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On the Degree of the Discriminant Locus of a Smooth Sectional Surface of a $(n+2)$-fold with Nonnegative Kodaira Dimension

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Abstract. Let X be a smooth sectional surface of an $(n+2)$-fold with nonnegative Kodaira dimension. In this paper we improve Lanteri and Sommese estimates of the degree of the discriminant locus of X when $n \geqslant 2$.

1. Let L_S be a very ample line bundle on a smooth, projective surface S. In this paper we investigate the problem of finding an upper and lower bound for the degree δ of the discriminant locus of singular elements of $|L_S|$. In [6] A. Lanteri has proved that if S is of general type then

$$\delta > 3d + 17 \tag{1.1}$$

where $d = L_S \cdot L_S$.

A. J. Sommese in [11] has shown that

$$\delta < 48\chi(\mathcal{O}_S) \tag{1.2}$$

if S is a smooth element of $|L|$, where L is a very ample line bundle on a threefold X with nonnegative Kodaira dimension. Furthermore, Sommese has found that S is of general type and that there is an ample line bundle $\hat{L}$ on an algebraic manifold $\hat{X}$ such that S is the blow up $\pi: X \to \hat{X}$ of $\hat{X}$ at a finite set F of points and $\pi_S: S \to \hat{S}$ is the map of S onto its minimal model $\hat{S} \in |\hat{L}|$.

In [2] the author improved (1.1) and (1.2) in the case in which S satisfies the same hypothesis as in [11]. Namely

$$6d \leqslant \delta \leqslant 36\chi(\mathcal{O}_S) \quad \text{if} \quad K_{\hat{X}} \text{ is nef} \tag{1.3}$$

$$16/3\,d + 6 \leqslant \delta \leqslant 48\chi(\mathcal{O}_S) - 10 \quad \text{if} \quad K_{\hat{X}} \text{ is not nef.} \tag{1.4}$$

Moreover (1.3) is sharp.

In this paper we generalize (1.3) and (1.4) in the case S is the intersection of n smooth transverse elements $A_1, \ldots, A_n$ of $|L|$, where L is a very ample line bundle of an $(n+2)$-dimensional manifold X with nonnegative Kodaira dimension. In particular we prove that

$$\delta \leqslant \begin{cases} \frac{1}{3}(n+3)^2 d + \frac{n+10}{3} & \text{if } K_{\hat{X}} \text{ is not nef} \\ \frac{(n+2)(n+3)}{2} d & \text{if } K_{\hat{X}} \text{ is nef.} \end{cases} \tag{1.5}$$

If $n = 2$ then

$$\delta \leqslant \begin{cases} \frac{75}{4} \chi(\mathcal{O}_S) - \frac{9}{4} & \text{if } K_{\hat{X}} \text{ is not nef} \\ \frac{120}{7} \chi(\mathcal{O}_S) & \text{if } K_{\hat{X}} \text{ is nef.} \end{cases} \tag{1.6}$$

If $n \geqslant 3$ then

$$\delta \leqslant 6 \frac{(n+3)^2}{(n+1)^2 + 2} \chi(\mathcal{O}_S) . \tag{1.7}$$

We study also the case in which the above inequalities are sharp. The case $n = 1$ has been studied in [2].

2. Let X be an $(n+2)$-dimensional, connected, projective manifold with nonnegative Kodaira dimension, i.e. $\Gamma(K_X^m) \neq 0$ for some $m > 0$. Let $A_1, \ldots, A_n$ be n smooth transverse elements of $|L|$, where L is a very ample line bundle on X. Let S be the intersection of all the A_j. We denote by L_S and $K_X|_S$ the restrictions to S of L and K_X respectively. Then $K_S = K_X|_S \otimes L_S^n = (K_X \otimes L^n) \cdot L$. We set $d = L_S \cdot L_S$, $\Delta = K_S \cdot L_S$ and $c_1^2 = K_S \cdot K_S$. Then we have (see [11], § 1, p. 27])

i) S is of general type

ii) There exists an ample line bundle $\hat{L}$ on an algebraic manifold $\hat{X}$ such that

a) X is the blow up $\pi: X \to \hat{X}$ of $\hat{X}$ at a finite set F of points

b) $\pi|_S: S \to \hat{S}$ is the map of S onto its minimal model $\hat{S}$.

iii) If we let $\hat{d} = \hat{L}_{\hat{S}} \cdot \hat{L}_{\hat{S}}$, $\hat{\Delta} = K_{\hat{S}} \cdot \hat{L}_{\hat{S}}$, $\hat{c}_1^2 = K_{\hat{S}} \cdot K_{\hat{S}}$ then

$$n^2 \hat{d} \leqslant n \hat{\Delta} \leqslant \hat{c}_1^2 \tag{2.1}$$

and equality occurs if and only if $K_{\hat{X}}^m$ is trivial for some $m \geqslant 1$.

