Commun. Math. Phys. 106, 691–704 (1986)

Communications in Mathematical Physics

Spinors and Diffeomorphisms

L. Dąbrowski* and R. Percacci

International School for Advanced Studies, SISSA, I-34014 Trieste, Italy

Abstract. We discuss the action of diffeomorphisms on spinors on an oriented manifold M. To do this, we first describe the action of the diffeomorphism group $D(M)$ on the set $\Pi = H^1(M, Z_2)$ of inequivalent spin structures and show that it is affine. We argue that in the presence of spinors the gauge group of gravity is a certain double cover of $D(M)$ which depends on the spin structure. We explicitly compute the action of $D(M)$ on Π when M is a closed Riemann surface; Π is seen to consist of exactly two orbits, corresponding to even and odd spin structures.

1. Introduction

It is often said that spinor fields transform as scalars under diffeomorphisms and as "spinors" under local rotations of the orthonormal frames. While this statement is true at the level of local components, it does not specify the transformation behaviour of the spinors regarded as intrinsic geometric objects. Therefore, in handling global geometric properties of the spinors, anomalies and similar problems, it is convenient to have a coordinate-free description of the action of the diffeomorphism group. This is more complicated than the action of the diffeomorphism group on tensors, for the following reason. The tensorfields on a manifold form an infinite dimensional linear space and diffeomorphisms transform this linear space into itself. On the other hand, the definition of spinorfields on a manifold M requires a previous specification of a metric tensor; for each metric tensor there is a distinct space of spinorfields. There is no natural way of identifying these spaces. Therefore, the configuration space of coupled spinors and metric tensors is not the cartesian product of the separate configuration spaces but rather an infinite dimensional vectorbundle $\mathscr{W}$ over the configuration space of the metric tensors. The fiber of this bundle over a particular metric tensor g is precisely the space of spinors for g. Since diffeomorphisms transform the metric tensor by

* On leave of absence from I.F.T., University of Wrocław, Poland

$f: g \to f^*g$, the linear space of spinors over g will be transformed into the linear space of spinors over f^*g. Therefore, each f will define a vectorbundle morphism of $\hat{\mathscr{W}}$ which covers the action of f on metrics.

Additional complications arise when M is multiply connected: for each metric tensor there are inequivalent spin structures, and we have to keep track of the way in which diffeomorphisms which are not connected to the identity permute the spin structures.

We denote $D(M)$ the group of orientation preserving diffeomorphisms of M and $D_0(M)$ the normal subgroup of diffeomorphisms which are homotopic to the identity. The quotient $\Omega(M) = D(M)/D_0(M)$ is the group of connected components of $D(M)$. Given a spin structure α, we will denote $D(M, \alpha)$ the subgroup of $D(M)$ consisting of diffeomorphisms which preserve α. Since $D_0(M) \subset D(M, \alpha)$ for all α, we can define $\Omega(M, \alpha) = D(M, \alpha)/D_0(M)$.

In Sect. 2 we discuss the transformation of spinors for the spin structure α under diffeomorphisms $f \in D(M, \alpha)$. In the case of isometries this has been discussed in [1, 2]. Our general framework for spin structures is that of [3–6], but in order to make the discussion as independent of the metric as possible, we prefer to talk about prolongations of the bundle of linear frames to the double cover of the linear group.

In Sect. 3 we generalize the transformation rule to arbitrary $f \in D(M)$. Although the notion of transformation of spin structure is fairly natural, to the best of our knowledge it has not been defined precisely anywhere in print. We prove that the action of a diffeomorphism on the set of inequivalent spin structures is given by an affine transformation. We also observe that the action of $D(M)$ on spinorfields yields in general only a projective representation. If the diffeomorphism group is thought of as the gauge group of pure gravity, then this observation shows that in the presence of spinors the gauge group, in general, has to be extended to a double cover which depends on the spin structure. In Sect. 4 we study the special case of compact oriented two dimensional manifolds without boundary. This is directly relevant to the study of spinning strings and superstrings [7, 8]. Using only elementary methods from fiber bundle theory, we determine the action of $\Omega(M)$ on the set of spin structures. We show that this action has two orbits, consisting of even and odd spin structures. We also determine the double cover of $D(M, \alpha)$ for any genus and any spin structure.

Throughout this paper we work in the category of real smooth manifolds. We will not be concerned with the functional analytic properties of the diffeomorphism group.

2. The Action of Spin Structure Preserving Diffeomorphisms

Let M be an n-dimensional oriented manifold and let F be the bundle of oriented linear frames on M. It is a principal bundle with structure group $GL^+(n)$. A riemannian structure g is equivalent to a principal $SO(n)$ subbundle $O_g \subset F$, namely the bundle of frames which are orthonormal with respect to g. For simplicity we shall only consider the positive definite case from now on, but similar considerations apply also to other signatures of the metric and also to nonorientable manifolds.

In order to define spinors for the metric g we first have to give a spin structure, i.e. a principal $Spin(n)$-bundle $\tilde{O}_g$ over M together with a principal bundle homomorphism $\eta_g: \tilde{O}_g \to O_g$ over the identity of M [4]. A spinor field (of type T) is a section of the vectorbundle associated to $\tilde{O}_g$ via the spinor representation T of $Spin(n)$ in the linear space V, or, equivalently [9], a T-equivariant map from $\tilde{O}_g$ to V. Typically, T comes from a representation of the Clifford algebra of R^n.

In order to avoid having to define a spin structure independently for each g, we make use of the following construction. Let $\varrho: \widetilde{GL}^+(n) \to GL^+(n)$ be the (unique) nontrivial double cover (universal for $n \geqq 3$). The kernel of ϱ is isomorphic to Z_2 and is contained in the center of $\widetilde{GL}^+(n)$. The inverse image of $SO(n)$ under ϱ is precisely $Spin(n)$. Let $(\tilde{F}, \eta)$ be a prolongation of F to $\widetilde{GL}^+(n)$, i.e. $\tilde{F}$ is a principal $\widetilde{GL}^+(n)$-bundle over M and $\eta: \tilde{F} \to F$ is a principal bundle morphism over the identity of M (η is a double cover). Given a riemannian structure g, define $\tilde{O}_g = \eta^{-1}(O_g)$ and $\eta_g = \eta | \tilde{O}_g$. It is clear that $\tilde{O}_g \subset \tilde{F}$ is a principal $Spin(n)$-bundle and η_g is a principal bundle morphism $\tilde{O}_g \to O_g$ over the identity of M, so $(\tilde{O}_g, \eta_g)$ is a spin structure for g.

We have seen that if F admits a prolongation to $\widetilde{GL}^+(n)$, then O_g admits a prolongation to $Spin(n)$, for any g. Conversely, if a spin structure $(\tilde{O}_g, \eta_g)$ exists for some metric tensor g, we can use the action of $Spin(n)$ on $\widetilde{GL}^+(n)$ to construct the associated bundle $\tilde{F} = \tilde{O}_g \times_{Spin(n)} \widetilde{GL}^+(n)$ and we can define $\eta: \tilde{F} \to F$ by $\eta: [\tilde{e}, \tilde{a}] \mapsto [\eta_g(\tilde{e}), \varrho(\tilde{a})]$, where $(\tilde{e}, \tilde{a}) \in \tilde{O}_g \times \widetilde{GL}^+(n)$, and we canonically identify F with $O_g \times_{SO(n)} GL^+(n)$. $(\tilde{F}, \eta)$ is a prolongation of F to $\widetilde{GL}^+(n)$. Therefore, the topological condition for the existence of the prolongation $(\tilde{F}, \eta)$ is the same as the condition for the existence of a spin structure, namely the vanishing of the second Stiefel-Whitney class [3].

