such that

$$d(x_m, x_n) < \lambda d(x, A) \leqslant \lambda d(x, x_l),\ l = 1, 2, \ldots;\ \forall\, m, n \geqslant N(x).$$

In particular

$$d(x_m, x_{N(x)}) < \lambda d(x, x_l),\ l = 1, 2, \ldots;\ \forall\, m \geqslant N(x). \tag{1}$$

By a similar reasoning there exists a least positive integer $n' = n'(n) > n$ such that

$$d(x_m, x_{n'}) < \lambda d(x_n, x_{n'}), \quad m \geqslant n'. \tag{2}$$

If $T: X \to X$ is defined as

$$T(x) = \begin{cases} x_{N(x)}, & x \notin A, \\ x_{n'}, & x (= x_n) \in A, \end{cases}$$

it is clear that T has no fixed-points since $x_{n'} \neq x_n$, $n = 1, 2, \ldots$. On the other hand, T satisfies conditions (i), (ii) of the theorem. (ii) being obvious from the definition of T, (i) is verified by writing $T(x) = x_n$, $T(y) = x_m$, and noting that

$$d(x_m, x_n) < \begin{cases} \lambda d(y, A - \{y\}), & n \geqslant m, \\ \lambda d(x, A - \{x\}), & n < m, \end{cases}$$

as is easily seen by using (1) and (2). This contradicts the hypothesis of the theorem and thereby establishes the same.

Remarks: KANNAN [6] and CHATTERJEA [2] have respectively shown that if (X,d) is a complete metric space and $T: X \to X$ is a mapping satisfying the condition

either (i′) $d(T(x), T(y)) \leqslant \lambda [d(x, T(x)) + d(y, T(y))],\ x, y \in X$ for a fixed λ, $0 < \lambda < \frac{1}{2}$,

or (i″) $d(T(x), T(y)) \leqslant \lambda [d(x, T(y)) + d(y, T(x))],\ x, y \in X$ for a fixed λ, $0 < \lambda < \frac{1}{2}$,

then T has a fixed-point. (i′) being weaker than (i) in Theorem 1, the class of mappings satisfying (i), (ii) can be replaced by the class of mappings satisfying (i′), (ii). In fact λ in (i′) can be any positive number and not necessarily less than $\frac{1}{2}$. From the proof of the theorem it is clear that (i) can be replaced, in the first instance, by

$$d(T(x), T(y)) \leqslant \lambda \max\{d(x, T(y)), d(y, T(x))\},\quad x, y \in X, \text{ for a fixed } \lambda > 0,$$

and thereafter by (i″) which is weaker than this condition. To summarize, the converses of the results of KANNAN [6] and CHATTERJEA [2] hold even in stronger forms.

Finally, once it is noted that the mapping T constructed in the proof of theorem 1 has no periodic point (i. e. for no $x \in X$ does there exist a positive integer K such that $T^K(x) = x$) and that (i) can be replaced by a more stringent condition, the truth of the following theorem is clear.

Theorem 2. *If (X, d) is a metric space in which every mapping satisfying*

either (ia) $d(T(x), T(y)) \leqslant \lambda \max\{\inf_K d(x, T^K(x)), \inf_K d(y, T^K(y))\},\ x, y \in X,$

or (ib) $d(T(x), T(y)) \leqslant \lambda \max\{\inf_K d(x, T^K(y)), \inf_K d(y, T^K(x))\},\ x, y \in X,$

for a fixed $\lambda > 0$, together with (ii) *of Theorem* 1 *has a periodic point, then the space is complete.*

§ 3. Topological Completeness

We point out in this section that a *metric space need not be metrically topologically complete even if every homeomorphism has a fixed-point.* We begin with a general observation: if A and B are two separated subsets of a topological space such that A is a non-trivial connected set having fixed-point property for homeomorphisms and B is a totally disconnected set then $A \cup B$ in the relative topology has fixed-point property for homeomorphisms. For, let f be any homeomorphism of $A \cup B$ into itself; since $f(A) \cup f(B) \subseteq A \cup B$ where $f(A)$ is connected and A, B are separated, $f(A)$ has to be contained in A or B. But B is totally disconnected so that $f(A) \subseteq A$. Thus by assumption f has a fixed-point.

The following example proves the assertion made in the beginning of this section. Let now A be the interval $[0,1]$ and B be the set $\{x \mid x \text{ rational and } \geqslant 2\} \cup \{x \mid x \text{ irrational and } < -1\}$. Then $A \cup B$ in the usual topology cannot be G_δ in its completion. For, otherwise $\{x \mid x \text{ rational and } \geqslant 2\}$ would be G_δ in $[2, \infty)$ contradicting that $[2, \infty)$ is of the second category in itself. Since metrically topologically complete spaces are precisely absolutely G_δ spaces (see [7], p. 207, K((a) to (c)), $A \cup B$ cannot be metrically topologically complete. (Incidentally $A \cup B$ is not an absolutely F_σ space too, since $\{x \mid x \text{ irrational}, x < -1\}$ is not F_σ in $(-\infty, 1]$.) That completeness does not insure the fixed-point property for homeomorphisms is readily seen by considering the map $x \to x+2$ on the real line.

§ 4. A Fixed-Point Theorem

Theorem 3. *Let X be a topological space having fixed-point property for continuous functions and Y be a densely ordered, order complete, chain with its order topology which is bounded below by y_0. Let $X_1 \subseteq X$, $M \subseteq X \times Y$ and the map a: $X \to Y$ be such that*

(i) $M = \bigcup_{x \in X_1} \{x\} \times [y_0, a(x)] \cup (X - X_1) \times \{y_0\}$

(ii) $\{x\} \times (y_0, a(x)]$ *is open in the topology of M relative to that of $X \times Y$.*

Then M has fixed-point property for continuous maps.