Let $c_i(X)$ be the i-th Chern class of X. We put $c_i = c_i(S)$ and $\hat{c}_i = c_i(\hat{S})$, $i = 1, 2$. Then we have:

$$\begin{aligned} c_1^2(X) \cdot \underbrace{L \cdot \ldots \cdot L}_{n\text{-times}} &= c_1^2 = 2n\Delta + n^2 d \\ c_1(X) \cdot \underbrace{L \cdot \ldots \cdot L}_{(n+1)\text{-times}} &= -\Delta + nd \\ c_2(X) \cdot \underbrace{L \cdot \ldots \cdot L}_{n\text{-times}} &= c_2 - n\Delta + \frac{n(n-1)}{2} d. \end{aligned} \tag{2.2}$$

Assume that $K_{\hat{X}}$ is nef. Since we have (see [9])

$$(3 c_2(\hat{X}) - c_1^2(\hat{X})) \cdot \underbrace{\hat{L} \cdot \ldots \cdot \hat{L}}_{n\text{-times}} \geqslant 0,$$

(2.2) implies that

$$3 \hat{c}_2 \geqslant \hat{c}_1^2 + n \hat{\Delta} + \frac{(3-n)n}{2} \hat{d}. \tag{2.3}$$

Using (2.1) we obtain that

$$\hat{c}_2 \geqslant \frac{n(n+1)}{2} \hat{d}. \tag{2.4}$$

Moreover, since $\hat{S}$ is of general type we have (see [8])

$$3 \hat{c}^2 \geqslant \hat{c}_1^2 \tag{2.5}$$

which holds even if $K_{\hat{X}}$ is not nef. (2.5) and Riemann—Roch Theorem imply that $\hat{c}_1^2 \leqslant 9 \chi(\mathcal{O}_{\hat{S}})$.

Let $J_1(X, L)$ be the first jet bundle of L in X. We have

$$c_1(J_1(X, L)) = -c_1(X) + (n+3) L$$

$$c_2(J_1(X, L)) = c_2(X) - (n+2) c_1(X) \cdot L + \frac{(n+2)(n+3)}{2} L \cdot L$$

and

$$c_1^2(J_1(X, L)) \cdot \underbrace{L \cdot \ldots \cdot L}_{n\text{-times}} = c_1^2 + 6\Delta + 9d$$

$$c_2(J_1(X,L))\cdot \underbrace{L\cdot\ldots\cdot L}_{n\text{-times}} = c_2 + 2\Delta + 3d\,. \tag{2.6}$$

Since $J_1(X,L)$ is spanned by global section we have (see [5, p. 216])

$$c_1^2(J_1(X,L)) - c_2(J_1(X,L)))\cdot \underbrace{L\cdot\ldots\cdot L}_{n\text{-times}} \geqslant 0\,.$$

Hence

$$c_1^2 + 4\Delta + 6d \geqslant c_2 \tag{2.7}$$

which implies that

$$\hat{c}_1^2 + 4\hat{\Delta} + 6\hat{d} \geqslant \hat{c}_2\,. \tag{2.8}$$

Moreover, the degree $\delta = \delta(S)$ of the discriminant locus of singular sections of L_S is given by (see [4])

$$\delta = c_2(J_1(S,L_S)) = c_2(J_1(X,L)) = c_2 + 2\Delta + 3d\,. \tag{2.9}$$

3. Assume that $X, \hat{X}, L$ and $\hat{L}$ are as in §2. If we denote with $r = \#|F|$. Then we have $\hat{c}_1^2 = c_1^2 + r$, $\hat{d} = d + r$, $\hat{\Delta} = \Delta - r$ and by (2.1) it follows that

$$\frac{n\Delta - c_1^2}{n+1} \leqslant r \leqslant \frac{\Delta - nd}{n+1}\,. \tag{3.1}$$

We set

$$a = \Delta - nd,\quad b = c_1^2 - n^2 d,\quad h = 9\chi(\mathcal{O}_S) - c_1^2\,.$$

We have

$$\begin{aligned} a &\geqslant (n+1)r \geqslant 0\\ b &\geqslant (n^2-1)r \geqslant 0\\ h &\geqslant 0. \end{aligned} \tag{3.2}$$

