Commun. Math. Phys. 118, 385–400 (1988)

Communications in Mathematical Physics

On the Asymptotic Behaviour of the L^2-Norm of Suitable Weak Solutions to the Navier–Stokes Equations in Three-Dimensional Exterior Domains

Paolo Maremonti

Departimento di Mat. ed Applicazioni "R. Caccioppoli," Via Mezzocannone 8, I-80134 Napoli, Italy

Abstract. We prove L^2-decay rates of suitable weak solutions to the Navier–Stokes equations in exterior domains. The results for the order of decay are the same as for the solutions to the Cauchy problem of the Navier–Stokes equations. Finally in the case of $\Omega = R^3$ the decay rate order is sharp in the class of solutions considered by us.

1. Introduction

Recently, the problem of the asymptotic behaviour of the kinetic energy of an incompressible viscous fluid, governed by the Navier–Stokes equations, when the region of motion is unbounded in all directions, has been studied by several authors, cf. [3, 4, 7, 13, 14, 16, 19, 20, 23, 24]. Formally, this question is reduced to the asymptotic behaviour of the L^2-norm of solutions to the Navier–Stokes equations. The results of [3, 4, 7, 13, 14, 16, 19, 20, 23, 24] can be essentially devided in two groups. In [3, 4, 13, 14, 16] the asymptotic behaviour of the L^2-norm of solutions is obtained when the region Ω of motion of the fluid is an exterior domain, while the other works concern the asymptotic behaviour of the L^2-norm of solutions to a Cauchy problem for the Navier–Stokes equations. As regards the case of an exterior domain, in [13] the asymptotic behaviour of the L^2-norm of weak solutions to the Navier–Stokes equations is proved when the weak solutions verify the energy inequality in the "strong" form:

$$|\mathbf{v}(t)|^2 + 2\int_s^t |\nabla \mathbf{v}(\tau)|^2 d\tau \leqq |\mathbf{v}(s)|^2 \quad \forall t \geqq s \quad \text{and} \quad \text{a.e.} \quad \text{for} \quad s \geqq 0, \tag{I}$$

($|\cdot|$ is the L^2-norm of solution $\mathbf{v}$). However, relation (I) is not an a priori estimate for weak solutions to the Navier–Stokes equations on exterior domains. This fact makes formal the results obtained in [13] except that in the particular cases of a Cauchy problem and of initial-boundary value problem in exterior domains, where the initial data of the solutions are "small" in a suitable sense (global solution of the type furnished in [5, 9]). Subsequently, in [4] the relation (I) is determined for a suitable class of weak solutions. However the initial data $\mathbf{v}_0$ of the weak solutions must verify some hypotheses of summability $\mathbf{v}_0 \in X$, where X is a suitable Banach space. The condition $\mathbf{v}_0 \in X$ implies in particular that $\mathbf{v}_0 \in L^{5/4}(\Omega)$. In [4], if we assume $\mathbf{v}_0 \in X \cap L^q(\Omega)$, for some $q \in (1, 5/4)$, then there exists a weak solution $\mathbf{v}$ to the Navier–Stokes equations, which verifies relation (I) and

$$|\mathbf{v}(t)| = O(t^{-\beta}), \quad \text{where} \quad \beta = (2-q)/4q. \tag{II}$$

If we compare the above results, concerning exterior domains, with the one concerning the three-dimensional Cauchy problem of the Navier–Stokes equations, obtained in [7, 14, 20], we can notice two differences: the former concerns the choice of the initial data, the latter the asymptotic behaviour of the L^2-norm of solutions. In fact, for the Cauchy problem in [7, 20, 24], it is possible to choose $\mathbf{v}_0 \in L^2(R^3) \cap L^r(\Omega)$ for some $r \in [1, 2)$, therefore there is not the bound $r \leqq 5/4$. Moreover, for $\mathbf{v}_0 \in L^2(R^3) \cap L^r(R^3)$ it is possible to furnish a weak solution $\mathbf{v}$ to the Navier–Stokes equations such that

$$|\mathbf{v}(t)| = O(t^{-\beta}), \quad \text{where} \quad \beta = 3(2-r)/4r. \tag{III}$$

This order of decay is better than the order established in (II), when for $\mathbf{v}_0$ $r \in (1, 5/4]$. Finally if $\mathbf{v}_0 \in L^2(R^3)$, then a weak solution corresponding to $\mathbf{v}_0$ is such that

$$\lim_{t \to \infty} |\mathbf{v}(t)| = 0. \tag{IV}$$

The aim of this work is to bridge the difference between the case of the three-dimensional exterior domain and the Cauchy problem. We prove that for an initial-boundary value problem in exterior domains for the Navier–Stokes equations, we can obtain weak solutions $\mathbf{v}$ corresponding to an initial data $\mathbf{v}_0 \in L^2(\Omega) \cap L^q(\Omega)$, for some $q \in (1, 2)$, such that

$$|\mathbf{v}(t)| = O(t^{-\beta}), \quad \text{where} \quad \beta = 3(2-q)/4q. \tag{V}$$

If $\mathbf{v}_0 \in L^2(\Omega)$, then there exists a weak solution such that

$$\lim_{t \to \infty} |\mathbf{v}(t)| = 0.$$

Moreover, we prove that for the three-dimensional Cauchy problem the behaviour (III) and the limit (IV) are sharp, in the sense that if we choose $\mathbf{v}_0 \in L^2(R^3) \cap L^q(R^3)$, for some $q \in (1, 2]$, the exponent β cannot be improved to $\beta + \mu$, for any $\mu > 0$. This result is obtained by the help of a result due to G. H. Knightly in [10]. We like here to note explicitly that in [10], although for a particular class of solutions to the Navier–Stokes equations, the asymptotic behaviour of the L^2-norm of solutions of a three-dimensional Cauchy problem is obtained for the first time. The order of decay obtained for $|\mathbf{v}(t)|$ in [10] is the same as the one found in (III).

We conclude this introduction with the following remark. The estimate (III) obtained in [7, 20, 24] is uniform with respect to time (that is $\forall t \geqq 0$), there is a constant C depending only on the L^2 and L^p norms of the initial data. Instead, our estimate (V) holds uniformly only for $t \geqq T_0$ (for a suitable $T_0 > 0$) and we determine a constant C depending on the L^2 and L^p norms of the initial data and also on several norms of derivatives of the solution computed for $t \geqq T_0$. In fact our solutions becomes regular for $t \geqq T_0$.

Some results of this work were communicated by the author in [13], others are here improved.

The plan of the work is as follows. In Sect. 2, after introducing some mathematical preliminaries and notations, we give the statement of the theorems. In Sect. 4 we give the proof of the theorems, after proving some preliminary lemmas in Sect. 3.

2. Preliminaries and Statement of the Theorems

In this work, Ω is a domain of the three-dimensional Euclidean space R^3, exterior to v ($v \geqq 0$) compact subregions, whose boundaries are supposed C^3-smooth. For $p \in [1, \infty]$, with $L^p(\Omega)$ we denote the Lebesgue space of functions on Ω. The norm of a function of $L^p(\Omega)$ is indicated by $|\cdot|_p$, in the case $p=2$ we put $|\cdot|_2 = |\cdot| \cdot W^{m;p}(\Omega)$ denotes the usual Sobolev space of (m,p)-order of functions on Ω and $|\cdot|_{m;p}$ is its associated norm. $\mathscr{C}_0(\Omega)$ denotes the set of functions $\boldsymbol{\Phi}$ on Ω with vector values in R^3, with components $\Phi_i \in C_0^\infty(\Omega)$ $(i=1,2,3)$ and such that $\nabla \cdot \boldsymbol{\Phi} = 0$. The following completion spaces are considered: $J^p(\Omega) \equiv$ completion of $\mathscr{C}_0(\Omega)$ in $L^p(\Omega)$; $\mathring{J}^{1,p}(\Omega) \equiv$ completion of $\mathscr{C}_0(\Omega)$ in $W^{1,p}(\Omega)$. Finally, by $L^p((0,s);X)$ we denote the set of functions $\boldsymbol{\Phi}$ from $(0,s)$ into X, where X is a Banach space, such that $\int_0^s |\boldsymbol{\Phi}(\tau)|_X^p \, d\tau < \infty$ ($|\cdot|x$ is X-norm); analogously, by $C((0,s);X)$ we indicate the set of functions $\boldsymbol{\Phi}$ from $(0,s)$ into X which are continuous from I into X, with norm $|\boldsymbol{\Phi}|_c \equiv \max_{[0,s]} |\boldsymbol{\Phi}|x$. By the symbol $(\boldsymbol{\Phi}, \boldsymbol{\Psi})$ we mean

$$(\boldsymbol{\Phi}, \boldsymbol{\Psi}) \equiv \int_\Omega \boldsymbol{\Phi}(x) \cdot \boldsymbol{\Psi}(x) \, dx,$$

for any $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$ such that the integral is finite. By $\boldsymbol{\Phi}_n = J_{1/n} * \boldsymbol{\Phi}$ we mean a spatial "mollification" of a function $\boldsymbol{\Phi}$. In this work the symbol C denotes a generic constant whose numerical value is inessential to our aims, and it may have several different values in a single computation.