Two spin structures $(\tilde{O}_g, \eta_g)$ and $(\tilde{O}'_g, \eta'_g)$ are said to be equivalent if there exists a principal $Spin(n)$-bundle isomorphism $\beta_g: \tilde{O}_g \to \tilde{O}'_g$ such that $\eta'_g \circ \beta_g = \eta_g$. Similarly, two prolongations $(\tilde{F}, \eta)$ and $(\tilde{F}', \eta')$ are said to be equivalent if there exists a principal $\widetilde{GL}^+(n)$-bundle isomorphism $\beta: \tilde{F} \to \tilde{F}'$ such that $\eta' \circ \beta = \eta$. It can be seen that if two $\widetilde{GL}^+(n)$-prolongations $(\tilde{F}, \eta)$ and $(\tilde{F}', \eta')$ are equivalent, the spin structures $(\tilde{O}_g, \eta_g)$ and $(\tilde{O}'_g, \eta'_g)$ constructed as above are equivalent for each g. Conversely, if two spin structures for a fixed metric g are equivalent, the prolongations of F obtained by association as described above are also equivalent. Therefore, the set of equivalence classes of prolongations of F, denoted Π, is in bijective correspondence with the set Σ_g of equivalence classes of spin structures for any riemannian metric g.

It is known [6] that the cohomology group $H^1(M, Z_2)$ acts freely and transitively on Σ_g, and hence on Π. Therefore, Π is an affine space for the vectorspace $H^1(M, Z_2)$. If $(\tilde{F}, \eta)$ is a prolongation of F and $\alpha \in H^1(M, Z_2)$, then acting by α on $(\tilde{F}, \eta)$ we get another prolongation $(\tilde{F}', \eta')$ in the following way. Choose a good cover $\{U_A\}$ on M and a bundle atlas for $\tilde{F}$ with bundle charts $\tilde{\psi}_A: U_A \times \widetilde{GL}^+(n) \to \tilde{F}|_{U_A}$ and transition functions $\tilde{\varphi}_{AB}: U_A \cap U_B \to \widetilde{GL}^+(n)$; also, represent α by a Čech 1-cocycle $\alpha_{AB}: U_A \cap U_B \to Z_2$. Then $\tilde{F}'$ is constructed à la Steenrod [10] from transition functions

$$\tilde{\varphi}'_{AB} = \tilde{\varphi}_{AB} \cdot \alpha_{AB}, \tag{2.1}$$

and $\eta': \tilde{\psi}'_A(x, a) \mapsto \psi_A(x, \varrho(a))$, where ψ_A are bundle charts for F such that $\eta: \tilde{\psi}_A(x, a) \mapsto \psi_A(x, \varrho(a))$.

If we pick up an arbitrary element of Π, represented by some prolongation $(\tilde{F}_0, \eta_0)$ and denote $(\tilde{F}_\alpha, \eta_\alpha)$ the prolongation obtained acting with α on $(\tilde{F}_0, \eta_0)$, then we have a bijection $H^1(M, Z_2) \to \Pi$ by $\alpha \mapsto$(the equivalence class of $(\tilde{F}_\alpha, \eta_\alpha)$).

Now let $f \in D(M)$; f has a natural lift to an automorphism Tf of F. If we call $g' = f^* g$, then the automorphism Tf of F maps orthonormal frames for g' to orthonormal frames for g, and therefore defines an isomorphism of principal $SO(n)$ bundles $O_{g'} \to O_g$ over f, which we still denote Tf. Let us assume that the $GL^+(n)$-automorphism Tf of F lifts to a $\widetilde{GL}^+(n)$-automorphism $\tilde{Tf}$ of $\tilde{F}$ such that $\eta \circ \tilde{Tf} = Tf \circ \eta$. In this case we say that the diffeomorphism f does not change the spin structure (this is certainly the case if f is homotopic to the identity; we discuss the general condition for the existence of the lift in the next section). Restricting $\tilde{Tf}$ to $\tilde{O}_{g'} \subset \tilde{F}$ induces an isomorphism of the spin structures associated to g' and g denoted again $\tilde{Tf}: \tilde{O}_{g'} \to \tilde{O}_g$, such that $\eta_g \circ \tilde{Tf} = Tf \circ \eta_{g'}$.

The transformation law of a spinor field, regarded as an equivariant map $\phi: \tilde{O}_g \to V$, is $\phi' = \phi \circ \tilde{Tf}$. If the spinorfield is regarded as a section ψ of the associated bundle $S_g = \tilde{O}_g \times_{Spin(n)} V$, then ψ' is a section of $S_{g'} = \tilde{O}_{g'} \times_{Spin(n)} V$ given by $\psi' = \widehat{Tf}^{-1} \circ \psi \circ f$, where $\widehat{Tf}: S_{g'} \to S_g$ is the isomorphism defined by $\widehat{Tf}[\tilde{e}, v] = [\tilde{Tf}(\tilde{e}), v]$, for $\tilde{e} \in \tilde{O}_{g'}$, $v \in V$. We check that this action is consistent with the statement that "spinors are scalars under diffeomorphisms." Let e be a local field of orthonormal frames for g on an open set $U \subset M$, i.e. e is a section $e: U \to O_{g'}$ and let $\tilde{e}$ be a local field of spin frames for g on U, i.e. $\tilde{e}$ is a section $\tilde{e}: U \to \tilde{O}_g$ and $\eta_g \circ \tilde{e} = e$. The local representative of the spinor field on U is $\varphi: U \to V$ defined by $\varphi(x) = \phi(\tilde{e}(x))$. As before, $e' = Tf^{-1} \circ e \circ f$ is a local field of orthonormal frames for $g' = f^* g$ on $U' = f^{-1}(U)$ and $\tilde{e}' = \tilde{Tf}^{-1} \circ \tilde{e} \circ f$ is a local field of spin frames for g' on U' such that $\eta_{g'} \circ \tilde{e}' = e'$. We cannot give the components of the transformed spinorfield in the old basis $\tilde{e}$, because $\tilde{e}$ is not a spin frame for g'. The components of the transformed spinor fields in the transformed spin frame is $\varphi': U' \to V$ given by $\varphi'(x) = \phi'(\tilde{e}'(x)) = \phi(\tilde{e}(f(x))) = \varphi(f(x))$.

3. Diffeomorphisms Which Change the Spin Structure

We now generalize the discussion of the previous section to diffeomorphisms which change the spin structure. For each pair of prolongations $(\tilde{F}, \eta)$ and $(\tilde{F}', \eta')$ and each $f \in D(M)$, we define a class $\kappa \in H^1(M, Z_2)$ which gives the obstruction to the lifting of Tf to an isomorphism $\tilde{Tf}: \tilde{F}' \to \tilde{F}$ such that

$$\eta \circ \tilde{Tf} = Tf \circ \eta' . \tag{3.1}$$

Let $\{U_A\}$ be a good cover of M, and $\tilde{\psi}_A$ be bundle charts for $\tilde{F}$ with transition functions $\tilde{\varphi}_{AB}$. There is a bundle atlas for F with bundle charts ψ_A such that $\eta: \tilde{\psi}_A(x, \tilde{a}) \mapsto \psi_A(x, \varrho(\tilde{a}))$ and transition functions $\varphi_{AB} = \varrho \circ \tilde{\varphi}_{AB}$. We construct another bundle atlas for F: the new cover is $\{U'_A\}$ with $U'_A = f^{-1}(U_A)$, the new bundle charts are ψ'_A such that $\psi'_A(f^{-1}(x), a) = Tf^{-1} \psi_A(x, a)$ and the new transition functions are $\varphi'_{AB}(f^{-1}(x)) = \varphi_{AB}(x)$. It is always possible to find a bundle atlas $\{U'_A, \tilde{\psi}'_A\}$ for $\tilde{F}'$ (not unique) such that $\eta': \tilde{\psi}'_A(f^{-1}(x), a) \mapsto \psi'_A(f^{-1}(x), \varrho(a))$ for $x \in U_A$; the transition functions of this atlas are denoted $\tilde{\varphi}'_{AB}$ and they satisfy: $\varrho \circ \tilde{\varphi}'_{AB} = \varphi'_{AB}$. We define a Čech 1-cochain on the cover U_A by

$$\kappa_{AB}(x)=\tilde{\varphi}'_{AB}(f^{-1}(x))\cdot\tilde{\varphi}_{AB}(x)^{-1}\,. \tag{3.2}$$