Moreover

$$d = \frac{1}{n^2}(9\chi(\mathcal{O}_S) - h - b)$$

$$\Delta = \frac{1}{n}(9\chi(\mathcal{O}_S) - h - b) + a = nd + a$$

$$c_1^2 = 9\chi(\mathcal{O}_S) - h = n^2 d + b$$

$$c_2 = 3\chi(\mathcal{O}_S) + h = \frac{1}{3}n^2 d + \frac{1}{3}b + \frac{4}{3}h$$

$$\delta = \frac{3}{n^2}(n+3)^2 \chi(\mathcal{O}_S) - \frac{1}{n^2}[(3-n)(n+1)h + (2n+3)b - 2n^2 a] =$$

$$= \frac{1}{3}(n+3)^2 d + \frac{1}{3} b + \frac{4}{3} h + 2a \, .$$

With the new notations (3.1) becomes

$$\frac{na-b}{n+1} \leqslant r \leqslant \frac{a}{n+1} \, . \tag{3.3}$$

Therefore

$$\gamma_0 = b - (n-1)a \geqslant 0 \, . \tag{3.4}$$

Moreover $9\chi(\mathcal{O}_S) \leqslant \hat{c}_1^2 = c_1^2 + r = 9\chi(\mathcal{O}_S) - h + r$ implies that

$$h \geqslant r \geqslant \frac{na-b}{n+1} \tag{3.5}$$

and hence

$$\gamma_1 = (n+1)h - na + b \geqslant 0 \, . \tag{3.6}$$

Setting

$$\alpha_n = \frac{1}{n^2}[(3-n)(n+1)h + (2n+3)b - 2n^2 a]$$

and

$$\beta_n = 1/3\,(b + 4h + 6a)$$

we have

$$\alpha_n = \frac{1}{n^2}[(3-n)\gamma_1 + 3n\gamma_0] \, . \tag{3.7}$$

Hence $\beta_n \geqslant 0$ and if $n \leqslant 3$ then $\alpha_n \geqslant 0$.

(3.8) Lemma. *Let $X, \hat{X}, L$ and $\hat{L}$ be as above. Assume that $K_{\hat{X}}^m \not\equiv \mathcal{O}_{\hat{X}}$ for every $m \geqslant 1$. Then*

i) $a > (n+1)r$, ii) $h > r$, iii) $h \geqslant 1$, iv) $a \geqslant 1$,
v) $\gamma_0 = b - (n-1)a \geqslant 1$, vi) $\gamma_1 = (n+1)h - na + b \geqslant n+1$.

Proof. i) By (3.2) it follows that $a \geqslant (n+1)r$. If $a = (n+1)r$ then

$$\hat{c}_1^2 = n^2 d + \frac{a}{n+1} + b$$

$$\hat{\Delta} = nd + \frac{n}{n+1}a = n\left(d + \frac{a}{n+1}\right)$$

$$\hat{d} = d + \frac{1}{n+1} a .$$

Hence $\hat{\Delta} = n\hat{d}$, $\hat{c}_1^2 = n^2 \hat{d} + \gamma_0$. If $\gamma_0 \geqslant 1$ then $\hat{c}_1^2 \cdot \hat{d} \geqslant \hat{\Delta}^2$ which contradicts the Algebraic Index Theorem. If $\gamma_0 = 0$ then $\hat{c}_1^2 = n\hat{\Delta} = n^2 \hat{d}$ implies that $K_{\hat{X}}^m \equiv \mathcal{O}_{\hat{X}}$ for some m. Therefore $a > (n+1) r$.

ii) By (3.5) it follows that $h \geqslant r$. Assume that $r = h$. Then $\hat{c}_1^2 = 9\chi(\mathcal{O}_{\hat{S}})$ which is impossible (see [12], [1, p. 39—40], [10, Example 1.3, p. 244] and [11, Remark 1.9.4]). Hence $r < h$.

iii) As in ii) $h \geqslant 1$ otherwise $9\chi(\mathcal{O}_S) = c_1^2$.

iv) If $a = 0$, then we have $c_1^2 = n^2 d + b$, $\Delta = nd$. If $b \geqslant 1$ then $c_1^2 \cdot d > \Delta^2$ which contradicts the Algebraic Index Theorem. If $b = 0$, then $c_1^2 = n\Delta = n^2 d$ which together with (3.1) imply that $r = 0$ and $K_{\hat{X}}^m \equiv \mathcal{O}_{\hat{X}}$ for some m. Hence $a \geqslant 1$.