By a weak solution of the initial boundary value problem of the Navier–Stokes equations,

$$\begin{gathered}
\mathbf{v}_t(x,t) + \mathbf{v}(x,t) \cdot \nabla \mathbf{v}(x,t) = -\nabla \pi(x,t) + \Delta \mathbf{v}(x,t) \quad \text{in} \quad \Omega x(0,T), \\
\nabla \cdot \mathbf{v}(x,t) = 0 \quad \text{in} \quad \Omega x(0,T), \\
\mathbf{v}(x,0) = \mathbf{v}_0, \quad \mathbf{v}(x,t)_{|\partial\Omega} = 0 \quad \text{and} \quad \mathbf{v}(x,t) \to 0 \quad \text{for} \quad |x| \to \infty,
\end{gathered} \tag{2.1}$$

we mean a function $\mathbf{v}(x,t)$ defined as follows.

Definition 1. A field $\mathbf{v}: \Omega x(0,T) \to R^3$ ($\forall T > 0$) is such that

i) $$\mathbf{v} \in L^2((0,T); \mathring{J}^{1,2}(\Omega)) \cap L^\infty((0,T); J^2(\Omega)) \quad \forall T > 0,$$

$$|\mathbf{v}(t)|^2 + 2\int_0^t |\nabla \mathbf{v}(\tau)|^2 \, d\tau \leqslant |\mathbf{v}_0|^2 \quad \forall t \geqq 0;$$

ii) $$\int_0^t \{(\mathbf{v}(\tau), \boldsymbol{\Phi}_\tau(\tau)) - (\nabla \mathbf{v}(\tau), \nabla \boldsymbol{\Phi}(\tau)) - (\mathbf{v}(\tau)\cdot\nabla\mathbf{v}(\tau), \boldsymbol{\Phi}(\tau))\}\,d\tau$$
$$= (\mathbf{v}(t), \boldsymbol{\Phi}(t)) - (\mathbf{v}_0, \boldsymbol{\Phi}(0)) \quad \forall t \geqq 0$$

and $\forall \boldsymbol{\Phi} \in C([0, T); \mathring{J}^{1,2}(\Omega))$ with $\boldsymbol{\Phi}_t \in L^2((0, T); J^2(\Omega))$;

(iii) $$\lim_{t\to 0^+} |\mathbf{v}(t) - \mathbf{v}_0| = 0.$$

Remark 1. As is well known, Hopf, [6], has furnished an existence theorem of weak solutions to system (2.1) for a general I.B.V.P. by the well known Faedo–Galerkin method. However, in this work, to prove the asymptotic behaviour of the L^2-norm of solutions, we construct weak solutions by a suitable approximation. This process of approximation was introduced by Leray in [11], and retaken in [1, 4, 22]. If we settle the initial and boundary conditions a priori in (2.1), our weak solution $\mathbf{v}$ cannot be assumed to coincide with another weak solution $\mathbf{w}$ to system (2.1), since a uniqueness theorem for these solutions is not known.

Theorem 1. *Let be $\mathbf{v}_0 \in J^2(\Omega)$. Then there exists a weak solution $\mathbf{v}$ to system (2.1) such that*

a) $\mathbf{v} \in C([T_0, T); \mathring{J}^{1,2}(\Omega)) \cap L^\infty((T_0, +\infty); \mathring{J}^{1,2}(\Omega)) \quad \forall T \geqq T_0$, $\mathbf{v} \in L^2((T_0, T); W^{2,2}(\Omega) \cap \mathring{J}^{1,2}(\Omega))$, *and* $\mathbf{v}_t \in L^2((T_0, T); J^2(\Omega)) \quad \forall T \geqq T_0$, *where* $T_0 \leqq (C|\mathbf{v}_0|^4 \exp(C|\mathbf{v}_0|^2 + 1))$, *moreover* $\mathbf{v}$ *verifies system (2.1) a.e. for* $t \geqq T_0$;
b) $\mathbf{v} \in L^r((0, T); J^s(\Omega))\ \forall T \geqq 0, \forall s \geqq 2$ *with* $1/r + 3/2s > 1$;
c) *If* $\mathbf{v}_0 \in J^2(\Omega) \cap J^p(\Omega)$ *for some* $p \in (1, 2)$, *then* $\mathbf{v} \in L^\infty((0, T);\ J^p(\Omega))\ \forall T \geqq 0$ *if* $p \in (1, 3/2]$, *otherwise* $\mathbf{v} \in L^r((0, T); J^p(\Omega))\ \forall T \geqq 0$ *for* $1/r + 3/2p > 1$.

Theorem 2. *Let be $\mathbf{v}_0 \in J^2(\Omega) \cap J^p(\Omega)$ for some $p \in (1, 2)$. Then there exists a weak solution to system (2.1) corresponding to $\mathbf{v}_0$, such that*

$$|\mathbf{v}(t)| = O(t^{-3(2-p)/4p}). \tag{2.2}$$

Theorem 3. *Let be $\mathbf{v}_0 \in J^2(\Omega)$. Then there exists a weak solution $\mathbf{v}$ to system (2.1) corresponding to $\mathbf{v}_0$, such that*

$$\lim_{t\to\infty} |\mathbf{v}(t)| = 0. \tag{2.3}$$

Theorem 4. *Let be $\Omega \equiv R^3$. Then $\forall p \in (1, 2]$ there exists an initial data $\mathbf{v}_0 \in J^2(R^3) \cap J^p(R^3)$ such that a unique solution $\mathbf{v}$ to system (2.1) corresponds to $\mathbf{v}_0$ with*

$$|\mathbf{v}(t)| \geqq Kt^{-(3(2-p)/4p+\mu)} \quad \forall \mu > 0 \quad \textit{and} \quad t \textit{ sufficiently large.}$$

Therefore the order of decay determined in (2.2) and the limit property (2.3) are sharp.

Remark 2. Point a) of Theorem 1 ensures that a weak solution becomes sufficiently smooth for $t \geqq T_0$. We deduce the result of point a) by "a priori estimates." The result of point a) is analogous to the result stated in the "Théorèm de structure" [5, 11]. Actually, the "Théorème de structure" holds if we prove relation (I) for weak solutions and we can prove relation (I) only for particular initial data $\mathbf{v}_0$ when Ω is an exterior domain ([4, 22]), while we now assume only $\mathbf{v}_0 \in J^2(\Omega)$.

Point b) of Theorem 1 is a new estimate for weak solutions; however, this estimate is not sufficient to inform us on the regularity of a weak solution. Point c) is the sufficient condition to deduce the results of Theorem 2 and Theorem 3.

Remark 3. Theorem 2 and Theorem 3 furnish the asymptotic behaviour of the L^2-norm of a weak solution $\mathbf{v}$. Relation (2.2) is the same as the one we can deduce for weak solution to the Cauchy problem of (2.1) (cf. [7, 20, 24]).

Theorem 4 makes sharp the order of decay obtained in (2.2) and the limit property (2.3) in the following sense. In the class of solution considered by us or in [7, 20, 24], we cannot to improve the exponent $3(2-p)/4p$ to exponent $\mu+3(2-p)/4p$ $\forall \mu>0$. The result of Theorem 4 is connected only to the chosen of the initial data. That is, it is not connected either to the fact that we consider weak solutions, or to the fact that we consider solutions to the Navier–Stokes equations. In fact Theorem 4 also holds for solutions to the heat equation. For this equation and (2.3) see also [20].

Theorem 4 is also an answer to the following question. $\forall \mathbf{v}_0 \in L^p(\Omega) \cap L^2(\Omega)$, for some $p \in [1, 2]$, is it possible to prove for a corresponding solution $\mathbf{v}(x, t)$, that $\mathbf{v}(x, \bar{t})$ belongs to L^q with $q<p$ in a certain instant $\bar{t}>0$? This problem was posed also in [8] Remark 1.1. Theorem 4 gives a negative answer to the above question. In fact if we assume that in an instant $\bar{t}$ $\mathbf{v}(x, \bar{t}) \in L^q(R^3)$ for some $q<p$, we have by Theorem 2 $|\mathbf{v}(t)| = O\,(t^{-3(2-q)/4q})$ with $3(2-q)/4q > 3(2-p)/4p$, which is absurd by virtue of Theorem 4. By these considerations we can deduce that the dissipation of the fluid works in the time but not in the space.

