STRONG AND WEAK CONSTRUCTIVIZATION AND COMPUTABLE FAMILIES

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In this paper we introduce the concept of a computable family of constructive models and we then study the constructivization of strongly constructivizable model $\mathfrak{A}$. Sufficient and necessary conditions are adduced for the nonautostability of $\mathfrak{A}$ with respect to strong constructivizations and for the existence of weak constructivizations. The possible numbers of strong and weak constructivizations are determined.

§0. DEFINITIONS AND NOTATION

Let $\mathfrak{A} = \langle A, \sigma \rangle$ be a countable model of language σ and let $\sigma \cap N = \emptyset$. We call mapping ν of the set N of natural numbers onto A an enumeration of model $\mathfrak{A}$ while the pair $(\mathfrak{A}, \nu)$ is called an enumerated model. If $(\mathfrak{A}, \nu)$ is an enumerated model then $\mathfrak{A}_\nu$ is the model obtained from $\mathfrak{A}$ by adding to its signature all the natural numbers which, with this, are then interpreted by means of enumeration ν. Enumerated model $(\mathfrak{A}, \nu)$ is constructive if the set $D(\mathfrak{A}_\nu)$ of all atomic, and negations of atomic, assertions which are true in model $\mathfrak{A}_\nu$ is a computable set, and is strongly constructive if $Th(\mathfrak{A}_\nu)$ is computable, this being the theory of model $\mathfrak{A}_\nu$. In these cases, ν is called, respectively, a constructivization and a strong constructivization. A constructivization which is not strong we shall call weak. Constructivizations ν_1 and ν_2 of model $\mathfrak{A}$ are auto-equivalent if there exists an automorphism θ of model $\mathfrak{A}$ and a g.r.f (general recursive function) f such that the following diagram is commutative:

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "\nu_1"'] & N \arrow[d, "\nu_2"] \\ \mathfrak{A} \arrow[r, "\theta"] & \mathfrak{A} \end{tikzcd}$$

Let K be some class of constructivizations of model $\mathfrak{A}$. Model $\mathfrak{A}$ is autostable with respect to K if any two constructivizations of K are auto-equivalent. Apparently, from the point of view of the theory of constructive models, auto-equivalent enumerations should not be considered as distinct, just as, in ordinary model theory, one does not distinguish isomorphic models.

If we are given an arbitrary class $\mathfrak{K}$ of constructive models of signature σ, we may be interested in the question of the possibility of specifying this class by means of a single effective process. In this case $\mathfrak{K}$ is called computable.

Definition 1. Class $\mathfrak{K}$ is (strongly) computable if there exists a g.r.f. $f(x,y)$ such that, for each fixed x_0, the function $f(x_0, y)$ computes $(Th(\mathfrak{A}_\nu))\, D(\mathfrak{A}_\nu)$, where $(\mathfrak{A}, \nu)$ is auto-equivalent to some model of family $\mathfrak{K}$ and, for each model of $\mathfrak{K}$, there exists a corresponding x_0.

This definition can be specialized in various ways. For example, if $\mathfrak{K}$ is specified effectively and indexed by the natural numbers, it can then be required that one find effectively from x_0 the corresponding model, and conversely. The usual diagonal argument shows that the class of all constructive models of given infinite signature is not computable. Certain properties of strongly computable classes are obtained in the first theorem.

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Everywhere in the sequel, $(\mathfrak{A},\mu)$ is strongly constructive (s.c.), $\mu(n)=a_n$, $T=Th(\mathfrak{A})$, and f_s is a single-place p.r.f. (partial recursive function) with ordinal number s, while f_s^t is the portion of it computed up to step t ; $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ is the Gödel sequence of all assertions of language $\sigma \cup N$; m, n, i, p, q, r, and s are natural numbers; x and y are variables; $\overline{m}_{(i)}$, $\overline{a}_{(i)}$ and $\overline{x}_{(i)}$ $(\overline{n}_{(i)}, \overline{b}_{(i)}$, and $\overline{y}_{(i)})$ are finite sequences of natural numbers, elements of model $\mathfrak{A}$, and variables of identical length; $i: N \to N$ is the identity mapping.