It is easy to see that $\varrho\circ\kappa_{AB}=1$, so κ_{AB} has values in $Z_2\subset\widetilde{GL}^+(n)$ and is constant on $U_A\cap U_B$. Furthermore, using the fact that Z_2 is in the center of $\widetilde{GL}^+(n)$, we have for $x\in U_A\cap U_B\cap U_C$, $\kappa_{BC}(x)\kappa_{AC}(x)^{-1}\kappa_{AB}(x)=1$, and therefore κ_{AB} defines a Čech cohomology class $\kappa\in H^1(M,Z_2)$. If $\kappa=0$, there exist a 0-cochain $\lambda_A:U_A\to Z_2$ such that $\kappa_{AB}=\lambda_B\lambda_A^{-1}$, and in this case we can define $\tilde{T}f:\tilde{F}'\to\tilde{F}$ by $\tilde{T}f:\tilde{\psi}'_A(f^{-1}(x),a)\mapsto\tilde{\psi}_A(x,\lambda_A(x)a)$. It can be seen that this definition is independent of the trivialization and that $\eta\circ\tilde{T}f=Tf\circ\eta'$. Conversely, if such a $\tilde{T}f$ exists, then we can define $\lambda_A:U_A\to\widetilde{GL}^+(n)$ by $\tilde{T}f(\tilde{\psi}'_A(f^{-1}(x),a)=\tilde{\psi}_A(x,a)\cdot\lambda_A(x)$; applying η to both sides, we find $\varrho\circ\lambda_A=1$, so λ_A has values in Z_2. Comparing the definitions of λ_A and λ_B on $U_A\cap U_B$ and using the transition functions twice, we get $\varphi_{AB}(x)\lambda_B(x)=\varphi'_{AB}(f^{-1}(x))\lambda_A(x)$ from which it follows that $\kappa_{AB}=\lambda_B\cdot\lambda_A^{-1}$, so $\kappa=0$. We have thus proven

Proposition 1. *Given $f\in D(M)$ and two prolongations (F,η) and (F',η') of F, the automorphism Tf of F lifts to an isomorphism $\tilde{T}f:F'\to F$ with $\eta\circ\tilde{T}f=Tf\circ\eta'$ if and only if $\kappa=0$.*

If κ is nonzero, we can use it to pick a unique, up to equivalence, prolongation (F'',η'') such that Tf lifts to an isomorphism $\tilde{T}f:F''\to F$ with $\eta\circ\tilde{T}f=Tf\circ\eta''$. The bundle F'' is constructed à la Steenrod [10] on the cover $f^{-1}(U_A)$ from transition functions $\tilde{\varphi}''_{AB}:f^{-1}(U_A\cap U_B)\to\widetilde{GL}^+(n)$ defined by $\tilde{\varphi}''_{AB}(f^{-1}(x))=\kappa_{AB}(x)^{-1}\tilde{\varphi}'_{AB}(f^{-1}(x))=\tilde{\varphi}_{AB}(x)$. The homomorphism $\eta'':F''\to F$ is defined by $\eta''(\tilde{\psi}''_A(f^{-1}(x),a)=\psi'_A(f^{-1}(x),\varrho(a))$. The obstruction class for the lifting of Tf to $\tilde{T}f:F''\to F$, defined by the cocycle $\tilde{\varphi}''_{AB}(f^{-1}(x))\cdot\tilde{\varphi}_{AB}(x)^{-1}$, vanishes. We observe that $\tilde{F}''$ is the pullback of $\tilde{F}$ by f and, denoting $\iota:\tilde{F}''\to\tilde{F}$ the canonical map which arises in the pullback construction, $\eta''=Tf^{-1}\circ\eta\circ\iota$. Therefore the prolongation $(\tilde{F}'',\eta'')$ can be regarded as the pullback by Tf of the prolongation $(\tilde{F},\eta)$.

We define the action $p(f)$ of a diffeomorphism f on the set Π of equivalence classes of prolongations by assigning to the class $[\tilde{F},\eta]$ the unique class $[\tilde{F}',\eta']$ for which there exists a lift $\tilde{T}f:\tilde{F}'\to\tilde{F}$ with $\eta\circ\tilde{T}f=Tf\circ\eta'$. Since $p(f^{-1})\circ p(f)=Id$ and $p(f)\circ p(f^{-1})=\mathrm{Id}$, $p(f)$ is a permutation of Π. Furthermore $p(f_1\circ f_2)=p(f_1)\circ p(f_2)$ and it is clear that if f_1 is homotopic to f_2, $p(f_1)=p(f_2)$.

Having fixed an origin in the set Π, we have a bijection between Π and $H^1(M,Z_2)$, and hence we can regard $p(f)$ as a permutation of $H^1(M,Z_2)$. As in the previous section we denote $(\tilde{F}_\alpha,\eta_\alpha)$ a fixed representative in the equivalence class corresponding to $\alpha\in H^1(M,Z_2)$; given α, β, and f let $\kappa_f(\beta,\alpha)$ denote the obstruction class for the lifting of Tf to $\tilde{T}f:\tilde{F}_\beta\to\tilde{F}_\alpha$. Notice in particular $\kappa_{Id(M)}(0,\alpha)=-\alpha$. From (2.1) and (3.2) it is easy to see, that

$$\kappa_f(\alpha+\beta,\gamma+\delta)=\kappa_f(\alpha,\gamma)-\delta+(f^{-1})^*\beta\,. \tag{3.3}$$

The action of a diffeomorphism f on a Čech cohomology class ω is defined as follows: if ω is represented by a cocycle ω_{AB} on the covering $\{U_A\}$, then $f^*\omega$ is represented by the cocycle $(f^*\omega_{AB})(f^{-1}(x))=\omega_{AB}(x)$ on the covering $\{f^{-1}(U_A)\}$. If in the previous discussion we put $(\tilde{F},\eta)=(\tilde{F}_\alpha,\eta_\alpha)$, $(\tilde{F}',\eta')=(\tilde{F}_\beta,\eta_\beta)$, then it is clear that the prolongation $(\tilde{F}'', \eta'')$, which by definition is equivalent to $(\tilde{F}_{p(f)(\alpha)}, \eta_{p(f)(\alpha)})$, is obtained from $(\tilde{F}_\beta, \eta_\beta)$ by subtracting the pullback of the obstruction cocycle $\kappa_f(\alpha, \beta)_{AB}$. Therefore, at the level of cohomology classes

$$(p(f))(\alpha) = \beta - f^* \kappa_f(\beta, \alpha). \tag{3.4}$$

From (3.3) and (3.4) it follows that

$$(p(f))(\alpha) = f^*\alpha - f^*\kappa_f(0,0). \tag{3.5}$$

We can summarize these results in

Proposition 2. *The map $f \mapsto p(f)$ defines a homomorphism from $\Omega(M)$ to the group of affine transformations of Π.*

We can now give the transformation rule for spinors under arbitrary diffeomorphisms. Let ϕ be a spinorfield for the riemannian metric g and a spin structure $(\tilde{O}^\alpha_g, \eta^\alpha_g)$ with $\tilde{O}^\alpha_g \subset \tilde{F}_\alpha$, i.e. ϕ is an equivariant map $\tilde{O}^\alpha_g \to V$. Let f be a diffeomorphism and $\beta \in H^1(M, Z_2)$ be such that Tf lifts to an isomorphism $\tilde{T}f : \tilde{F}_\beta \to \tilde{F}_\alpha$ with $\eta_\alpha \circ \tilde{T}f = Tf \circ \eta_\beta$ i.e. $\beta = p(f)(\alpha)$. With $g' = f^*g$ we construct the spinstructure $(\tilde{O}^\beta_{g'}, \eta^\beta_{g'})$, with $\tilde{O}^\beta_{g'} \subset \tilde{F}_\beta$; $\tilde{T}f$ restricts to an isomorphism $\tilde{O}^\beta_{g'} \to \tilde{O}^\alpha_g$. The transformed spinorfield is a spinorfield for the pullback metric g' and the transformed spin structure β, defined by $\phi' = \phi \circ \tilde{T}f$.