3. Preliminary Results

As is well known $L^p(\Omega) \equiv J^p(\Omega) \oplus G^p(\Omega)$ for $p>1$, where

$$G_p(\Omega) = \{\nabla p : \nabla p \in L^p(\Omega) \quad \text{and} \quad p \in L^p_{loc}(\Omega)\}.$$

By P_p we denote the projection operator from $L^p(\Omega)$ into $J^p(\Omega)$. For $p=2$ we set $P_2 = P$. We have $\forall \boldsymbol{\Phi} \in J^p(\Omega)$, $\gamma_{\vec{n}}(\boldsymbol{\Phi}) = 0$, where $\gamma_{\vec{n}}$ is the trace operator of $\boldsymbol{\Phi}\cdot\vec{n}$ to $\partial\Omega$ and

$$(\boldsymbol{\Phi}, \nabla p) = 0 \quad \forall \nabla p \in G^q(\Omega), \quad \text{if } q \text{ is such that } 1/p + 1/q = 1.$$

For the elementary properties of the space introduced above, we refer the reader to [18, 21].

Let $\boldsymbol{\Phi} \in W^{2,\,2}(\Omega) \cap \mathring{J}^{1,2}(\Omega)$, then

$$|D^2\boldsymbol{\Phi}| \leqq C(|P\Delta\boldsymbol{\Phi}| + |\nabla\boldsymbol{\Phi}|), \tag{3.1}$$

$$|\nabla\boldsymbol{\Phi}|_3 \leqq C(|P\Delta\boldsymbol{\Phi}|^{1/2}|\nabla\boldsymbol{\Phi}|^{1/2} + |\nabla\boldsymbol{\Phi}|). \tag{3.2}$$

For inequalities (3.1)–(3.2) cf. [5] Lemma 1.

Let $\boldsymbol{\Phi} \in C_0^\infty(R^3)$, $1 \leqq q,\, r \leqq \infty$, j and m two integers such that $0 \leqq j \leqq m$. Then

$$|D^j\boldsymbol{\Phi}|_p \leqq C|D^m\boldsymbol{\Phi}|_r^a|\boldsymbol{\Phi}|_q^{1-a} \quad \text{for} \quad a \in [j/m, 1], \tag{3.3}$$

where

$$1/p = j/3 + a(1/r - 2/3) + (1-a)1/q,$$

provided that $m - j - 3/r < 0$, otherwise $a = j/m$, cf. [2] Theorem 9.3.

The following lemma proves that $\mathscr{C}_0(\Omega)$ is dense in $J^p(\Omega)\cap J^q(\Omega)$. The result of the lemma is trivial when Ω is bounded. In the case in which Ω is an exterior domain the proof of the lemma is a consequence of standard arguments we use to prove that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)\cap L^q(\Omega)$. Professor G. P. Galdi communicated to the author that he found an analogous result.

Lemma 1. *$\mathscr{C}_0(\Omega)$ is dense in $J^p(\Omega)\cap J^q(\Omega)$ for any $p,q>1$.*

Proof. Let $\boldsymbol{\Phi}\in J^p(\Omega)\cap J^q(\Omega)$. We consider the function $\tilde{\boldsymbol{\Phi}}$ defined a.e. in R^3 by

$$\tilde{\boldsymbol{\Phi}}(x)=\begin{cases}\boldsymbol{\Phi}(x) & \text{if } x\in\Omega\\ 0 & \text{if } x\in R^3-\bar{\Omega},\end{cases}$$

to define the functions $\boldsymbol{\Phi}_n(x)=\int_{R^3} J_{1/n}(x-y)\tilde{\boldsymbol{\Phi}}(y)\,dy$. Now, we consider r and s such that $1/r+1/p=1/s+1/q=1$. We have for $\nabla\pi\in L^r(R^3)$ and $\nabla q\in L^s(R^3)$,

$$\int_{R^3}\tilde{\boldsymbol{\Phi}}(x)\cdot\nabla\pi(x)\,dx=(\boldsymbol{\Phi},\nabla\pi)=\lim_n(\boldsymbol{\Phi}'_n,\nabla\pi)=I_1,$$

$$\int_{R^3}\tilde{\boldsymbol{\Phi}}(x)\cdot\nabla q(x)\,dx=(\boldsymbol{\Phi},\nabla q)=\lim_n(\boldsymbol{\Phi}''_n,\nabla q)=I_2,$$

where $\{\boldsymbol{\Phi}'_n\}_{n\in N}\subseteq\mathscr{C}_0(\Omega)$ and $\boldsymbol{\Phi}'_n\to\boldsymbol{\Phi}$ in $J^p(\Omega)$, $\{\boldsymbol{\Phi}''_n\}_{n\in N}\subseteq\mathscr{C}_0(\Omega)$ and $\boldsymbol{\Phi}''_n\to\boldsymbol{\Phi}$ in $J^q(\Omega)$. Integrating by parts we have $I_1=I_2=0$. Therefore $\tilde{\boldsymbol{\Phi}}(x)$ is divergence free in the distributional sense, and since $\boldsymbol{\Phi}\in L^p(\Omega)\cap L^q(\Omega)$, it follows that $\tilde{\boldsymbol{\Phi}}(x)\in J^p(R^3)\cap J^q(R^3)$. Consequently $\boldsymbol{\Phi}_n(x)$ also is divergence free $\forall n\in N$. In fact, since $\tilde{\boldsymbol{\Phi}}(x)\in J^p(R^3)\cap J^q(R^3)$ there exists a sequence $\{\boldsymbol{\Phi}_k\}_{k\in N}\subseteq\mathscr{C}_0(R^3)$ such that $\boldsymbol{\Phi}_k\to\tilde{\boldsymbol{\Phi}}$ in $J^p(R^3)$ or in $J^q(R^3)$, therefore

$$\nabla\cdot\boldsymbol{\Phi}_n(x)=\lim_k\int_{R^3}\nabla J_{1/n}(x-y)\cdot\boldsymbol{\Phi}_k(y)\,dy=0.$$

Now, we consider a sequence $\{K_h\}_{h\in N}$ of compacts expanding in Ω, with $K_h\subseteq K_{h+1}$ and $\bigcup_{h\in N}K_h=\Omega$. to define

$$\boldsymbol{\Phi}_{n,h}(x)=\begin{cases}\boldsymbol{\Phi}_n(x) & \text{if } x\in K_h\\ 0 & \text{if } x\in R^3-K_h\end{cases}.$$

$\boldsymbol{\Phi}_{n,h}\in J^p(R^3)\cap J^q(R^3)$, since $\nabla\cdot\boldsymbol{\Phi}_{n,h}=0\ \forall x\in R^3$ and $\boldsymbol{\Phi}_{n,h}\in L^p(R^3)\cap L^q(R^3)$. Therefore there exists $\{\boldsymbol{\Phi}_{n,h,i}\}_{i\in N}\subseteq\mathscr{C}_0(R^3)$ such that $\boldsymbol{\Phi}_{n,h,i}\to\boldsymbol{\Phi}_{n,h}$ in $J^p(R^3)$ or in $J^q(R^3)$. We set, $\forall j\in N$ such that $1/j<\operatorname{dist}(K_h,\partial\Omega)$,

$$\boldsymbol{\Phi}_{n,h,j}(x)=\int_{R^3}J_{1/j}(x-y)\boldsymbol{\Phi}_{n,h}(y)\,dy.$$

Then $\boldsymbol{\Phi}_{n,h,j}\in\mathscr{C}_0(\Omega)$, since $\boldsymbol{\Phi}_{n,h,j}\in C_0^\infty(\Omega)$ and

$$\nabla\cdot\boldsymbol{\Phi}_{n,h,j}(x)=\int_{R^3}\nabla J_{1/j}(x-y)\cdot\boldsymbol{\Phi}_{n,h}(y)\,dy=\lim_i\int_{R^3}\nabla J_{1/j}(x-y)\cdot\boldsymbol{\Phi}_{n,h,i}(y)\,dy=0.$$

Now, it is very simple to verify that $\boldsymbol{\Phi}_{n,h,j}\to\boldsymbol{\Phi}$ in $L^p(\Omega)\cap L^q(\Omega)$. Therefore, the lemma is completely proved.

We consider the linear Navier–Stokes system:

$$\Delta \mathbf{w}(x,t) + \nabla p(x,t) = \mathbf{w}_t(x,t) \quad \text{in} \quad \Omega x(0,T),$$
$$\nabla \cdot \mathbf{w}(x,t) = 0 \quad \text{in} \quad \Omega x(0,T),$$
$$\mathbf{w}(x,0) = \mathbf{w}_0 \in \mathscr{C}_0(\Omega), \quad \mathbf{w}(x,t)_{|\partial\Omega} = 0 \quad \text{and} \quad \mathbf{w}(x,t) \to 0 \quad \text{for} \quad |x| \to \infty. \tag{3.4}$$

The following theorem holds for system (3.4).

Theorem 3.1. *Let* $\mathbf{w}_0 \in \mathscr{C}_0(\Omega)$. *Then there exists a unique solution* $\mathbf{w}(x,t)$ $\forall t \geqq 0$ *to the system* (*3.4*) *with*

$$\mathbf{w}(x,t) \in L^q((0,T); W^{2,q}(\Omega) \cap \mathring{J}^{1,q}(\Omega)), \quad \textit{and} \quad \mathbf{w}_t(x,t) \in L^q((0,T); J^q(\Omega)) \quad \forall q > 1, \tag{3.5}$$

moreover for $q \geqq p > 1$ *we have*

$$|\mathbf{w}(t)|_q \leqq C|\mathbf{w}_0|_p \exp(Ct) t^{-3(1/p-1/q)/2} \quad \forall t \geqq 0. \tag{3.6}$$

Proof. For any fixed q, existence and uniqueness of solution $\mathbf{w}$ is proved in [21] Theorem 4.1. Property (3.5) is proved in [4] Lemma 1.2. Property (3.6) is proved in [21] Lemma 5.1.