Remarks: (i) Theorem 3 includes CONNELL's example mentioned earlier in the assertion that the (metrically incomplete but metrically topologically complete) space of his example has the fixed-point property for continuous maps. (The space considered by CONNELL is G_δ in its completion and hence by a theorem of HAUSDORFF [4] is topologically complete.) In fact to obtain CONNELL's example we choose $X = Y = [0,1] \subseteq R$, $X_1 = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $a(x) = 1$ for each $x \in X_1$.

(ii) From [7], p. 58, I(d) it follows that Y is precisely a chain which is connected in the order topology.

(iii) Y can be different structurally from a real interval (bounded below), for example the product of $[0,1]$ with itself, ordered lexicographically ([7], p. 164, J).

Proof: If $f: M \to M$ is continuous, let f_1 be its restriction to $X \times \{y_0\}$. Since X is homeomorphic to $X \times \{y_0\}$, if $p_1: \langle x,y\rangle \to \langle x,y_0\rangle$, $p_1 \circ f_1$ has a fixed-point $\langle x_0,y_0\rangle$. From the definition of p_1 and f_1 it follows that $f\langle x_0,y_0\rangle = \langle x_0,z_0\rangle$. If $x_0 \in X - X_1$, then by the definition of M, $z_0 = y_0$ so that $\langle x_0,z_0\rangle$ is a fixed-point of f. If $x_0 \in X_1$ and $z_0 = y_0$ again f has a fixed-point.

If possible suppose $z_0 \neq y_0$. Then $z_0 > y_0$. Let S be the set $\{y \in Y \mid f\langle x_0,y\rangle = \langle x_0,z\rangle, z > y\}$. Since $x_0 \in X_1$, $\{x_0\} \times [y_0, a(x_0)]$ is a subset of M. So S is a bounded non-void subset of Y. As Y is order-complete, $\sup S = s$ for some $s \in [y_0, a(x_0)]$.

The set S with the order $\leqslant$ is a net and it converges to s, for any neighbourhood of s contains some interval (a,b) with $a < s < b$. If no x of S lies in (a,b) then each element of S is less than a or greater than b. The latter possibility is ruled out as $x \leqslant s$, for each $x \in S$. The former possibility, too is ruled out, as $\sup S = s$, so that there exists at least one $s_1 \in S$, $a < s_1 \leqslant s$ for any $a < s$, $a \in S$.

So the net $(f\langle x_0,y\rangle)_{y\in S}$ converges to $f\langle x_0,s\rangle$, as f is continuous. From the definition of S it follows that $f\langle x_0,y\rangle = \langle x_0,z\rangle$, $z \in Y$, for each $y \in S$. So $f\langle x_0,s\rangle = \langle x_0,t\rangle$ for some $t \in Y$, by the continuity of f and the X-projection.

If $t < s$, then as the ordering is dense there exists t' such that $t < t' < s$. Because the net S converges to s, S lies eventually in $(t', a(x_0)]$. Further $(f\langle x_0,y\rangle)_{y\in S}$ lies eventually in the interval $\{x_0\} \times (y_0,t')$. (This follows from the continuity of f and the fact that $\{x\} \times (a,b)$ is open in M, for any $a,b \in [y_0, a(x)]$.) Thus $f\langle x_0,y\rangle = \langle x_0,y'\rangle$ with $y_0 < y' < t'$. But this contradicts that $y \in S$! So $t \geqslant s$.

If $t > s$ then the interval $T = (s,t_1)$ is a net with the order α defined as $x \alpha y$ if and only if $y \leqslant x$ where $s < t_1 < t$. Clearly as the ordering is dense, T converges to s. Hence $(\langle x_0,y\rangle)_{y\in T}$ converges to $\langle x_0,s\rangle$ and $(f\langle x_0,y\rangle)_{y\in T}$ converges to $f\langle x_0,s\rangle$, f being continuous.

Now $\{x_0\} \times (y_0,t_1)$ is a neighbourhood of the point $\langle x_0,s\rangle$ while $\{x_0\} \times (t_1, a(x_0))$ is a neighbourhood of $\langle x_0,t\rangle$. Since $(f\langle x_0,y\rangle)_{y\in T}$

converges to $\langle x_0, t\rangle$ this net lies eventually in $\{x_0\}\times(t_1, a(x_0))$. So $f\langle x_0, y\rangle = \langle x_0, w(y)\rangle$ with $w(y)\in(t_1, a(x_0))$, after some stage for $y\in T$. But $s<y<t_1$ and $w(y)>t_1$. This means that $y(>s=\sup S)\in S$, a contradiction. Hence $t=s$ and $\langle x_0, s\rangle$ is a fixed point of f.

The proof is complete.

References

[1] BESSAGA, C.: On the converse of Banach fixed-point principle. Colloq. Math. **7**, 41—43 (1959).

[2] CHATTERJEA, S. K.: Some theorems on fixed-points. Research Report No. 2, Centre of Advanced Study in Appl. Math., University of Calcutta. 1971.

[3] CONNELL, E. H.: Properties of fixed-point spaces. Proc. Amer. Math. Soc. **10**, 974—979 (1959).

[4] HAUSDORFF, F.: Die Mengen G_δ in vollständigen Räumen. Fund. Math. **6**, 146—148 (1924).

[5] HU, T. K.: On a fixed-point theorem for metric spaces. Amer. Math. Monthly **74**, 436—437 (1967).

[6] KANNAN, R.: Some results on fixed-points. Bull. Calcutta Math. Soc. **60**, 71—76 (1968).

[7] KELLEY, J. L.: General Topology. New York: Van Nostrand. 1955.

P. V. SUBRAHMANYAM
Department of Mathematics
Indian Institute of Technology
Madras 600036, India