The lift $\tilde{T}f$ of a diffeomorphism f is not unique. Assume that there are two isomorphisms u and u' from $\tilde{F}'$ to $\tilde{F}$ such that $\eta \circ u = Tf \circ \eta'$ and $\eta \circ u' = Tf \circ \eta'$. It follows that $u' \circ u^{-1}$ is an automorphism of $\tilde{F}$ over the identity and $u' \circ u^{-1} \circ \psi_A(x, a) = \psi_A(x, a\lambda_A)$, where λ_A is a constant map from U_A to Z_2. The choice of λ_A in Z_2 over U_A determines λ_B for all other charts U_B. Indeed, it follows from the definition of λ_A and λ_B that $\lambda_A = \lambda_B$ if $U_A \cap U_B \neq \emptyset$. Therefore, there are exactly two lifts of f and they differ by the automorphism γ of $\tilde{F}$, where γ is the multiplication by the generator of $Z_2 \subset \widetilde{GL}^+(n)$.

In general, it is not possible to choose consistently one lift $\tilde{T}f$ for all $f \in D(M)$, i.e. the composition rule holds only up to $Z_2 = \{Id_F, \gamma\}$. For simplicity consider the group $D(M, \alpha)$ preserving the spin structure α. In general, $D(M, \alpha)$ will only act projectively on the spinors in the spin structure α. In order to have a group which acts on spinors in the ordinary sense, we have to go to a double cover. Let $\tilde{D}(M, \alpha)$ be the group of $\widetilde{GL}^+(n)$-automorphisms (u, f) of $\tilde{F}$ such that $\eta_\alpha \circ u = Tf \circ \eta_\alpha$. The homomorphism $\tilde{D}(M, \alpha) \to D(M, \alpha)$ defined by $(u, f) \to f$ is a double cover and the map $(u, f) : \phi \mapsto \phi' = \phi \circ u$ is a true (not a projective) representation of $\tilde{D}(M, \alpha)$ on spinorfields.

4. Spin Structures and Diffeomorphisms on Riemann Surfaces

We now apply our general formalism to the case when M is a compact connected oriented 2-dimensional manifold without boundary, i.e. the real manifold underlying a closed Riemann surface. Topologically, M is entirely characterized by its genus g (number of handles). In most of this section we consider the case $g \geq 1$. We visualize the surface M as in Fig. 1.

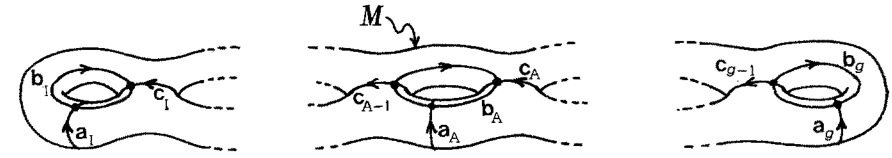

Fig. 1

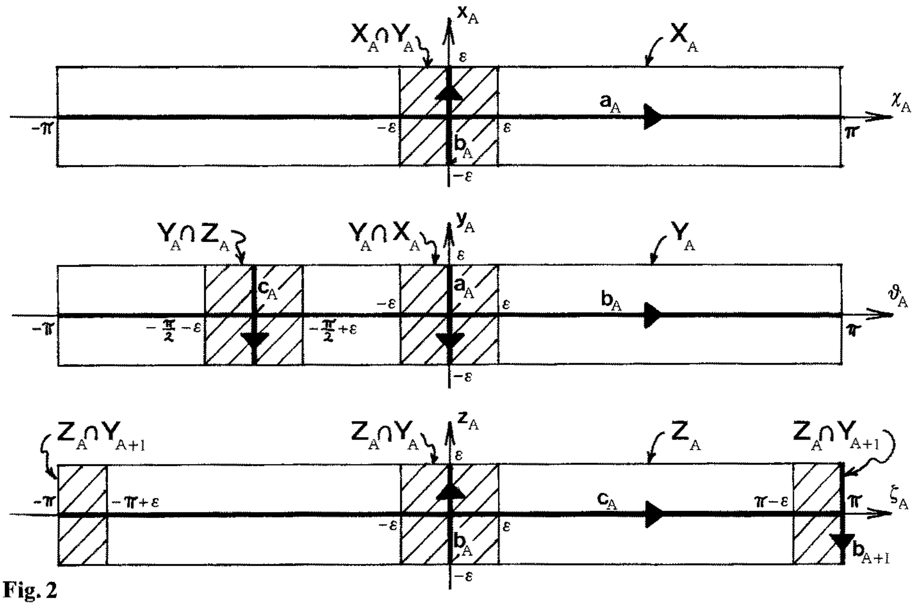

Fig. 2

The loops a_A and b_A for $A=1, \ldots, g$ generate the group $H_1(M, Z)=(Z)^{2g}$. The Euler characteristic of M is $\chi=2-2g$; the second Stiefel-Whitney class of M is the Euler class mod2, so M admits spin structures. Since $H^1(M, Z_2)=(Z_2)^{2g}$, there are 2^{2g} inequivalent spin structures. In order to describe them, we start by introducing coordinates (χ_A, x_A), (ϑ_A, y_A), and (ζ_A, z_A) in open neighbourhoods X_A, Y_A, and Z_A, each with the topology of a cylinder, of the loops a_A, b_A, and c_A respectively. The coordinates χ_A, ϑ_A, and ζ_A are periodic with period 2π, and we take the fundamental domain to be $(-\pi, \pi]$; the coordinates x_A, y_A, and z_A run from $-\varepsilon$ to ε with ε sufficiently small. On the intersection regions, the coordinate transformations are given by the following table:

$$\text{on } X_A \cap Y_A \qquad \begin{array}{lll} \chi_A=-y_A & \text{for} & -\varepsilon<\chi_A<\varepsilon \\ x_A=\vartheta_A & \text{for} & -\varepsilon<\vartheta_A<\varepsilon, \end{array} \tag{4.1}$$

$$\text{on } Y_A \cap Z_A \qquad \begin{array}{lll} \vartheta_A=-\frac{\pi}{2}+z_A & \text{for} & -\frac{\pi}{2}-\varepsilon<\vartheta_A<-\frac{\pi}{2}+\varepsilon \\ y_A=-\zeta_A & \text{for} & -\varepsilon<\zeta_A<\varepsilon, \end{array} \tag{4.2}$$

$$\text{on } Y_{A+1} \cap Z_A \qquad \begin{array}{lll} \zeta_A=-\pi+y_{A+1} & \text{for} & -\pi<\zeta_A<-\pi+\varepsilon \\ \zeta_A=\pi+y_{A+1} & \text{for} & \pi-\varepsilon<\zeta_A<\pi \\ z_A=\frac{\pi}{2}-\vartheta_{A+1} & \text{for} & \frac{\pi}{2}-\varepsilon<\vartheta_{A+1}<\frac{\pi}{2}+\varepsilon. \end{array} \tag{4.3}$$