§1. STRONG CONSTRUCTIVIZATION

The theorem of this section gives a necessary and sufficient condition for the existence of nonautoequivalent strong constructivizations of given model $\mathfrak{A}$.

THEOREM 1. Let $(\mathfrak{A},\mu)$ be a strongly constructive model of complete theory T. Then, the following conditions are equivalent:

(1) $\mathfrak{A}$ is not autostable with respect to strong constructivizations;

(2) there does not exist a finite sequence $\overline{a}_0$ of elements of model $\mathfrak{A}$ such that $\mathfrak{A}$ is a simple model of $T(a_0)$ and the family of sets of atoms of the Boolean algebras $F_n(T(\overline{a}_0))$ is computable;

(3) there exists a strongly computable family of models $(\mathfrak{A},\mu_0), \ldots, (\mathfrak{A},\mu_s), \ldots$ whose terms are pairwise elementarily constructively nonembeddable in one another;

(4) there does not exist a strongly computable family of models of signature σ containing all the strong constructivizations of model $\mathfrak{A}$.

Proof. Evidently, (3) $\longrightarrow$ (1), (4) $\longrightarrow$ (1), and (1) $\longrightarrow$ (2) $\Longleftrightarrow$ $\neg$ (2) $\longrightarrow$ $\neg$ (1).

We now prove $\neg$ (2) $\longrightarrow \neg$ (1). For $\overline{a}_0$, let $\mathfrak{A}$ be a simple model of $T(\overline{a}_0)$ and let the family of sets of atoms of algebra $F_n(T(\overline{a}_0))$ be computable. For the proof, it suffices, from two simple models of $T(\overline{a}_0)$, $\mathfrak{N}_1$, and $\mathfrak{N}_2$, with basic set N, such that $(\mathfrak{N}_1, i)$ and $(\mathfrak{N}_2, i)$ are s.c. models, to be able to construct a general recursive elementary mapping $f: \mathfrak{N}_1 \xrightarrow{\text{onto}} \mathfrak{N}_2$. Temporarily, we denote by $[\overline{m}]_i$ the atom of algebra $F_n(T(\overline{a}_0))$ satisfied by sequence $\overline{m}$ of length n in model $\mathfrak{N}_i$ (found effectively, by hypothesis). We define $f=\{\langle m_0,n_0\rangle, \ldots, \langle m_t,n_t\rangle, \ldots\}$. By Vaught's Theorem [7], f is an isomorphism if, for any t, $[\langle m_0,\ldots,m_t\rangle]_1=[\langle n_0,\ldots,n_t\rangle]_2$. By induction we set $m_{2t}=\min(N^1\setminus\{m_0,\ldots,m_{2t-1}\})$ and

$$n_{2t}=\min\{n:[\langle m_0,\ldots,m_{2t}\rangle]_1=[\langle n_0,\ldots,n_{2t-1},n\rangle]_2\};$$

$$n_{2t+1}=\min(N\setminus\{n_0,\ldots,n_{2t}\})$$

and

$$m_{2t+1}=\min\{m:[\langle m_0,\ldots,m_{2t},m\rangle]_1=[\langle n_0,\ldots,n_{2t+1}\rangle]_2.$$

We find n_{2t} and m_{2t+1} effectively since $(\mathfrak{N}_1,i)$ and $(\mathfrak{N}_2,i)$ are strongly constructive. Obviously, f maps N onto N.

We now prove (2) $\longrightarrow$ (3). The steps by t we define by an effective process. On step $t+1$ we construct the finite enumerations $\mu_0^{t+1},\ldots,\mu_t^{t+1}$, and the finite sets $T_0^{t+1},\ldots,T_t^{t+1}$ of assertions in the language $\sigma \cup N$. Moreover, for $s\leq t$, in some μ_s^t ordinal numbers are labeled by the triples $\langle p_0,q_0,r_0\rangle,\ldots,\langle p_t, q_t,r_t\rangle$ of the set, ordered by type ω, of triples of natural numbers, where $p_i\neq q_i$. The process will possess properties 1°-8°:

1°. The triple $\langle p,q,r\rangle$ on step t can label some μ_p^t and some μ_q^t ordinal numbers.