v) By (3.4) it follows that $\gamma_0 \geqslant 0$. Assume that $\gamma_0 = 0$, then (3.3) implies that $\frac{1}{n+1} a \leqslant r < \frac{1}{n+1} a$ which is impossible. Therefore $\gamma_0 \geqslant 1$.

vi) Since $\frac{na - b}{n+1} \leqslant r < h$, it follows that $\gamma_1 \geqslant 1$. If $\gamma_1 = s$ with $s = 0, \ldots, n$ then $\frac{na - b}{n+1} = h - \frac{s}{n+1}$.

Thus $h - \frac{s}{n+1} \leqslant r \leqslant h$ which gives a contradiction. □

(3.9) Theorem. *Let $X, \hat{X}, L$ and $\hat{L}$ be as above. Then for $n \geqslant 2$*

a) $\delta \geqslant \frac{1}{3}(n+3)^2 d + \frac{n+10}{3}$ *if $K_{\hat{X}}$ is not nef*

b) $\delta \geqslant \frac{(n+2)(n+3)}{2} \hat{d} \geqslant \frac{(n+2)(n+3)}{2} d$ *if $K_{\hat{X}}$ is nef.*

If $n = 2$

c) $\delta \leqslant \frac{75}{4} \chi(\mathcal{O}_S) - \frac{9}{4}$ *if $K_{\hat{X}}$ is not nef*

d) $\delta \leqslant \frac{120}{7} \chi(\mathcal{O}_S)$ *if $K_{\hat{X}}$ is nef.*

If $n = 3$

e) $\delta \leqslant 12 \chi(\mathcal{O}_S)$

If $n \geqslant 4$

f) $\delta \leqslant 6 \frac{(n+3)^2}{(n+1)^2 + 2} \chi(\mathcal{O}_S)$.

Moreover, in b) *and in* d) *the equality occurs if and only if $\hat{X}$ has a torus as finite cover and in* e) *the equality occurs if and only if $K_{\hat{X}}^m \equiv \mathcal{O}_{\hat{X}}$ for some $m \geqslant 1$.*

(3.10) *Remark.* The case $n = 1$ has been studied in [2].

Proof of Theorem (3.9). a) and c) are a direct consequence of Lemma (3.8) and the fact that

$$\delta = \frac{3}{n^2}(n+3)^2 \chi(\mathcal{O}_S) - \alpha_n = \frac{1}{3}(n+3)^2 d + \beta_n .$$

e) In this case we use (2.1) to get $\delta \leqslant \hat{c}_2 + \hat{c}_1^2 = 12 \chi(\mathcal{O}_{\hat{S}})$. The equality occurs if and only if in (2.1) we have equalities which happen if and only if $K_{\hat{X}}^m \equiv \mathcal{O}_{\hat{X}}$ for some $m \neq 0$.

b) follows directly by (2.1) and (2.4).

d) Using (2.3) and (2.1) we obtain that for any $\lambda \geqslant 0$

$$\delta = \hat{c}_2 + 2\hat{\Delta} + 3\hat{d} \leqslant (1+\lambda)\hat{c}_2 + (7/4 - 3/4\,\lambda)\hat{c}_1^2$$

and in particular when $\lambda = 3/7$ we get d).

f) Using (2.7) and (2.1) we obtain that for any $1 > \lambda \geqslant 0$

$$\delta = \hat{c}_2 + 2\hat{\Delta} + 3\hat{d} \leqslant (1-\lambda)\hat{c}_2 + \frac{1}{n^2}[(1+2\lambda)(2n+3) + \lambda n^2]\hat{c}_1^2$$

and in particular when $\lambda = \frac{1}{2}\frac{n^2 - 2n - 3}{n^2 + 2n + 3}$ we get f).

Moreover in b) and in d) we have equality if and only if

$$\begin{cases} \hat{c}_1^2 = n\hat{\Delta} = n^2 \hat{d} \\ \hat{c}_2 = \frac{n(n+1)}{2}\hat{d} \end{cases} \tag{3.11}$$

By (2.2) and the ampleness of $\hat{L}$, (3.11) is equivalent to

$$\begin{cases} c_1(\hat{X}) \equiv 0 \\ c_2(\hat{X}) = 0 \end{cases} \tag{3.12}$$

which happens if and only if $\hat{X}$ has a torus as finite cover (see [2, Corollary 2, p. 5]. □

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