In the following we shall regard the loops a_A, b_A, and c_A as the images of maps $S^1 \to M$, which we denote by the same symbols. In local coordinates $-\pi < s < \pi$ on S^1 and on X_A, Y_A, and Z_A we have $a_A : s \mapsto (\chi_A, x_A) = (s, 0)$, $b_A : s \mapsto (\vartheta_A, y_A) = (s, 0)$, and $c_A : s \mapsto (\zeta_A, z_A) = (s, 0)$, respectively (see Fig. 2).
We now describe the bundle of frames F. It is sufficient to consider a cover of M consisting of two open sets U and U', where

$$U = \left(\bigcup_{A=1}^{g} X_A\right) \cup \left(\bigcup_{A=1}^{g} Y_A\right) \cup \left(\bigcup_{A=1}^{g-1} Z_A\right),$$

$$U' = M \setminus \left(\bigcup_{A=1}^{g-1} c_A\right).$$

We fix the local trivializations of F by giving smooth fields of frames $e = (e_1, e_2)$ on U and $e' = (e'_1, e'_2)$ on U'. The open set U' is topologically a punctured torus; a simple form for e' is

$$e' = \left(\frac{\partial}{\partial \chi_A}, \frac{\partial}{\partial x_A}\right) \quad \text{on } X_A, \tag{4.4}$$

$$e' = \left(-\frac{\partial}{\partial y_A}, \frac{\partial}{\partial \vartheta_A}\right) \quad \text{on } Y_A, \tag{4.5}$$

$$e' = \left(\frac{\partial}{\partial \zeta_A}, \frac{\partial}{\partial z_A}\right) R\left(-\lambda\left(\frac{\zeta_A - \varepsilon}{\pi - 2\varepsilon}\right) - \lambda\left(\frac{\zeta_A + \pi - \varepsilon}{\pi - 2\varepsilon}\right) + \pi\right) \quad \text{on } Z_A^-, \tag{4.6}$$

$$e' = \left(\frac{\partial}{\partial \zeta_A}, \frac{\partial}{\partial z_A}\right) R\left(\lambda\left(\frac{\zeta_A - \varepsilon}{\pi - 2\varepsilon}\right) + \lambda\left(\frac{\zeta_A + \pi - \varepsilon}{\pi - 2\varepsilon}\right) - \pi\right) \quad \text{on } Z_A^+. \tag{4.7}$$

Here $Z_A^{\pm} = \{p \in Z_A | z_A \gtrless 0\}$, $R(s)$ is the matrix representing an $SO(2)$ notation by the angle s in the counterclockwise direction and λ is a smooth real function such that $\lambda(s) = 0$ for $s \leqq 0$ and $\lambda(s) = \pi$ for $s \geqq 1$. The region of U' where e' is not specified by (4.4)–(4.7) consists of g disjoint disks; the precise form of e' on this region is irrelevant for our purposes.

We choose the field of frames e on U as follows:

$$e = \left(\frac{\partial}{\partial \chi_A}, \frac{\partial}{\partial x_A}\right) \quad \text{on } X_A, \tag{4.8}$$

$$e = \left(-\frac{\partial}{\partial y_A}, \frac{\partial}{\partial \vartheta_A}\right) \quad \text{on } Y_A, \tag{4.9}$$

$$e = \left(\frac{\partial}{\partial \zeta_A}, \frac{\partial}{\partial z_A}\right) R\left(-\lambda\left(\frac{\zeta_A - \varepsilon}{\pi - 2\varepsilon}\right) - \lambda\left(\frac{\zeta_A + \pi - \varepsilon}{\pi - 2\varepsilon}\right) + \pi\right) \quad \text{on } Z_A. \tag{4.10}$$

The intersection $U \cap U'$ is $U \setminus \left(\bigcup_A c_A\right)$. The transition function $\varphi = \varphi_{UU'} : U \cap U' \to GL^+(2)$ defined by $e' = e\varphi$ is $\mathbb{1}$ everywhere except on each Z_A^+, where

$$\varphi(\zeta_A, z_A) = R\left(2\lambda\left(\frac{\zeta_A - \varepsilon}{\pi - 2\varepsilon}\right) + 2\lambda\left(\frac{\zeta_A + \pi - \varepsilon}{\pi - 2\varepsilon}\right)\right). \tag{4.11}$$

The rotation matrix on the right-hand side of (4.10) is important. In working with spin structures it is convenient to have a bundle atlas for F whose transition functions are of the form $\varphi_{AB}=\varrho\circ\tilde{\varphi}_{AB}$, where $\tilde{\varphi}_{AB}$ are transition functions for $\tilde{F}$. This implies in particular that for any loop $\ell:S^1\to U_A\cap U_B$ the map $\varphi_{AB}\circ\ell$ has even winding number. The rotation in (4.10) ensures that the function $\varphi\circ\ell$ always has even winding number [e.g. winding number two on $\ell:s\mapsto(\zeta_A,z_A)=(s,\varepsilon/2)$].

The maximal compact subgroup $SO(2)$ of $GL^+(2)$ is isomorphic to $U(1)$, so F is topologically characterized by its first Chern class. Since the transition function (4.11) has winding number two on each of the $g-1$ regions Z_A^+, $c_1(F)=2(g-1)$. The maximal compact subgroup $Spin(2)$ of $\widetilde{GL}^+(2)$ is also isomorphic to $U(1)$ and the covering homomorphism ϱ can be written $\varrho:e^{i\alpha}\mapsto e^{2i\alpha}$. Therefore, $\tilde{F}$ is topologically characterized by $c_1(\tilde{F})=g-1$. This shows that we can use the same principal bundle $\tilde{F}$ to describe all prolongations. $\tilde{F}$ can be trivialized on the open cover $\{U,U'\}$. We call $\tilde{e}$ and $\tilde{e}'$ local sections of $\tilde{F}$ over U and U' respectively; the transition function $\tilde{\varphi}=\tilde{\varphi}_{UU'}:U\cap U'\to\widetilde{GL}^+(2)$ defined by $\tilde{e}'=\tilde{e}\tilde{\varphi}$, satisfies $\varrho\circ\tilde{\varphi}=\varphi$. The inequivalent prolongations arise from inequivalent bundle homomorphisms $\tilde{F}\to F$, which we now describe.

For each $\underline{i}=(i_1,\dots,i_g)\in(Z_2)^g$ and $\underline{j}=(j_1,\dots,j_g)\in(Z_2)^g$ with $Z_2=\{0,1\}$, we define the homomorphism $\eta_{\underline{ij}}:\tilde{F}\to F$ by its action on local sections

$$\begin{aligned}\eta_{\underline{ij}}(\tilde{e}(p))&=e(p)r_{\underline{ij}}(p)\quad\text{for}\quad p\in U\,,\\ \eta_{\underline{ij}}(\tilde{e}'(p))&=e'(p)\quad\text{for}\quad p\in U'\backslash(U'\cap U)\,,\end{aligned}\tag{4.12}$$

and extending it by equivariance. The function $r_{\underline{ij}}:U\to SO(2)\subset GL^+(2)$ is given by

$$r_{\underline{ij}}(p)=1\quad\text{for}\quad p\in U\backslash(X_A\cup Y_A)\,,\tag{4.13}$$

$$r_{\underline{ij}}(p)=R\left(2i_A\lambda\left(\frac{y_A}{\varepsilon}\right)\right)\quad\text{for}\quad p=(\vartheta_A,y_A)\in Y_A\backslash(X_A\cap Y_A)\,,\tag{4.14}$$

$$r_{\underline{ij}}(p)=R\left(2j_A\lambda\left(\frac{x_A}{\varepsilon}\right)\right)\quad\text{for}\quad p=(\chi_A,x_A)\in X_A\backslash(X_A\cap Y_A)\,,\tag{4.15}$$

$$r_{\underline{ij}}(p)=R\left(2i_A\lambda\left(\frac{y_A}{\varepsilon}\right)+j_A\lambda\left(\frac{x_A}{\varepsilon}\right)\right)\quad\text{for}\quad p=(x_A,y_A)\in X_A\cap Y_A\,.\tag{4.16}$$