2°. If $T_p^t=T_p^t(\overline{m})$, then μ_p^t is defined on the numbers of m and $\mathfrak{A} \overset{p}{\models} \& [T_p^t(\mu_p^t(\overline{m}))]$.

3°. If, on step t, we have triple $\langle p,q,r\rangle$ then, among the μ_p^t and μ_q^t ordinal numbers labeled by it we can find sequences $\overline{m}$ and $\overline{n}$ such that, for some formula $\varphi(\overline{x})$ of language σ, there holds

$$\neg\varphi(\bar m)\in T_p^t, \quad \varphi(\bar n)\in T_q^t \text{ and } f_z^t:\bar m\longrightarrow\bar n.$$

Thus, f_z will not, henceforth, define the elementary embedding of $(\mathfrak{A},\mu_p)$ in $(\mathfrak{A},\mu_q)$.

We now fix step t, enumeration μ_p^t, and set of assertions $T_p^t=T_p^t(\bar m_0,\bar m_1,\bar m_2)$. Let $\mu_p^t:\bar m_0\longrightarrow\bar a_0$. The type of sequence $\bar m_1$ in μ_p^t relative to $\bar m_0$ is what we call formula $\Phi(\bar a_0,\bar x_1)=\exists\bar x_2[\&[T_p^t(\bar a_0,\bar x_1,\bar x_2)]]$ of language $\sigma\cup\bar a_0$. If, on step t, triple $<p,q,z>$ is not established while f_z^t is defined on the numbers of $\bar m_0$ and $\bar m_1$, where $\bar m_0$ consists of all the μ_p^t ordinal numbers labeled by smaller triples, and assertion $\varphi(\bar m_0,\bar m_1,\bar m_2)$ is such that

$$T(\bar a_0)\vdash\ \exists\bar x_1\bar x_2[\&T_p^t(\bar a_0,\bar x_1,\bar x_2)\&\varphi(\bar a_0,\bar x_1,\bar x_2)],$$

we then define the action of φ on the triple $<p,q,z>$ in μ_p^t:

1. φ establishes in μ_p^t the triple $<p,q,z>$, if $\varphi-\varphi(\bar m_0,\bar m_1)$;

$$T(\bar a_0)\vdash\ \exists\bar x_1\bar x_2[\&T_p^t(\bar a_0,\bar x_1,\bar x_2)\&\ \varphi(\bar a_0,\bar x_1)]$$

and

$$T(\bar a_0)\vdash\ \exists\bar x_1\bar x_2[\&T_p^t(\bar a_0,\bar x_1,\bar x_2)\&\neg\varphi(\bar a_0,\bar x_1)].$$

2. In μ_p^t, φ has no effect on triple $<p,q,z>$, if

$$T(\bar a_0)\vdash\exists\bar x_2[\&T_p^t(\bar a_0,\bar x_1,\bar x_2)]\longrightarrow\exists\bar x_2[\&T_p^t(\bar a_0,\bar x_1,\bar x_2)\&\varphi(\bar a_0,\bar x_1,\bar x_2)].$$

3. In μ_p^t, φ does have an effect on $<p,q,z>$, if 2 does not hold. It is understandable that if $\varphi(\bar m_0,\bar m_1,\bar m_2)$ has an effect on $<p,q,z>$, then $\exists\bar x_2[\&T(\bar m_0,\bar m_1,\bar x_2)\&\ \varphi(\bar m_0,\bar m_1,\bar x_2)]$ establishes $<p,q,z>$, and if φ establishes $<p,q,z>$, then too $\neg\varphi$ establishes $<p,q,z>$.