Lemma 2. *Let be* $\Phi(t) \in C^1(t_0, +\infty)$ $(t_0 \geqq 0)$ *with* $\Phi(t) \geqq 0$. *Let us assume that*

$$\Phi'(t) \leqq g(\Phi) \quad \forall t \geqq t_0,$$

moreover $g(\Phi) \leqq \alpha\Phi^2$ *for* $\Phi \leqq \beta$, *when* $\alpha, \beta > 0$ *are given real numbers. Let us assume that* $\int_{t_0}^{\infty} \Phi(t)dt \leqq M$. *Then for* $t \geqq (M/\beta)\exp(\alpha M)$ *we have*

$$\Phi(t) \leqq (\exp(\alpha M) - 1)/\alpha t \quad \forall t \geqq t_0.$$

Proof. cf [5] Lemma 6.

The following system is important for our aims:

$$\mathbf{v}_t(x,t) - \Delta \mathbf{v}(x,t) = -J_{1/n}(\mathbf{v}) \cdot \nabla \mathbf{v}(x,t) + \nabla p(x,t) \quad \text{in} \quad \Omega x(0,T),$$
$$\nabla \cdot \mathbf{v}(x,t) = 0 \quad \text{in} \quad \Omega x(0,T),$$
$$\mathbf{v}(x,0) = \mathbf{u}_0, \quad \mathbf{v}(x,t)_{|\partial\Omega} = 0 \quad \text{and} \quad \mathbf{v}(x,t) \to 0 \quad \text{for} \quad |x| \to \infty. \tag{3.7}$$

Lemma 3. *Let* $\mathbf{u}_0 \in \mathscr{C}_0(\Omega)$. *Then there exists* $\forall t \geqq 0$ *a unique solution* $\mathbf{v}$ *to system* (*3.7*) *for any fixed* n, *with*

$$\mathbf{v} \in L^2((0,T); W^{2,2}(\Omega) \cap \mathring{J}^{1,2}(\Omega)) \quad \textit{and} \quad \mathbf{v}_t \in L^2((0,T); J^2(\Omega) \quad \forall T \geqq 0, \tag{3.8}$$

moreover

$$|\mathbf{v}(t)|^2 + 2\int_s^t |\nabla \mathbf{v}(\tau)|^2 d\tau = |\mathbf{v}(s)|^2 \quad \forall t \geqq s \geqq 0. \tag{3.9}$$

Proof. The existence of local (in time) solution $\mathbf{v}$, verifying (3.8)–(3.9), can be proved by the well known Galerkin method, in the way suggested in [5] for exterior domains. As proved in [5], to obtain a global (in time) solution it is sufficient to prove that $|\mathbf{v}(t)| + |\nabla \mathbf{v}(t)|$ is uniformly bounded in time. The boundedness of $|\mathbf{v}(t)|$ is a consequence of (3.9). To obtain the boundedness of $|\nabla \mathbf{v}(t)|$, we multiply $(3.7)_1$ by $P\Delta \mathbf{v}$ in $L^2(\Omega)$; integrating by parts, we obtain:

$$(1/2)\frac{d}{dt}|\nabla \mathbf{v}(t)|^2 + |P\Delta \mathbf{v}(t)|^2 = (J_{1/n}(\mathbf{v})\cdot\nabla \mathbf{v}, P\Delta \mathbf{v})^1 \quad \text{in} \quad (0, T).$$

On the other hand $|J_{1/n}(\mathbf{v}(x,t))| \leqq C(n)|\mathbf{v}(t)|_2 \ \forall x \in R^3$ and $\forall t \geqq 0$. Therefore employing the Schwartz inequality and the Cauchy inequality we obtain

$$\frac{d}{dt}|\nabla \mathbf{v}(t)|^2 + |P\Delta \mathbf{v}(t)|^2 \leqq C^2(n)|\mathbf{u}_0|^2|\nabla \mathbf{v}(t)|^2 \quad \text{in} \quad (0, T),$$

which implies $|\nabla \mathbf{v}(t)|^2 \leqq |\nabla \mathbf{u}_0|^2 + C^2(n)|\mathbf{u}_0|^4 \ \forall t \geqq 0$.

The uniqueness of solutions is a consequence of energy equality written for the difference of two solutions and of the regularity of the solutions.

Let $\mathbf{v}_0 \in J^2(\Omega) \cap J^p(\Omega)$ for some $p \in (1, 2]$. We denote by $\{\boldsymbol{\Phi}_n\}_{n \in N} \subseteq \mathscr{C}_0(\Omega)$ a sequence such that $\boldsymbol{\Phi}_n \to \mathbf{v}_0$ in $J^2(\Omega) \cap J^p(\Omega)$ and $|\boldsymbol{\Phi}_n| \leqq 2|\mathbf{v}_0| \forall n \in N$. $\forall n \in N$ Lemma 3 ensure, set $\mathbf{v}_n(x, 0) = \boldsymbol{\Phi}_n(x)$, the existence of the solution $\mathbf{v}_n$ to system $(3.7)_n$. System $(3.7)_n$ is the system obtained from (3.7) varying n for $J_{1/n}$. However, Lemma 3 does not give the validity of (3.8) uniformly with respect to n. The following lemma gives a partial result in this sense. The proof of the lemma is the standard proof of the "Théorème de structure" in the case of the exterior domain, (cf. [5]), however for the sake of completeness we propose it.

Lemma 4. *For any $n \in N$, let $\mathbf{v}_n$ be the solution to the system $(3.7)_n$ assuming as initial data $\mathbf{v}_n(x, 0) = \boldsymbol{\Phi}_n(x)$. Then*

$$|\mathbf{v}_n(t)|^2 + 2\int_0^t |\nabla \mathbf{v}_n(\tau)|^2 d\tau = |\boldsymbol{\Phi}_n|^2 \leqq 2|\mathbf{v}_0|^2 \quad \forall t \geqq 0$$

$$\text{and uniformly with respect to } n; \tag{3.10}$$

moreover, there exists an instant T_0, with $T_0 \leqq C|\mathbf{v}_0|^4 \exp(C|\mathbf{v}_0|^2 + 1)$, such that

$$\mathbf{v}_n \in C((T_0, \infty); \mathring{J}^{1,2}(\Omega)) \cap L^\infty((T_0, \infty); \mathring{J}^{1,2}(\Omega)),$$

$$D^2\mathbf{v}_n, \mathbf{v}_{nt} \in L^2((T_0, \infty); J^2(\Omega)) \quad \text{uniformly with respect to } n. \tag{3.11}$$

Proof. Let $\mathbf{v}_n$ be the solution corresponding to $\boldsymbol{\Phi}_n$. Inequality (3.10) is an immediate consequence of (3.9) and of the choice of $\boldsymbol{\Phi}_n$. To obtain (3.11), we multiply $(3.7)_n$ first by $P\Delta \mathbf{v}_n$ in $L^2(\Omega)$, then by $\mathbf{v}_{nt}$ in $L^2(\Omega)$. Last, integrating by parts, we obtain

$$(1/2)\frac{d}{dt}|\nabla \mathbf{v}_n(t)|^2 + |P\Delta \mathbf{v}_n(t)|^2 = (J_{1/n}(\mathbf{v}_n)\cdot\nabla \mathbf{v}_n, P\Delta \mathbf{v}_n), \tag{3.12}$$

$$(1/2)\frac{d}{dt}|\nabla \mathbf{v}_n(t)|^2 + |\mathbf{v}_{nt}(t)|^2 = (J_{1/n}(\mathbf{v}_n)\cdot\nabla \mathbf{v}_n, \mathbf{v}_{nt}), \tag{3.13}$$

Applying the Hölder inequality with exponents $1/6 + 1/3 + 1/2 = 1$ to the right-hand side of (3.12) and (3.13), we obtain

$$|(J_{1/n}(\mathbf{v}_n)\cdot\nabla \mathbf{v}_n, P\Delta \mathbf{v}_n)| \leqq |J_{1/n}(\mathbf{v}_n)|_6 |\nabla \mathbf{v}_n|_3 |P\Delta \mathbf{v}_n|,$$

$$|(J_{1/n}(\mathbf{v}_n)\cdot\nabla \mathbf{v}_n, \mathbf{v}_{nt})| \leqq |J_{1/n}(\mathbf{v}_n)|_6 |\nabla \mathbf{v}_n|_3 |\mathbf{v}_{nt}|.$$

---

[1] We recall that from (3.8) it is possible to deduce that $\mathbf{v} \in C((0, T); \mathring{J}^{1,2}(\Omega))$; this fact is tacitly assumed.