The homomorphisms $\eta_{\underline{ij}}$ and $\eta_{\underline{i}'\underline{j}'}$ define equivalent prolongations if and only if there exists an automorphism β of $\tilde{F}$ such that $\eta_{\underline{ij}}\circ\beta=\eta_{\underline{i}'\underline{j}'}$. In a local trivialization this implies

$$\varrho(\tilde{h})=(r_{\underline{ij}})^{-1}r_{\underline{i}'\underline{j}'}\,,\tag{4.17}$$

where $\beta(\tilde{e})=\tilde{e}\tilde{h}$. Now, compose both sides of (4.17) with the loop $a_A:S^1\to U$. From (4.1), (4.15), and (4.16) we have $r_{\underline{i}'\underline{j}'}\circ a_A(s)=R(i'_A\lambda(s))$, so this map has winding number i'_A. Similarly $r_{\underline{ij}}\circ a_A(s)$ has winding number i_A. Since $\varrho\circ\tilde{h}\circ a_A$ has always an even winding number, $i'_A=i_A$. Repeating this reasoning for all homology generators, we find $\underline{i}'=\underline{i}$ and $\underline{j}'=\underline{j}$. Therefore, $\tilde{F}$ together with homomorphisms $\eta_{\underline{ij}}$ provide all 2^{2g} inequivalent prolongations on the Riemann surface of genus g.

Note that the labels $(\underline{i},\underline{j})\in Z^{2g}$ can be regarded as the homomorphism from $H_1(M,Z_2)$ to Z_2 which assigns to a loop ℓ the (mod 2) winding number of $r_{\underline{ij}}\circ\ell$.

Since $\mathrm{Hom}(H_1(M,Z_2),Z_2)\approx H^1(M,Z_2)$ the relation with the Čech cohomological classification of prolongations in Sect. 3 is evident.

The group of connected components $\Omega(M)$ (the mapping class group) is generated by $3g-1$ Dehn twists associated to the loops a_A, b_A, and c_A [11, 12] (see Fig. 1). These diffeomorphisms we take in the form

$$f_{a_B}:(\chi_B,x_B)\mapsto\left(\chi_B+2\lambda\left(\frac{x_B+\varepsilon}{\varepsilon}\right),x_B\right)\quad\text{on } X_B, \tag{4.18}$$

$$f_{b_B}:(\vartheta_B,y_B)\mapsto\left(\vartheta_B+2\lambda\left(\frac{y_B+\varepsilon}{\varepsilon}\right),y_B\right)\quad\text{on } Y_B, \tag{4.19}$$

$$f_{c_B}:(\zeta_B,z_B)\mapsto\left(\zeta_B+2\lambda\left(\frac{z_B+\varepsilon}{\varepsilon}\right),z_B\right)\quad\text{on } Z_B, \tag{4.20}$$

and equal to the identity elsewhere.

We determine the action of the Dehn twists on the set of equivalence classes of prolongations (spin structures) by generalizing the method used to prove that $\eta_{\underline{i}\underline{j}}$ and $\eta_{\underline{i}'\underline{j}'}$ are equivalent if and only if $\underline{i}=\underline{i}'$, $\underline{j}=\underline{j}'$.

Let f be a diffeomorphism of M whose support is contained in the domain U of a local trivialization of F (this is the case for all Dehn twists). Let $h:U\to GL^+(2)$ and $\tilde{h}:U\to\widetilde{GL}^+(2)$ be local descriptions of Tf and $\tilde{T}f$ respectively, i.e. for $p\in U$, $Tf(e(p))=e(f(p))h(p)$, and $\tilde{T}f(\tilde{e}(p))=\tilde{e}(f(p))\tilde{h}(p)$. Equation (3.1) implies that

$$r_{\underline{i}\underline{j}}(f(p))\varrho(\tilde{h}(p))=h(p)r_{\underline{i}'\underline{j}'}(p)\,. \tag{4.21}$$

Given $\underline{i},\underline{j}$; $\underline{i}',\underline{j}'$ and f, the function $\tilde{h}$ exists only if the map

$$s\mapsto(r_{\underline{i}\underline{j}}\circ f\circ\ell(s))^{-1}\cdot(h\circ\ell(s))\cdot(r_{\underline{i}'\underline{j}'}\circ\ell(s)) \tag{4.22}$$

has even winding number for any loop $\ell:S^1\to U$. Applying this condition to all the cohomology generators will be sufficient to determine completely $\underline{i}',\underline{j}'$ in terms of f and $\underline{i},\underline{j}$.

We begin by applying the method to f_{a_B} for a fixed B and a loop a_A for $A\neq B$. The map (4.22) takes the form

$$s\mapsto(r_{\underline{i}\underline{j}}\circ a_A(s))^{-1}\cdot(r_{\underline{i}'\underline{j}'}\circ a_A(s))\,,$$

so its winding number is i'_A-i_A. This implies $i'_A=i_A$. Similarly, replacing a_A by b_A, for $A\neq B$, we get $j'_A=j_A$.

Next we apply the method to the Dehn twist f_{a_B} and the loop a_B. From (4.15) and (4.16), the winding number of $r_{\underline{i}'\underline{j}'}\circ a_B$ is i'_B. Since a_B is fixed by the Dehn twist f_{a_B}, the winding number of $r_{\underline{i}\underline{j}}\circ f_{a_B}\circ a_B$ is i_B. From (4.2), and (4.18) it follows that

$$h(\chi_B,x_B)=\begin{pmatrix}1 & \frac{2}{\varepsilon}\lambda'\left(\frac{x_B+\varepsilon}{\varepsilon}\right)\\ 0 & 1\end{pmatrix}$$

on X_B, where $\lambda'(s)=d\lambda/ds$. Thus, the winding number of $h\circ a_B$ is zero. Altogether, the right-hand side of Eq. (4.22) has the winding number $i_B-i'_B$, so we find again $i'_B=i_B$. For the Dehn twist f_{a_B} and the loop b_B, the third term on the right-hand side of Eq. (4.22) contributes winding number j'_B. The second term is

$$h \circ b_B(s) = \begin{pmatrix} 1 & \frac{2}{\varepsilon}\lambda'\left(\frac{s+\varepsilon}{\varepsilon}\right) \\ 0 & 1 \end{pmatrix}$$

and has winding number zero. Consider now the first term. For $-\pi \leqq s \leqq -\varepsilon$, $r_{\underline{i}\underline{j}} \circ f_{a_B} \circ b_B(s) = 1$; for $-\varepsilon \leqq s \leqq 0$ we have in coordinates (χ_B, x_B) $r_{\underline{i}\underline{j}} \circ f_{a_B} \circ b_B(s) = r_{\underline{i}\underline{j}}\left(2\lambda\left(\frac{s+\varepsilon}{\varepsilon}\right), s\right)$ so there is a contribution i_B to the winding number; for $0 \leqq s \leqq \varepsilon$ there is a contribution j_B to the winding number and for $\varepsilon \leqq s \leqq \pi$, $r_{\underline{i}\underline{j}} \circ f_{a_B} \circ b_B(s) = 1$. Altogether the map $r_{\underline{i}\underline{j}} \circ f_{a_B} \circ b_B$ has winding number $i_B + j_B$. The total winding number on the right-hand side of Eq. (4.22) is $j'_B + j_B + i_B$. So we find $j'_B = j_B + i_B$. Thus, the transformation rule of spin structures under the Dehn twist f_{a_B} is $(\underline{i}, \underline{j}) \mapsto (\underline{i}', \underline{j}')$ with $i'_A = i_A$, $j'_A = j_A + \delta_{AB} i_B$, where δ_{AB} is the Kronecker symbol.