<u>The Inductive Definition. Step $t+1$.</u> For $s\le t$ we define first the enumeration $\mu_s^{t'}$, then $\mu_s^{t+1}=\mu_s^{t'}\cup\{<m_s,a_{n_s}>\}$, where $m_s=\min\{m: m$ is not a $\mu_s^{t'}$ ordinal number$\}$, while $n_s=\min\{n: a_n$ does not have a $\mu_s^{t'}$ ordinal number$\}$.

Let $\varphi_{<t>_1^3}=\varphi(\bar m)$ and $<t>_2^3=p$. We proceed differently in the following situations:

1. In μ_p^t, $\varphi(\bar m)$ establishes the triple $<p,q,z>$ on the set of the first t triples and does not affect smaller triples;

$f_z^t:\bar m\longrightarrow\bar n$ and $\mathfrak{A}\vDash\ \varphi(\mu_q^t(\bar n))$, while $\varphi(\bar n)$ in μ_q^t has no effect on any triple less than $<p,q,z>$.

In this case we set

$$T_p^{t+1}=T_p^t\cup\{\neg\varphi(\bar m)\}\quad\text{and}\quad T_q^{t+1}=T_q^t\cup\{\varphi(\bar n)\};$$

enumeration $\mu_p^{t'}$ on $\bar m_0$ coincides with μ_p^t and on $\bar m_1\wedge\bar m_2$ is such that $\mathfrak{A}\vDash\&T_p^{t+1}(\mu_p^{t'}(\bar m))$. Moreover, all elements of $\mathfrak{A}$ which, under that renumbering, lose their own μ_p^t ordinal numbers are furnished with new $\mu_p^{t'}$ ordinal numbers. For $s\ne p$, we have $\mu_p^{t'}=\mu_s^t$.

All the μ_p^{t+1} and μ_q^{t+1} ordinal numbers are labeled by the triple $<p,q,z>$.

2. $\mathfrak{A}\vDash\ \varphi(\mu_p^t(\bar m))$ and $\varphi(\bar m)$ in μ_p^t has no effect on any of the first t triples.

We set

$$T_p^{t+1} = T_p^t \cup \{\varphi(\bar{m})\}.$$

3. In any other case, $\mu_s^{t'} = \mu_s^t$, $T_s^{t+1} = T_s^t$, and no new triples are established. The definition is completed.

Obviously, for any s and t we have $T_s^t \subseteq T_s^{t+1}$. We set $T_s = \bigcup_t T_s^t$. Properties 1°-3° are verified immediately.

It follows from the inductive definition that in our construction, each element of model $\mathfrak{A}$ obtains some μ_s^t-ordinal number and each natural number becomes a μ_s^t ordinal number. More than that,

4°. For any m_0, a_1, and s there exist step t', element a_{n_0}, and number m_1 such that, $t \geq t'$:

$$\mu_s^t(m_0) = a_{n_0} \Leftrightarrow \mu_s(m_0),$$

$$\mu_s^t(m_1) = a_{n_1} = \mu_s(m_1).$$

Indeed, if $\mu_s^t(m_0) \neq \mu_s^{t+1}(m_0)$ and, on step t, the μ_s^t-ordinal number of m had not been labeled then, on step $t+1$, it is labeled; if, however, on step t m_0 had already been labeled then, on step $t+1$, it is labeled by a triple less than all the triples labeling it on step t. Such a situation can arise only a finite number of times. And, if the μ_s^t- and μ_s^{t+1}-ordinal numbers of element a_{n_1} are different, the analogous situation supervenes.

5°. For each i there exists a step t_i beginning with which new triples from the set $\{\langle p_0, q_0, z_0\rangle, \ldots, \langle p_i, q_i, z_i\rangle\}$ are not established.

6°. If, on step t, assertion ψ establishes in μ_p^t the triple $\langle p, q, z\rangle$, then on some step $t'(\geq t)$, there actually arises a triple less than, or equal to, $\langle p, q, z\rangle$.

Proof is by induction on the magnitude of the triple $\langle p, q, z\rangle$. Indeed, on a sufficiently large step if we take as φ either assertion ψ or its negation then case 1 of the induction hypothesis occurs. Otherwise, on this step we can find an assertion which establishes a strictly smaller triple, and then induction is applicable.