Since $|J_{1/n}(\mathbf{v}_n)|_6 \leqq |\mathbf{v}_n|_6$, from (3.2) and (3.3), we have

$$|J_{1/n}(\mathbf{v}_n)|_6 |\nabla \mathbf{v}_n|_3 |P\Delta \mathbf{v}_n| \leqq C|\nabla \mathbf{v}_n|^6 + C|\nabla \mathbf{v}_n|^4 + 1/3|P\Delta \mathbf{v}_n|^2, \tag{3.14}$$

$$|J_{1/n}(\mathbf{v}_n)|_6 |\nabla \mathbf{v}_n|_3 |\mathbf{v}_{nt}| \leqq C|\nabla \mathbf{v}_n|^6 + C|\nabla \mathbf{v}_n|^4 + 1/6|P\Delta \mathbf{v}_n|^2 + 1/2|\mathbf{v}_{nt}|^2, \tag{3.15}$$

where increasing we have employed the Cauchy inequality with a suitable factor. Summing (3.12) and (3.13), and increasing by (3.14)–(3.15), we deduce the following differential inequality:

$$\frac{d}{dt}|\nabla \mathbf{v}_n(t)|^2 + |P\Delta \mathbf{v}_n(t)|^2 + |\mathbf{v}_{nt}(t)|^2 \leqq C|\nabla \mathbf{v}_n(t)|^6 + C|\nabla \mathbf{v}_n(t)|^4. \tag{3.16}$$

Since $\int_0^{+\infty} |\nabla \mathbf{v}_n(\tau)|^2 d\tau \leqq 2|\mathbf{v}_0|^2 = M \forall n \in N$, set $\alpha = C + 1/M$ and $\beta = 1/MC$, from Lemma 2 we have $|\nabla \mathbf{v}_n(t)| \leqq K$ (K suitable constant) $\forall t \geqq T_0$. From this last inequality, after integrating with respect to time (3.16), we deduce (3.11).

Lemma 5. *Let* $\mathbf{v}_0 \in J^2(\Omega) \cap J^p(\Omega)$, *for some* $p \in (1, 2]$, *and* $\{\boldsymbol{\Phi}_n\}_{n\in N} \subseteq \mathscr{C}_0(\Omega)$ *with* $\boldsymbol{\Phi}_n \to \mathbf{v}_0$ *in* $J^2(\Omega) \cap J^p(\Omega)$. *We denote by* $\mathbf{v}_n$ *the solution to system* (3.7)$_n$ *corresponding to* $\boldsymbol{\Phi}_n$, $\forall n \in N$. *Then*

$$\textit{if } \mathbf{v}_0 \in J^p(\Omega) \textit{ for some } p \in (1, 3/2]\ \mathbf{v}_n \in L^\infty((0, T); J^p(\Omega))\ \forall T > 0 \quad \textit{uniformly with respect to } n; \tag{3.17}$$

$$\textit{if } \mathbf{v}_0 \in J^p(\Omega) \textit{ for some } p \in (3/2, 2)\ \mathbf{v}_n \in L^r((0, T); J^p(\Omega)) \quad \textit{with } 1/r + 3/2p > 1, \quad \forall T > 0 \quad \textit{uniformly with respect to } n; \tag{3.18}$$

$$\textit{if } \mathbf{v}_0 \in J^2(\Omega)\ \mathbf{v}_n \in L^r((0, T); L^s(\Omega)) \textit{ with } 1/r + 3/2s > 1\ s \geqq 2 \quad \textit{and} \quad \forall T > 0 \quad \textit{uniformly with respect to } n. \tag{3.19}$$

Proof. For $p \in (1, 3/2]$, it is possible to obtain (3.17) modifying in a suitable way the proof of Lemma 3.2 of [4]. Therefore we omit the proof. Now, let $p > 3/2$ and consider the solution $\mathbf{w}(x, t)$ to system (3.4) corresponding to $\mathbf{w}(x, 0) = \mathbf{w}_0 \in \mathscr{C}_0(\Omega)$. Set $\boldsymbol{\Theta}(x, \tau) = \mathbf{w}(x, h - \tau)$. By the properties of regularity of $\mathbf{w}$, we can multiply (3.7)$_n$ by $\boldsymbol{\Theta}$ in $L^2(\Omega)$ and integrating by parts over $\Omega x(0, h)$, we obtain

$$(\mathbf{v}(h), \mathbf{w}_0) = (\boldsymbol{\Phi}_n, \mathbf{w}(h)) + \int_0^h (J_{1/n}(\mathbf{v}_n) \cdot \nabla \mathbf{v}_n, \boldsymbol{\Theta}) d\tau. \tag{3.20}$$

Applying the Hölder inequality with exponents $1/6 + 1/2 + 1/3 = 1$, from (3.3) and (3.6), it follows for $1/q + 1/p = 1$

$$|(\mathbf{v}_n(h), \mathbf{w}_0/|\mathbf{w}_0|_q)| \leqq C|\mathbf{v}_0|_p + C\int_0^h |\nabla \mathbf{v}_n(\tau)|^2 (h - \tau)^{-(3-q)/2q} d\tau,$$

$$\forall h \geqq 0 \quad \text{and} \quad \mathbf{w}_0 \in \mathscr{C}_0(\Omega),$$

which implies

$$|\mathbf{v}_n(h)|_p \leqq C|\mathbf{v}_0|_p + C\int_0^h |\nabla \mathbf{v}_n(\tau)|^2 |h - \tau|^{-(3-q)/2q} d\tau \quad \forall h \geqq 0. \tag{3.21}$$

Since $|\nabla \mathbf{v}_n(\tau)|^2 \in L^1(0, \infty)$, we deduce (3.18) from (3.21). To prove (3.19) we reason in the same way up to relation (3.20). We increase the right-hand side of (3.20) by the Schwartz inequality and with (3.6) applied to the first term, while the integral term is treated in the same way shown above. From (3.6) we have for $1/q+1/p=1$,

$$|\mathbf{v}_n(h)|_p \leqq C|\mathbf{v}_0|h^{-3(2-q)/4q} + C\int_0^h |\nabla \mathbf{v}_n(\tau)|^2 |h-\tau|^{-(3-q)/2q} d\tau \quad \forall h \geqq 0. \quad (3.22)$$

Now, it is easy to deduce (3.19) from (3.22).

4. Proof of Theorems

Lemma 4 and Lemma 5 ensure the existence of a sequence of solutions $\{\mathbf{v}_n\}_{n\in N}$ to system $(3.7)_n$, with integral estimates for $\mathbf{v}_n$ uniform with respect to n. Therefore, if $\mathbf{v}_0 \in J^2(\Omega)\cap J^p(\Omega)$, for some $p\in(1,2]$, it is routine to deduce the existence of a weak solution $\mathbf{v}$ to system (3.1) with properties a)–c) of Theorem 1. Theorem 1 can be considered acquired.

Remark 4. For our weak solution $\mathbf{v}$ it is possible to deduce results of partial regularity in the sense of [1]. However, since we are in a different context, we omit for the sake of brevity these results and we refer the reader to the works of [1, 15, 22].

If we take into account the regularity results of a solution to the Navier–Stokes equations obtained in [5] (cf. Theorem 3 and Theorem 5), we can consider the weak solution $\mathbf{v}$ sufficiently smooth $\forall t > T_0$, in such a way that we can consider the L^2-norms of derivatives of $\mathbf{v}$, $D_t^k D_x^h \mathbf{v}$ $h=1,2$ and $\forall K \in N$ and $\forall t > T_0$. *This last regularity of solution* $\mathbf{v}$ *we take into account in the next lemmas.*

We must preface the proof of Theorem 2 by some lemmas.