Repeating the same arguments for the Dehn twist f_{b_B}, we get $i'_A = i_A + \delta_{AB} j_B$, $j'_A = j_A$.

Since only the loops b_B and b_{B+1} intersect the support of the Dehn twist f_{c_B}, it is clear from the previous considerations that $i'_A = i_A$ for all A and $j'_A = j_A$ for $A \neq B$ and $A \neq B+1$. Consider now the Dehn twist f_{c_B} and the loop $b_B : S^1 \to U$. The functions $r_{\underline{i}'\underline{j}'} \circ b_B$ and $r_{\underline{i}\underline{j}} \circ f_{c_B} \circ b_B$ have winding numbers j'_B and $j_B + i_B + i_{B+1}$, respectively. This time however, the term $h \circ b_B(s)$ has a nonvanishing winding number due to the nontrivial rotation we had to introduce in (4.10). From (4.10) and (4.20) we have

$$h(\zeta_A, z_A) = R\left[\lambda\left(\frac{\zeta_A + 2\lambda\left(\frac{z_A+\varepsilon}{\varepsilon}\right) - \varepsilon}{\pi - 2\varepsilon}\right) + \lambda\left(\frac{\zeta_A + 2\lambda\left(\frac{z_A+\varepsilon}{\varepsilon}\right) + \pi - \varepsilon}{\pi - 2\varepsilon}\right) + \pi\right]$$
$$\times \begin{bmatrix} 1 & \frac{2}{\varepsilon}\lambda'\left(\frac{z_A+\varepsilon}{\varepsilon}\right) \\ 0 & 1 \end{bmatrix} \times R\left[-\lambda\left(\frac{\zeta_A - \varepsilon}{\pi - 2\varepsilon}\right) - \lambda\left(\frac{\zeta_A + \pi - \varepsilon}{\pi - 2\varepsilon}\right) - \pi\right]. \tag{4.23}$$

For $-\frac{\pi}{2} - \varepsilon \leqq s \leqq -\frac{\pi}{2}$, $b_B : s \mapsto (\zeta_A, z_A) = (0, s + \frac{\pi}{2})$; therefore the last term of (4.23) is constant, the second has winding number 0 and the first 1.

The total winding number on the right-hand side of (4.22) is $j_B + j'_B + i_B + i_{B+1} + 1$, therefore we find $j'_B = j_B + i_B + i_{B+1} + 1$. Exactly in the same way we find for the Dehn twist f_{c_B} and the loop b_{B+1}, $j'_{B+1} = j_{B+1} + i_B + i_{B+1} + 1$.

We summarize the transformation rule of spinstructures under Dehn twists:

$$p(f_{a_B}) : \begin{array}{l} i_A \to i_A \\ j_A \to j_A + \delta_{AB} i_B \end{array},$$
$$p(f_{b_B}) : \begin{array}{l} i_A \to i_A + \delta_{AB} j_B \\ j_A \to j_A \end{array}, \tag{4.24}$$
$$p(f_{c_B}) : \begin{array}{l} i_A \to i_A \\ j_A \to j_A + (\delta_{AB} + \delta_{A,B+1})(i_B + i_{B+1} + 1) \end{array}.$$

Consider the quadratic function on Π with values in Z_2 [13]

$$\varphi(\underline{i},\underline{j})=\sum_{A=1}^{g}(i_A+1)(j_A+1). \tag{4.25}$$

A prolongation (spin structure) is said to be even (respectively odd) if $\varphi(\underline{i},\underline{j})$ is $0 \bmod 2$ (respectively $1 \bmod 2$). The prolongation is even if an even number of terms in (4.25) is one, and this can happen in $\sum_{(k\ \text{even})}\binom{n}{k}3^{g-k}=2^{g-1}(2^g+1)$ ways. Similarly, the prolongation is odd if an odd number of terms in (4.25) is one, and this can happen in $\sum_{(k\ \text{odd})}\binom{n}{k}3^{g-k}=2^{g-1}(2^g-1)$ ways. It is easily checked that φ is invariant under the transformations (4.24); therefore the property of a prolongation of being even or odd is preserved under arbitrary diffeomorphisms.

We will now show that the Dehn twists act transitively on the sets Π^+ and Π^- of even and odd prolongations.

Using Dehn twists it is possible to transform any prolongation $(i_1,\dots,i_g;j_1,\dots,j_g)$ to the standard form

$$(\underbrace{0,\dots,0}_{g};\underbrace{1,\dots,1}_{g}) \quad \text{if } \underline{i},\underline{j} \text{ is even}$$

and

$$(\underbrace{0,\dots,0}_{g};\underbrace{1,\dots,1}_{g-1},0) \quad \text{if } \underline{i},\underline{j} \text{ is odd}.$$

If $(i_1,j_1)=(0,1)$, $(1,0)$ or $(1,1)$, we apply Id_M, $f_{b_1}\circ f_{a_1}$ or f_{b_1} respectively and obtain $(0,i_2,\dots,i_g;1,j_2,\dots,j_g)$. If $(i_1,i_2,j_1,j_2)=(0,0,0,0)$ or $(0,0,0,1)$ we apply f_{c_1} and obtain

$$(0,0,i_3,\dots,i_g;1,1,j_3,\dots,j_g) \quad \text{or} \quad (0,0,i_3,\dots,i_g;1,0,j_3,\dots,j_g)$$

respectively. If $(i_1,i_2,j_1,j_2)=(0,1,0,0)$ or $(0,1,0,1)$ we apply $f_{c_1}\circ f_{b_2}\circ f_{a_2}$ or $f_{c_1}\circ f_{b_2}$ and obtain $(0,0,i_3,\dots,i_g;1,0,j_3,\dots,j_g)$. In all cases, we are able to bring the couple (i_1,j_1) to the form $(0,1)$. Iterating this procedure $g-1$ times we can bring $(\underline{i},\underline{j})$ to the form $(0,\dots,0,\hat{i}_g;1,\dots,1,\hat{j}_g)$ for some $\hat{i}_g,\hat{j}_g$. If $(\underline{i},\underline{j})$ was an odd prolongation, since the Dehn twists preserve the even or odd character of the prolongations, we must have $(\hat{i}_g,\hat{j}_g)=(0,0)$. If $(\underline{i},\underline{j})$ was even, then $(\hat{i}_g,\hat{j}_g)=(0,1)$, $(1,0)$ or $(1,1)$, and using the Dehn twists f_{a_g}, f_{b_g} we can bring them to the form $(0,1)$. This completes the proof.