Remark 1. If, on step t_i, triple $\langle p_i, q_i, z_i\rangle$ is not established while function $f_{z_i}^{t_i}$ is defined on the $\mu_{p_i}^{t_i}$-ordinal numbers of $\bar{m}_0$, labeled by lesser triples, then on it no assertion can have any effect. Starting with step t_i, sequence $\bar{m}_0$ consists of $\mu_{p_i}^t$ ordinal numbers of one and the same sequence of elements $\bar{a}_0$. If, for $t'(\geq t_i)$, function $f_{z_i}^{t'}$ is defined on $\mu_{p_i}^t$ ordinal numbers of a sequence of length κ and $\bar{m}_1$ and $\bar{m}_0 \cap \bar{m}_1 = \phi$, then the type of $\bar{m}_1$ with respect to $\bar{m}_0$ in $\mu_p^{t'}$ is the atom $F_\kappa(T(\bar{a}_0))$.

7°. If function f_{z_i} is general recursive then, on step t_i, triple $\langle p_i, q_i, z_i\rangle$ stands.

The proof is by contradiction.

We assume that, on step t_i, triple $\langle p_i, q_i, z_i\rangle$ does not label anything, and we let function $f_{z_i}^{t_i}$ be defined on the $\mu_{p_i}^{t_i}$-ordinal numbers of $\bar{m}_0$, labeled by smaller triples. Two cases are possible:

a) There exists a sequence of elements of model $\mathfrak{A}$ $\bar{a}$, generating a nonprincipal type over $\bar{a}_0$.

By property 4°, from some step $t'(\geq t_i)$ a_1 has a constant sequence of $\mu_{p_i}^t$ ordinal numbers $\bar{m}_1$ and function $f_{z_i}^t$ is defined on the numbers of $\bar{m}_1$. By the previous remark, sequence $\bar{m}_1$ with respect to $\bar{m}_0$ in $\mu_{p_i}^{t'}$ has the definite type $\Phi(\bar{a}_0, \bar{x}_1)$ which must be an atom. We arrive at a contradiction.

b) The family of sets of atoms of algebra $F_n(T(\bar{a}))$ is not computable.

Let $\varphi(\bar{a}_0, \bar{x}_1)$ be an arbitrary, but compatible with $T(\bar{a}_0)$, formula of language $\mathfrak{G} \cup \bar{a}_0$ which is not satisfied by sequences containing elements of $\bar{a}_0$ and $\bar{x}_1$ of length κ. By using induction, starting with step t_i, we will know whether $\varphi(\bar{a}_0, \bar{x}_1)$ is an atom of algebra $F_\kappa(T(\bar{a}_0))$. In model $\mathfrak{A}$ there is a sequence of elements a_1 such that

$$\mathfrak{A} \models \varphi(\bar{a}_0, \bar{a}_1).$$

We can find step $t'(>t_i)$ for which $\bar{a}_1$ has a $\mu^t_{p_i}$-ordinal number of $\bar{m}_1$, while $\bar{m}_0$ and $\bar{m}_1$ enter into the domain of definition of function $f^{t'}_{\tau_i}$. By the preceding remark, the type of $\bar{m}_1$ with respect to $\bar{m}_0$ in $\mu^{t'}_{p_i}$ is an atom compatible with $\varphi(\bar{a}_0, \bar{x}_1)$. Thus, continuing the process from step t_i, we can expect there to be a step t on which the type of some sequence of $\mu^t_{p_i}$-ordinal numbers, entering into the domain of definition of function $f^t_{\tau_i}$, with respect to $\bar{m}_0$ in $\mu^t_{p_i}$ is compatible with formula $\varphi(\bar{a}_0, \bar{x}_1)$, and we can compare them. If they coincide then $\varphi(\bar{a}_0, \bar{x}_1)$ is an atom of $F_\kappa(T(\bar{a}_0))$. Thus, the family of sets of atoms of algebra $F_n(T(\bar{a}_0))$ is computable. We arrive at another contradiction.