Lemma 6. *Let* $\mathbf{v}$ *be a weak solution to system (2.1) determined by Theorem 1. Assume that* $|\mathbf{v}(t)| = O(t^{-\alpha})$ *for some* $\alpha \geqq 0$ *and* $t \geqq T_0$. *Then*

$$|\nabla \mathbf{v}(t)| = O(t^{-\alpha-1/2}) \quad \forall t \geqq T_1 > T_0, \quad (4.1)$$

$$|P\Delta \mathbf{v}(t)| = O(t^{-\alpha-1}) \quad \forall t \geqq T_1 > T_0. \quad (4.2)$$

Proof. Since $\mathbf{v}$ is sufficiently smooth for $t > T_0$, we can consider system (2.1) in ordinary sense. Multiplying $(2.1)_1$ by $\mathbf{v}$ in $L^2(\Omega)$, we obtain, after integrating by parts,

$$(1/2)\frac{d}{dt}|\mathbf{v}(t)|^2 + |\nabla \mathbf{v}(t)|^2 = 0 \quad \forall t \geqq T_0, \quad (4.3)$$

which implies the following

$$|\nabla \mathbf{v}(t)|^2 \leqq |\mathbf{v}(t)|\,|\mathbf{v}_t(t)| \leqq Ct^{-\alpha}|\mathbf{v}_t(t)| \quad \forall t > T_0, \quad (4.4)$$

where we have taken into account that $|\mathbf{v}(t)| = O(t^{-\alpha})$ $\forall t \geqq T_0$. Deriving $(2.1)_1$ with respect to time and multiplying by $P\Delta \mathbf{v}_t$ we obtain

$$(1/2)\frac{d}{dt}|\nabla \mathbf{v}_t(t)|^2 + |P\Delta \mathbf{v}_t(t)|^2 = (\mathbf{v}\cdot\nabla \mathbf{v}_t, P\Delta \mathbf{v}_t) + (\mathbf{v}_t\cdot\nabla \mathbf{v}, P\Delta \mathbf{v}_t). \quad (4.5)$$

Applying the Hölder inequality we have

$$|(\mathbf{v}\cdot\nabla\mathbf{v}_t, PA\mathbf{v}_t)| \leqq \sup_\Omega |\mathbf{v}|\,|\nabla\mathbf{v}_t|\,|PA\mathbf{v}_t|,$$

$$|(\mathbf{v}_t\cdot\nabla\mathbf{v}, PA\mathbf{v}_t)| \leqq |\mathbf{v}_t|_6|\nabla\mathbf{v}|_3|PA\mathbf{v}_t|.$$

Since $\sup_\Omega |\mathbf{v}| \leqq C(|D^2\mathbf{v}| + |\nabla\mathbf{v}|)$, from (3.2) and (3.3) we can deduce

$$|(\mathbf{v}\cdot\nabla\mathbf{v}_t, PA\mathbf{v}_t)| + |(\mathbf{v}_t\cdot\nabla\mathbf{v}, PA\mathbf{v}_t)| \leqq C(|PA\mathbf{v}| + |\nabla\mathbf{v}|)|\nabla\mathbf{v}_t|\,|PA\mathbf{v}_t|. \tag{4.6}$$

Increasing the right-hand side of (4.5) by (4.6) and applying the Cauchy inequality,

$$\frac{d}{dt}|\nabla\mathbf{v}_t|^2 \leqq C|\nabla\mathbf{v}_t|^2(|PA\mathbf{v}|^2 + |\nabla\mathbf{v}|^2)$$

holds. Integrating the last inequality from $T_1 > T_0$, we obtain

$$\mathbf{v}_t \in L^\infty((T_1, \infty); L^2(\Omega)). \tag{4.7}$$

Now, we multiply $(2.1)_1$ by $\mathbf{v}_t$ in $L^2(\Omega)$, after integrating by parts, we deduce

$$2|\mathbf{v}_t|^2 = -\frac{d}{dt}|\nabla\mathbf{v}(t)|^2 + 2(\mathbf{v}_t\cdot\nabla\mathbf{v}_t, \mathbf{v}). \tag{4.8}$$

From (4.8), by application of the Schwartz inequality and the Hölder inequality with exponents $1/3 + 1/2 + 1/6 = 1$, it follows that

$$|\mathbf{v}_t|^2 \leqq |\nabla\mathbf{v}(t)|\,|\nabla\mathbf{v}_t| + |\mathbf{v}_t|_3|\nabla\mathbf{v}_t|\,|\mathbf{v}|_6,$$

which we can increase by (3.3) and (4.7) with

$$|\mathbf{v}_t|^2 \leqq C|\nabla\mathbf{v}|\,|\nabla\mathbf{v}_t|,$$

that implies by virtue of (4.4)

$$|\mathbf{v}_t|^3 \leqq Ct^{-\alpha}|\nabla\mathbf{v}_t|^2 \quad \forall t > T_1. \tag{4.9}$$

Deriving $(2.1)_1$ with respect to time and multiplying by $\mathbf{v}_t$ in $L^2(\Omega)$, after integrating by parts, we obtain

$$\frac{d}{dt}|\mathbf{v}_t|^2 + |\nabla\mathbf{v}_t|^2 \leqq |(\mathbf{v}_t\cdot\nabla\mathbf{v}, \mathbf{v}_t)| \quad \forall t > T_1.$$

Applying the Hölder inequality with exponents $1/3 + 1/2 + 1/6 = 1$, (3.3) and the Cauchy inequality, we have

$$\frac{d}{dt}|\mathbf{v}_t|^2 + |\nabla\mathbf{v}_t|^2 \leqq C|\mathbf{v}_t|^2|\nabla\mathbf{v}|^4 \quad \forall t > T_1.$$

By virtue of (4.4), (4.11) and $|\nabla\mathbf{v}(t)| \leqq C$ we deduce the differential inequality

$$\frac{d}{dt}|\mathbf{v}_t|^2 + C^{-1}t^\alpha|\mathbf{v}_t|^3 \leqq Ct^{-\alpha}|\mathbf{v}_t|^3 \quad \forall t > T_1.$$

Without loss of generality, we can assume that T_1 is such that $Ct^{-\alpha} \leqq C^{-1}t^\alpha/2$, so we obtain

$$\frac{d}{dt}|\mathbf{v}_t| + Ct^{\alpha}|\mathbf{v}_t|^2 \leqq 0,$$

which implies

$$|\mathbf{v}_t| = O(t^{-1-\alpha}) \quad \forall t \geqq T_1. \tag{4.10}$$

Inequalities (4, 4) and (4.10) imply (4.1). To obtain (4.2), we observe that multiplying $(2.1)_1$ by $P\Delta\mathbf{v}$ in $L^2(\Omega)$, we have

$$|P\Delta\mathbf{v}(t)|^2 \leqq |(\mathbf{v}\cdot\nabla\mathbf{v}, P\Delta\mathbf{v})| + |(\mathbf{v}_t, P\Delta\mathbf{v})|.$$

Applying the Schwartz inequality for $(\mathbf{v}_t, P\Delta\mathbf{v})$ and reasoning in the same way of (3.14) for $|(\mathbf{v}\cdot\nabla\mathbf{v}, P\Delta\mathbf{v})|$, we obtain

$$|P\Delta\mathbf{v}| \leqq C(|\nabla v|^3 + |\nabla\mathbf{v}|^2 + |\mathbf{v}_t|),$$

which implies (4.2) by (4.1) and (4.10).

Lemma 7. *Let* $\mathbf{v}$ *be a weak solution determined by Theorem 1. Then*

$$|\nabla\pi|_{p,\Omega} \leqq C(|\mathbf{v}\cdot\nabla\mathbf{v}|_p + |D^2\mathbf{v}|_{p,\Omega\cap\omega} \quad p\in(1, 6/5) \quad \forall t \geqq T_1, \tag{4.11}$$

where ω *is enclosed in* Ω *with* $\partial\Omega \cap \partial\omega = \phi$ *and* meas $\{\omega\} < \infty$.

Proof. From $(2.1)_1$ we deduce for π the following Neumann problem:

$$\Delta\pi = \nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) \quad \text{in} \quad \Omega \quad \forall t \geqq T_1,$$

$$\left.\frac{d\pi}{d\vec{n}}\right|_{\partial\Omega} = (\operatorname{rot}\operatorname{rot}\mathbf{v})\cdot\vec{n} - (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\vec{n}.$$

Inequality (4.11) is a consequence of Lemma 2.1 of [21] and Prop. 1.5 of [18]. In [21] there is the solution to the problem $\Delta q_1 = 0$ with $dq_1/d\vec{n} = (\operatorname{rot}\operatorname{rot}\mathbf{v})\cdot\vec{n}$ with $|\nabla q_1|_p \leqq C|D^2\mathbf{v}|_{L^p(\omega)}$ for $p\in(1, 6/5)$. In [18] there is the solution to the problem $\Delta q_2 = \nabla\cdot\mathbf{f}$ and $dq_2/d\vec{n} = \mathbf{f}\cdot\vec{n}$ with $|\nabla q_2|_p \leqq C|\mathbf{f}|_p$.

Remark 5. We can consider the boundary condition $d\pi/d\vec{n} = (\operatorname{rot}\operatorname{rot} v)\cdot \vec{n} - (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\vec{n}$, in the Neumann problem of π, by virtue of the regularity of $\mathbf{v}$ proved in [5]. In fact by the Remark on p. 665 of [5], we have $\mathbf{v}\in C^2$ if $\partial\Omega\in C^3$. As regards these properties of regularity see also [25].

We can increase (4.11) in the following way:

$$|\nabla\pi|_{p,\Omega} \leqq C(|\nabla\mathbf{v}|^{(4p-3)/p}|\mathbf{v}|^{(3-2p)/p} + |P\Delta\mathbf{v}| + |\nabla\mathbf{v}|) \quad \forall p\in(1, 6/5). \tag{4.12}$$

We obtain (4.12) applying the Hölder inequality to the right-hand side of (4.11) with exponents $p/2 + (2-p)/2 = 1$:

$$|\mathbf{v}\cdot\nabla\mathbf{v}|_p \leqq |\mathbf{v}|_{2p/(2-p)}|\nabla\mathbf{v}|$$

and

$$|D^2\mathbf{v}|_{p,\Omega\cap\omega} \leqq \{\text{meas}(\omega)\}^{(2-p)/2p}|D^2\mathbf{v}|,$$

taking into account of (3.3) for the norm $|\mathbf{v}|_{2p/(2-p)}$ and the mean $(\omega) < \infty$.