We conclude this section by discussing the structure of $\tilde{D}(M,\alpha)$. In the case $g=0$, M is the two-sphere S^2; there is only one spin structure which is preserved by all diffeomorphisms. $D(S^2)$ is retractable to $SO(3)$ [14] and it follows from the results in [15] that $\tilde{D}(S^2)$ is retractable to $Spin(3)$, so $\tilde{D}(S^2)$ is the nontrivial double cover of $D(S^2)$. In the case $g=1$, M is the torus T^2 and the mapping class group is $\Omega(T^2)=SL(2,Z)/Z_2$, where

$$SL(2,Z)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\middle|\, a,b,c,d\in Z;\ ad-bc=1\right\}$$

and

$$Z_2=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right\}.$$

This group is generated by the matrices $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\1&1\end{pmatrix}$, corresponding to the Dehn twists f_a and f_b respectively. The stabilizers of the spin structures are as follows: $\Omega(T^2;0,0)=\Omega(T^2)$; $\Omega(T^2;1,0)$ is the subgroup for which b is even; $\Omega(T^2;0,1)$ is the subgroup for which c is even; $\Omega(T^2;1,1)$ is the subgroup for which $a+c$ and $b+d$ are odd, or equivalently ab and cd are even [7]. All connected components of $\tilde{D}(T^2;i,j)$ are homeomorphic, so it will be sufficient to determine the double cover of $D_0(T^2)\subset D(T^2;i,j)$, which we denote $\tilde{D}_0(T^2;i,j)$. [Notice that in general $\tilde{D}_0(T^2;i,j)$ is not the identity connected component of $\tilde{D}(T^2;i,j)$.] $D_0(T^2)$ is retractable to the group $U(1)\times U(1)$ of translations of the form $f_{t,s}:(\chi,\vartheta)\mapsto(\chi+t,\vartheta+s)$, where (χ,ϑ) are periodic coordinates on T^2 with period 2π. For each $f_{s,t}$ there are two elements $u_{t,s}$ and $u_{t,s}\circ\gamma$ in $\tilde{D}$ such that $\eta_{ij}\circ u_{t,s}=Tf_{t,s}\circ\eta_{ij}$. Since for $Tf_{t,s}$ we have $h_{t,s}=1$, this condition reads in local coordinates

$$\varrho(\tilde{h}_{t,s}(\chi,\vartheta))=(r_{ij}(\chi+t,\vartheta+s))^{-1}r_{ij}(\chi,\vartheta).$$

Keeping s fixed and varying t from 0 to 2π, the right-hand side, regarded as a map from the first factor $U(1)$ to $SO(2)$, has winding number i, so the endpoints of the path $t\mapsto\tilde{h}_{t,s}(\chi,\vartheta)$ differ by $(-1)^i$. Similarly for t fixed and varying s, the endpoints of the path $s\mapsto\tilde{h}_{t,s}(\chi,\vartheta)$ differ by $(-1)^j$. This shows that for each spinstructure i,j there is a different double cover: $\tilde{D}_0(T^2;0,0)\to D_0(T^2)$ is retractable to $U(1)\times U(1)\times Z_2\to U(1)\times U(1)$ given by $(e^{i\alpha},e^{i\beta},\pm1)\mapsto(e^{i\alpha},e^{i\beta})$ and is therefore trivial; $\tilde{D}_0(T^2;0,1)\to D_0(T^2)$ is retractable to $U(1)\times U(1)\to U(1)\times U(1)$, given by $(e^{i\alpha},e^{i\beta})\mapsto(e^{i\alpha},e^{2i\beta})$; $\tilde{D}_0(T^2;1,0)\to D_0(T^2)$ is retractable to $U(1)\times U(1)\to U(1)\times U(1)$, given by $(e^{i\alpha},e^{i\beta})\mapsto(e^{2i\alpha},e^{i\beta})$ and $\tilde{D}_0(T^2;1,1)\to D_0(T^2)$ is retractable to $(U(1)\times U(1))/Z_2\to U(1)\times U(1)$ given by $[e^{i\alpha},e^{i\beta}]\mapsto(e^{2i\alpha},e^{2i\beta})$, where [ ] denotes the Z_2 equivalence class and $Z_2=\{(1,1),(-1,-1)\}$.

Finally, for $g\geqq 2$ $D_0(M)$ is contractible [14] so $\tilde{D}(M;i,j)$ is the trivial double cover for each spin structure i,j.

5. Final Remarks

The transformation properties of spin structures are relevant to the study of global diffeomorphism anomalies, i.e. noninvariance of the effective action under diffeomorphisms which are not homotopic to the identity. In gravity it is also important to look at the behaviour of the theory under rotations of frames which are not homotopic to the identity. Such global frame anomalies can arise when M admits inequivalent spin structures [16]. Therefore it is important to know the behaviour of spin structures under vertical automorphisms of F. A vertical automorphism $u:F\to F$ transforms the spin structure $(\tilde{F}_\alpha,\eta_\alpha)$ to the spin structure $(\tilde{F}_\beta,\eta_\beta)$ for which there is a principal bundle morphism $\tilde{u}:\tilde{F}_\beta\to\tilde{F}_\alpha$ such that $\eta_\alpha\circ\tilde{u}=u\circ\eta_\beta$. Given $(\tilde{F}_\alpha,\eta_\alpha)$ and $(\tilde{F}_\beta,\eta_\beta)$ there exists an automorphism u of F which transforms the spin structures one into the other if and only if $\tilde{F}_\alpha$ and $\tilde{F}_\beta$ are isomorphic as principal $\widetilde{GL}^+(n)$-bundles. In particular, on a Riemann surface the group of vertical automorphisms of F acts transitively on the set of spin structures.

The action of orientation preserving diffeomorphisms on spin structures on Riemann surfaces has been computed by R. Lee and E. Miller, as announced in [17], and has also been determined by L. Alvarez-Gaumé, G. Moore, and C. Vafa by using the results of the theory of theta functions [18].

Acknowledgements. We thank E. Gava and R. Iengo for useful discussions on spinors in string theory. The second author also wishes to thank D. Freed for a stimulating conversation in the early stages of this work.

References

1. Atiyah, M.F., Hirzbruch, F.: Spin-manifolds and group actions. In: Essays on topology and related topics, pp. 18–28. Berlin, Heidelberg, New York: Springer 1970
2. Chichilnisky, G.: Group actions on spin manifolds. Trans. Am. Math. Soc. **172**, 307 (1972)
3. Haeflieger, A.: Sur l'extension du groupe structural d'un espace fibré. C.R. Acad. Sci. Paris **243**, 558 (1956)
4. Milnor, J.: Spin structures on manifolds. L'enseignement Math. **9**, 198 (1963)
5. Isham, C.J.: Spinor fields in four dimensional space-time. Proc. R. Soc. Lond. A **364**, 591 (1978)
6. Greub, W., Petry, H.R.: On the lifting of the structure groups. In: Lecture Notes in Mathematics, Vol. 676, pp. 217–246. Berlin, Heidelberg, New York: Springer 1978
7. Nahm, W.: Mass spectra of dual strings. Nucl. Phys. B **114**, 174 (1976)
8. Witten, E.: Global anomalies in string theories. In: Symposium on anomalies geometry topology, pp. 61–99. Singapore: World Scientific 1985
9. Kobayashi, S., Nomizu, K.: Foundations of differential geometry. New York: Interscience 1963
10. Steenrod, N.: The topology of fibre bundles. Princeton: University Press 1951
11. Dehn, M.: Die Gruppe der Abbildungsklassen. Acta Math. **69**, 135 (1938)
12. Lickorish, W.B.R.: A finite set of generators for the homeotopy group of a 2-manifold. Proc. Camb. Phil. Soc. **60**, 769 (1964)
13. Atiyah, M.F.: Riemann surfaces and spin structures. Ann. Scient. Éc. Norm. Sup. 4e serie, t. **4**, 47 (1971)
14. Earle, C.J., Eells, J.: The diffeomorphism group of a compact Riemann surface. Bull. Am. Math. Soc. **73**, 557 (1967)
15. Dabrowski, L., Trautman, A.: Spinor structures on spheres and projective spaces. J. Math. Phys. **27**, 2022 (1986)
16. Avis, S.J., Isham, C.J.: Lorentz gauge invariant vacuum functionals for quantized spinor fields in non-simply connected space-times. Nucl. Phys. B **156**, 441 (1979)
17. Seiberg, N., Witten, E.: Spin structures in string theory. Princeton University preprint (1986)
18. Alvarez-Gaumé, L., Moore, G., Vafa, C.: Theta functions, modular invariance and strings. Commun. Math. Phys. **106**, 1–40 (1986)

Communicated by S.-T. Yau

Received April 24, 1986