Since, on each step, no more than one triple can be established then, by 7°, any sequence of μ^t_p-ordinal numbers for some i is labeled by some triple $\langle p_i, q_i, \tau_i \rangle$.

8°. Let $\mu_{p_i} : \bar{m} \to \bar{a}$ and $\mathfrak{A} \models \varphi(\bar{a})$. Then, $\varphi(\bar{m}) \in T$. We can find step t' $(\geq t_i)$ such that $\mu^{t'}_p(\bar{m}) = \bar{a}$ and $\varphi(\bar{m}) = \varphi_{\langle t \rangle^3_1}$. By definition 5° for step t_i, $\varphi(\bar{m})$ cannot exert any influence in $\mu^{t'}_{p_i}$ on triple $\langle p_i, q, \tau \rangle$. That is to say that case 2 of the inductive definition holds, and $\varphi(\bar{m}) \in T^{t+1}_{p_i}$.

It follows from property 4° that all the μ_p are enumerations of model $\mathfrak{A}$. They are strongly constructive, by virtue of 8°. It follows from 7° that all the models $(\mathfrak{A}, \mu_0), \ldots, (\mathfrak{A}, \mu_p), \ldots$ are pairwise elementarily constructively nonembeddable in one another.

(2) $\longrightarrow$ (4). Proof is by contradiction.

Let there be a strongly computable family containing all the strong constructivizations of model $\mathfrak{A}$. The inductive process of the proof of (2) $\longrightarrow$ (3) can be so defined that for $p \neq 0$ the strong constructivizations of our family are computed. Then, for $p = 0$, we compute the strong constructivization of model $\mathfrak{A}$, which is not equivalent to any of the constructivizations of this family. Theorem 1 is proven.

COROLLARY 1. Strongly constructivizable model $\mathfrak{A}$ is autostable with respect to strong constructivizations if and only if, for some sequence $\bar{a}$ of elements of $\mathfrak{A}$, the system $\mathfrak{A}$ is a simple model $Th(\mathfrak{A}, \bar{a})$ and the family of sets of atoms of algebra $F_n(Th(\mathfrak{A}, \bar{a}))$ is computable.

Example 1. If $\mathfrak{A}$ is saturated over finite sets, $Th(\mathfrak{A})$ is not a $\aleph_0$-category, and $\mathfrak{A}$ is strongly constructivizable, then $\mathfrak{A}$ has infinitely many strong constructivizations.

Example 1' [6]. A denumerable unsaturated model decidable by $\aleph_1, \neg\aleph_0$-categorical theory has infinitely many strong constructivizations.

Example 1" (A. I. Mal'tsev [4]). A complete torsion-free Abelian group of finite rank is autostable, while an infinite one is not autostable.

Example 2 (E. A. Palyutin, M. G. Peretyat'kin). If T is decidable and $\aleph_0$ categorical, then the family of sets of atoms of algebra $F_n(T)$ is not computable if and only if function $f(n) = |F_n(T)|$ is not general recursive. Examples of such theories were independently constructed by E. A. Palyutin and M. G. Peretyat'kin. By Theorem 1, denumerable models of these theories admit infinitely many strong constructivizations.

§2. WEAK CONSTRUCTIVIZATION

It is known [1] that a constructive model of a complete model–complete decidable theory is strongly constructive.

LEMMA 1. If, for some finite sequence $\bar{a}_0$ of elements of $\mathfrak{A}$, theory $Th(\mathfrak{A}, \bar{a}_0)$ is decidable and, for an arbitrary sequence $\bar{a}_1$, any formula $\varphi(\bar{a}_0, \bar{x}_1)$ which holds on $\bar{a}_1$ follows from some $\exists$-formula $\psi(\bar{a}_0, \bar{x}_1)$ which holds on $\bar{a}_1$, then each constructivization ν of model $\mathfrak{A}$ will be strong.