The following lemma improves an analogous lemma proved in [13]. However, both the lemmas have as a starting point the estimate in L^p of solutions to the Navier–Stokes equations given in the works by Galdi–Rionero (cf. [9]).

Lemma 8. *Let $\mathbf{v}_0 \in J^2(\Omega) \cap J^p(\Omega)$ be, for some $p \in (1,2)$ and $\mathbf{v}$ be a corresponding weak solution to system (2.1) of Theorem 1. Assume that $|\mathbf{v}(t)| = O(t^{-\alpha})(\alpha \geqq 0)$. Then there exists an instant T_2 such that $|\mathbf{v}(T_2)|_p < \infty$ and for $\alpha \neq (2-p)/2p$,*

$$|\mathbf{v}(t)|_p \leqq C(|\mathbf{v}(T_2)|_p) + [2C(2-p-2\alpha p)^{-1}|\mathbf{v}_0|(t^\beta - T_2^\beta)]^{1/p},$$
$$\text{where} \quad \beta = (1-\alpha p - p/2) \quad \text{and} \quad \forall t \geqq T_2, \tag{4.13}$$

and for $\alpha = (2-p)/2p$

$$|\mathbf{v}(t)|_p \leqq C(|\mathbf{v}(T_2)|_p) + C(|\mathbf{v}_0|)\log^{1/p}(t/T_2) \quad \forall t \geqq T_2. \tag{4.14}$$

Proof. Since $\mathbf{v} \in L^\infty((0,T); L^p(\Omega))$, if $p \in (1, 3/2]$ and $\mathbf{v} \in L^s((0,T), L^p(\Omega))$ for $p \in (3/2, 2)$ with $1/s + 3/2p > 1$ $\forall T > 0$, we can choose an instant $T_2 > T_1$ such that $|\mathbf{v}(T_2)|_p < \infty$. We multiply $(2.1)_1$ by $\Phi(r)\mathbf{v}/(\mathbf{v}^2(x,t)+\sigma)^{1-p/2}$ in $L^2(\Omega)$, where $\Phi(r) \in C^\infty(0,\infty)$ is a cut-off function such that $\Phi(r) = 1$ if $r \leqq R$ and $\Phi(r) = 0$ if $r \geqq 2R$ for $R > \operatorname{diam}(R^3 - \Omega)$, with $|\nabla\Phi(r)| \leqq C/R$ and $|\Delta\Phi(r)| \leqq C/R^2$, finally $\sigma = 1/R^4$. We integrate by parts over Ω:

$$\begin{aligned}
&(1/p)\frac{d}{dt}\int_\Omega (\mathbf{v}^2(x,t)+\sigma)^{\frac{p}{2}}\Phi(r)\,dx + \int_\Omega \nabla\mathbf{v}(x,t)\!:\!\nabla\mathbf{v}(x,t)(\mathbf{v}^2(x,t)+\sigma)^{\frac{p}{2}-1}\Phi(r)\,dx \\
&\quad + (p-2)\int_\Omega (\nabla\mathbf{v}(x,t)\cdot\mathbf{v}(x,t))^2(\mathbf{v}^2(x,t)+\sigma)^{\frac{p}{2}-2}\Phi(r)\,dx \\
&= 1/p\left(\int_\Omega (\mathbf{v}^2(x,t)+\sigma)^{\frac{p}{2}}(\Delta\Phi(r))\,dx + \int_\Omega (\mathbf{v}^2(x,t)+\sigma)^{\frac{p}{2}}\mathbf{v}(x,t)\cdot\nabla(\Phi(r))\,dx\right) \\
&\quad + \int_\Omega \nabla\pi(x,t)\cdot\mathbf{v}(x,t)(\mathbf{v}^2(x,t)+\sigma)^{\frac{p}{2}-1}\Phi(r)\,dx = \sum_{i=1}^3 I_i \quad \forall t \geqq T_2.
\end{aligned}$$

Since

$$\nabla\mathbf{v}(x,t);\nabla\mathbf{v}(x,t)(\mathbf{v}^2(x,t)+\sigma)^{\frac{p}{2}-1} + (p-2)(\nabla\mathbf{v}(x,t)\cdot\mathbf{v}(x,t))^2(\mathbf{v}^2(x,t)+\sigma)^{\frac{p}{2}-2} > 0$$
$$\forall (x,t) \in \Omega x(T_2,\infty),$$

we neglect the integrals with these terms. Now, applying the Hölder inequality with suitable exponents, we have

$$\begin{aligned}
I_1 &\leqq (C/R^2)\int_{R\leqq|x|\leqq 2R} |\mathbf{v}(x,t)|^p\,dx + (C/R^{2+2p}) \\
&\cdot \int_{R\leqq|x|\leqq 2R} dx \leqq (C/R^2)R^{3(2-p)/2}|\mathbf{v}(t)|^p + CR/R^{2p} \\
&\leqslant C(R)(|\mathbf{v}(t)|^p + 1) \quad \text{with} \quad C(R)\to 0 \quad \text{for} \quad R\to\infty; \\
I_2 &\leqq (C/R)\int_{R\leqq|x|\leqq 2R} |\mathbf{v}(x,t)|^{p+1}\,dx + (C/R^{2p+1}) \\
&\cdot \int_{R\leqq|x|\leqq 2R} dx \leqq (C/R)|\mathbf{v}(t)|_{p+1}^{p+1} + (C/R^{2p})R^2 \\
&\leqq C(R)(|\mathbf{v}(t)|_{p+1}^{p+1} + 1) \quad \text{with} \quad C(R)\to 0 \quad \text{for} \quad R\to\infty; \\
I_3 &\leqq C|\nabla\pi(t)|_{6/(7-p)}|\mathbf{v}(t)|_6^{p-1}.
\end{aligned}$$

Set $E_{\sigma,\Phi}(t)=\int_\Omega(\mathbf{v}^2(x,t)+\sigma)^{\frac{p}{2}}\Phi(r)\,dx$, taking into account of (4.12) and of (3.3) for $|\mathbf{v}(t)|_6$, we obtain

$$\frac{d}{dx}E_{\sigma,\Phi}(t)\leqq C|\nabla\mathbf{v}(t)|^p+C|\nabla\mathbf{v}(t)|^{p+1}+C|P\Delta\mathbf{v}(t)|\,|\nabla\mathbf{v}(t)|^{p-1}$$
$$+C(R)(|\mathbf{v}(t)|_{p+1}^{p+1}+1)+C(R)(|\mathbf{v}(t)|^{p+1}).$$

Integrating with respect to time and making $R\to\infty$, we deduce

$$E(t)\leqq E(T_1)+\sum_{i=1}^{3}J_i\quad\forall t\geqq T_2.$$

Since $p+1>2$ and $p-1>0$, from (4.1)–(4.2) we obtain $J_1+J_2\leqq C\ \forall t\geqq T_2$, while $J_3\leqq[2C/(2-p-2\alpha p)](t^\beta-T_2^\beta)$, where $\beta=1-\alpha p-p/2$ if $\alpha\neq(2-p)/2p$ and $J_3\leqq C\log(t/T_2)$ if $\alpha=(2-p)/2p$.

We are now in a position to prove Theorem 2. We obtain the result by Lemma 8. Set $\alpha=0$ in Lemma 8, from estimate (4.13) we deduce by interpolation the following estimate:

$$|\mathbf{v}(t)|_2\leqq|\mathbf{v}(t)|_p^{1-\theta}|v(t)|_6^\theta\leqq[C_1+C_2(t^\beta-T_2^\beta)^{1/p}]^{1-\theta}|v(t)|_6^\theta \tag{4.16}$$

with $\theta=3(2-p)/(6-p)\ \forall t\geqq T_2$. Taking into account the energy equality verified $\forall t\geqq T_2$, (4.16) and (3.3) for $|\mathbf{v}(t)|_6$, we obtain the differential inequality

$$\frac{d}{dt}|\mathbf{v}(t)|^2\leqq-C|\mathbf{v}(t)|^{2/\theta}[C_1+C_2(t^{1-p/2})^{1/p}]^{-2(1-\theta)/\theta}\quad\forall t\geqq T_2, \tag{4.17}$$

where we have taken into account that $1-p/2>0$ $p\in(1,2)$. On the other hand for $p\in(1,2)$ $(2-p)(1-\theta)/p\theta=2/3$, therefore from (4.17) we deduce

$$|\mathbf{v}(t)|=O(t^{(2-p)/4p})\quad\forall t\geqq T_2.$$

If we consider now (4.13) for $\alpha=(2-p)/4p$, (4.17) becomes

$$\frac{d}{dt}|\mathbf{v}(t)|^2\leqq-C|\mathbf{v}(t)|^{2/\theta}[C_1+C_2t^{(2-p)/4p}]^{-2(1-\theta)/\theta}\quad\forall t\geqq T_2,$$

where we have taken into account that $(2-p)/4>0$ $p\in(1.2)$. On the other hand it holds $\forall p\in(1,2)$ $(2-p)(1-\theta)/2p\theta=1/3$. Therefore, we deduce by analogous arguments that

$$|\mathbf{v}(t)|=O(t^{(2-p)/2p})\quad\forall t\geqq T_2. \tag{4.17}$$

Since in (4.17) $\alpha=(2-p)/2p$ we can consider (4.14), which implies

$$|\mathbf{v}(t)|=O(t^{-\alpha})\quad\forall t\geqq T_2\quad\text{and}\quad\alpha>(2-p)/2p.$$

Now for $\alpha>(2-p)/2p$, (4.13) is uniformly bounded, therefore (4.17) becomes

$$\frac{d}{dt}|\mathbf{v}(t)|^2\leqq-C|\mathbf{v}(t)|^{2/\theta}\quad\forall t\geqq T_2\quad\text{and}\quad\theta=3(2-p)/(6-\theta),$$

integrating this last differential inequality, we obtain (2.2).

The proof of Theorem 3 is quite analogous to the proof of Theorem 1.1 of [13], therefore it is omitted.

For the proof of Theorem 4 is important to premise the following theorem due to Knightly:

Theorem. *Let* $\mathbf{g}(x) = A\operatorname{rot}(0, F(x), H(x))$ *with*

$$F(x) = x_1(1+|x|^2)^{-(1+s)/2} \quad \text{if} \quad s \in [0,2),$$

$$H(x) = \begin{cases} -(1/2)\log(1+|x|^2) & \text{if} \quad s = 0 \\ (1/s)(1+|x|^2)^{-s/2} & \text{if} \quad s \in (0,2), \end{cases}$$

A is a suitable constant. Then there exists a unique solution $\mathbf{g}(x,t)$ *to system (2.1) corresponding to* $\mathbf{g}(x)$ *and defined* $\forall t \geqq 0$, *such that for* $p \in (1,2]$ $D_x^k D_t^h \mathbf{g}(x,t) \in J^2(R^3) \cap J^p(R^3)$ $\forall k,h \in N$ *if* $s \in ((3-p)/p, 2)$. *Moreover*

$$\sup_{\mathbb{R}^2} |\mathbf{g}(x,t)| \geqq |\mathbf{g}(0,t)| \geqq Ct^{-(1+s)/2} \text{ for } t \text{ sufficiently large.} \tag{4.18}$$

Proof. See [10] §.5 pp. 239–240.

We assume now that $\forall \mathbf{v}_0 \in J^2(R^3) \cap J^p(R^3)$, for some fixed $p \in (1,2]$, there exists a weak solution $\mathbf{v}$ corresponding to $\mathbf{v}_0$, such that

$$|\mathbf{v}(t)| = O(t^{-\mu-3(2-p)/4p}) \quad \text{for some} \quad \mu > 0.$$

By virtue of Lemma 6 we have $|P\Delta\mathbf{v}(t)| = |\Delta\mathbf{v}(t)| = O(t^{-1-\mu-3(2-p)/4p})$ and $|\nabla\mathbf{v}(t)| = O(t^{-\mu-1/2-3(2-p)/4p})$. Now, we consider (3.3) for $j=0$, $p=\infty$, $m=2$, $r=2$ and $q=6$. Therefore

$$\sup_{\mathbb{R}^3} |\mathbf{v}(x,t)| \leqq C|D^2\mathbf{v}(t)|^{1/2}|\mathbf{v}(t)|_6^{1/2}.$$

Since $|D^2\mathbf{v}(t)| \leqq |\Delta\mathbf{v}(t)|$ and $|\mathbf{v}(t)|_6 \leqq C|\nabla\mathbf{v}(t)|$, we can deduce

$$\sup_{\mathbb{R}^3} |\mathbf{v}(x,t)| \leqq O(t^{-\mu-3/4-3(2-p)/4p}). \tag{4.19}$$

If we observe that it is always possible to determine a $s = \varepsilon + (3-p)/p$ such that $\varepsilon/2 < \mu$ and $\mathbf{g}(x) \in J^2(R^3) \cap J^p(R^3)$, then from (4.18) and (4.19) we have

$$Ct^{-1/2(\varepsilon+3/p)} \leqq Ct^{-(\mu+3/2p)} \quad \text{for} \quad t \text{ sufficiently large},$$

which is absurd. This fact completes the proof of Theorem 4.

Acknowledgements. The author wishes to thank the referee for his valuable comments. This work was done under the auspices of GNFM of the Italian C.N.R. and on MPI 40% Contract.

References

1. Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations, Commun. Pure Appl. Math. **35**, 771–831 (1982)
2. Friedman, A.: Partial differential equations. New York: Holt, Rinehart and Winston 1969
3. Galdi, G. P., Rionero, S.: Weighted energy methods in fluid dynamics and elasticity. Lectures Notes in Mathematics, Vol. **1134**. Berlin, Heidelberg, New York: Springer 1985
4. Galdi, G. P., Maremonti, P.: Monotonic decreasing and asymptotic behaviour of the kinetic energy

for weak solutions of the Navier–Stokes equations in exterior domains. Arch. Rat. Mech. Anal. **94**, 253–266 (1986)
5. Heywood, J. G.: The Navier–Stokes equations: On the existence, regularity and decay of solutions. Indiana Univ. Math. J. **29**, 639–681 (1980)
6. Hopf, E.: Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen Math. Nachr. **4**, 213–231 (1951)
7. Kajikiva, R., Miyakawa, T.: On L^2 decay of weak solutions of the Navier–Stokes equations in R^n. Math. Z. **192**, 135–148 (1986)
8. Kato, T.: Strong L^p-solutions of the Navier–Stokes equation in R^n, with applications to weak solutions. Math. Z. **187**, 471–480 (1984)
9. Kiselev, A. A., Ladyzhenskaya, D. A.: On the existence and uniqueness of the solution of the nonstationary problem for a viscous incompressible fluid. Izv. Akad. Nauk SSSR Ser. Mat. **21**, 655–680 (1957)
10. Knightly, G. H.: On a class of global solutions of the Navier–Stokes equations. Arch. Rat. Mech. Anal. **21**, 211–245 (1966)
11. Leray, J.: Sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Math. **63**, 193–248 (1934)
12. Maremonti, P.: Asymptotic stability of incompressible viscous fluid motion in exterior domains. Rend. Sem. Mat. Univ. di Padova **71**, 35–72 (1984)
13. Maremonti, P.: Stabilità asintotica in media per moti fluidi viscosi in domini esterni., Ann. Mat. Pura Appl. **142**, 57–75 (1985)
14. Maremonti, P.: L^2 decay of suitable weak solutions to the Navier–Stokes equations in three-dimensional exterior domains. Proc. Workshop "Energy stability and convection", Capri May 1986, to be published by Longman
15. Maremonti, P.: Partial regularity of a generalized solution to the Navier–Stokes equations in exterior domain. Commun. Math. Phys. **110**, 75–87 (1987)
16. Masuda, K.: Weak solutions of the Navier–Stokes equations. Tohoku Math. J. **36**, 623–646 (1984)
17. Miranda, C.: Istituzioni di analisi funzionale lineare, Vol. **I–II**, Unione Matem. Italiana and C.N.R., Tip. Oderisi (ed.). Gubbio, Italy 1978
18. Miyakawa, T.: On nonstationary solutions of the Navier–Stokes equations in an exterior domain. Hiroshima Math. J. **12**, 115–140 (1982)
19. Schonbek, M. E.: L^2-decay for weak solutions of the Navier–Stokes equations. Arch. Rat. Mech. Anal. **89**, 209–222 (1985)
20. Schonbek, M. E.: Large time behaviour of solutions to the Navier–Stokes equations. Commun. Part. Differ. Equations **11**, 733–763 (1986)
21. Solonnikov, V. A.: Estimates for solutions of nonstationary Navier–Stokes equations. J. Sov. Math. **8**, 467–528 (1977)
22. Sohr, H., von Wahl, W.: On the regularity of the pressure of weak solutions of Navier–Stokes equations. Arch. Math. **46**, 428–439 (1986)
23. Sohr, H. von Wahl, W., Wiegner, M.: Zur asymptotik der gleichungen von Navier–Stokes. Nachr. Akad. Wiss. Göttingen **3**, 45–59 (1986)
24. Wiegner, M.: Decay results for weak solutions of the Navier–Stokes equations on R^n. J. Lond. Math. Soc. **35**, 303–313 (1987)
25. Galdi, G. P., Maremonti, P.: Sulla regolarità delle soluzioni deboli al sistema di Navier–Stokes in arbitrari domini di R^n, per per $n = 2, 3, 4$. (forthcoming)

Communicated by C. H. Taubes

Received November 9, 1987; in revised form March 16, 1988