Proof. It suffices for us to know whether, for arbitrary formula $\varphi(\bar{a}_0, \bar{x}_1)$ and sequence of numbers $\bar{m}_1$, $\mathfrak{A} \models \varphi(\bar{a}_0, \nu(\bar{m}_1))$ is true or not. By virtue of the constructiveness of enumeration ν we can effectively enumerate all $\exists$-formulas of theory $T(\bar{a}_0)$ which hold on sequence $\nu(\bar{m}_1)$. By using the decidability of $T(\bar{a}_0)$, we can also effectively enumerate all their consequences among which, by hypothesis, there necessarily occurs either $\varphi(\bar{a}_0, \bar{x}_1)$ or $\neg\varphi(\bar{a}_0, \bar{x}_1)$. This gives a decision procedure for $Th(\mathfrak{A}_\nu)$.

THEOREM 2. For strongly constructive model $(\mathfrak{A}, \mu)$ the following conditions are equivalent:

(1) $\mathfrak{A}$ has a weak constructivization;

(2) $\mathfrak{A}$ does not satisfy the conditions of Lemma 1;

(3) there exists a denumerable computable family of weakly constructive models $(\mathfrak{A}, \nu_0), \ldots, (\mathfrak{A}, \nu_p), \ldots$ the terms of which are pairwise elementarily constructively nonembeddable in one another.

Proof. (3) $\longrightarrow$ (1). Obvious. (1) $\longrightarrow$ (2). We assume the contrary and use Lemma 1. (1) $\longrightarrow$ (3). By induction on t we define an effective process during which, on each step t, there are defined finite enumerations of model $\mathfrak{A}$ $\nu_0^t, \ldots, \nu_s^t, \ldots$, and the corresponding diagrams $\mathcal{D}_0^t, \ldots, \mathcal{D}_s^t, \ldots$, i.e., if $\varphi(\bar{m})$ is either an atomic, or the negation of an atomic, assertion of language $\mathfrak{S} \cup N$, then $\varphi(\bar{m}) \in \mathcal{D}_s^t \iff$ the numbers of $\bar{m}$ are ν_s^t ordinal numbers and $\mathfrak{A} \models \varphi(\nu_s^t(\bar{m}))$. By type ω we order the set of all pairs $\langle p, z\rangle$ and triples $\langle p, q, z\rangle$ $(p \neq q)$ of the natural numbers and put them into a one-to-one correspondence with the marks $\boxed{0} < \ldots < \boxed{i} < \ldots$. The process to be defined will possess properties 1°–6°.

1°. For any s and t, it is true that $\mathcal{D}_s^t \subseteq \mathcal{D}_s^{t+1}$.

2°. On step t, the mark corresponding to the pair $\langle p, z\rangle$, can label some ν_p^t ordinal number. In this case, we can find an assertion of language $\mathfrak{S} \cup N$ $\varphi_K(\bar{m})$, such that all the numbers of $\bar{m}$ are labeled by this mark, $f_z^t(K) = 0$, but

$$\mathfrak{A} \models \neg\varphi(\nu_p^t(\bar{m})).$$

Thus, function f_z will not be the characteristic function of $Th(\mathfrak{A}_{\nu_p})$, if ν_p on $\bar{m}$ coincides with ν_p^t.

3°. On step t, the mark corresponding to triple $\langle p, q, z\rangle$ may (and only simultaneously) label some ν_p^t and ν_q^t ordinal numbers. In this case, among the ν_p^t and ν_q^t ordinal numbers labeled by it, one can find sequences $\bar{m}$ and $\bar{n}$ and formula $\varphi(\bar{x})$ of language $\mathfrak{S}$ such that

$$f_z^t : \bar{m} \longrightarrow \bar{n};$$
$$\mathfrak{A} \models \neg\varphi(\nu_p^t(\bar{m}));$$
$$\mathfrak{A} \models \varphi(\nu_q^t(\bar{n})).$$

That is, function f_z will not define an elementary embedding of model $(\mathfrak{A}, \nu_p)$ in $(\mathfrak{A}, \nu_q)$, if ν_p and ν_q on $\bar{m}$ and $\bar{n}$ coincide with ν_p^t and ν_q^t.

Inductive definition. Step 0. For any s, it is true that $\nu_s^0 = \varnothing$ and $\mathcal{D}_s^0 = \varnothing$.

Step $t+1$. For any $s \neq \langle t\rangle_1^4 = p$ we have $\nu_s^{t+1} = \nu_s^t$ and $\mathcal{D}_s^{t+1} = \mathcal{D}_s^t$.

Before defining enumeration ν^{t+1} and placing the marks, we adopt enumeration $\nu_p^{t'}$. Then, $\nu_p^{t+1} = \nu_p^{t'/p} \cup \{\langle m_0, a_{n_0}\rangle\}$, where $m_0 = \min\{m : m$ is not a $\nu_p^{t'}$ ordinal number$\}$ and $n_0 = \min_{t+1}\{n : a_n$ does not have $\nu_p^{t'}$ ordinal number$\}$, while $\mathcal{D}_p^{t+1}$ corresponds to ν_p^{t+1}: if the mark $\boxed{\langle t\rangle_2^4}$ is not placed on step t, all the ν_p^t ordinal numbers labeled by smaller marks enter into $\bar{m}_0$ and $\nu_p^t : \bar{m}_0 \longrightarrow \bar{a}_0$; $\mathcal{D}_p^t = \mathcal{D}(\bar{m}_0, \bar{m}_1)$, $\varphi_{\langle t\rangle_3^4} = \varphi(\bar{m}_0, \bar{m}_1)$, $\mathfrak{A} \models \varphi(\nu_p^t(\bar{m}_0 \wedge \bar{m}_1))$ and with $T(\bar{a}_0)$ the following formula is compatible:

$$\neg\varphi(\bar{a}_0,\bar{x}_1) \mathbin{\&} [\mathbin{\&}\mathcal{D}(\bar{a}_0,\bar{x}_1)],$$

we then verify the following condition:

A. Mark $\boxed{\langle t\rangle_2^4}$ corresponds to the pair $\langle p,\tau\rangle$

and

1. $f_\tau^t(\langle t\rangle_3^4) = 0$

or

2. $f_\tau^t(\langle t\rangle_3^4) \neq 0.$

B. Mark $\boxed{\langle t\rangle_2^4}$ corresponds to the triple $\langle p,q,\tau\rangle, f_\tau^t: \bar{m}_0\wedge\bar{m}_1 \to \bar{n}_0\wedge\bar{n}_1$

and

1. $\mathfrak{A} \models \varphi(\nu_q^{t+1}(\bar{n}_0\wedge\bar{n}_1))$

or

2. $\mathfrak{A} \models \neg\varphi(\nu_q^{t+1}(\bar{n}_0\wedge\bar{n}_1)).$

If A, 1 or B, 1 holds, then enumeration $\nu_p^{t'}$ on $\bar{m}_0$ coincides with ν_p^t and on $\bar{m}_1$ is such that

$$\mathfrak{A} \models \neg\varphi(\nu_p^{t'}(\bar{m}_0\wedge\bar{m}_1)) \mathbin{\&} [\mathbin{\&}\mathcal{D}(\nu_p^{t'}(\bar{m}_0\wedge\bar{m}_1))].$$

Moreover, all the elements of $\mathfrak{A}$ which, as the result of such an enumeration, lose their own ν_p^t ordinal numbers, are equipped with new $\nu_p^{t'}$ ordinal numbers.

All the ν_p^{t+1} ordinal numbers in case A, and all the ν_p^{t+1} and ν_q^{t+1} ordinal numbers in case B, are labeled by the marks $\boxed{\langle t\rangle_2^4}$, and all larger marks are taken down. In all other cases, $\nu_p^{t'} = \nu_p^t$, and all marks remain in their previous locations. This completes the definition.

Properties 1°-3° follow immediately from the inductive definition. Properties 4° and 5° are completely analogous to the properties from the proof of Theorem 1.

4°. For any natural numbers m_0, n_1, and p, we can find a sufficiently remote step t', and numbers n_0 and m_1, such that, for any $t > t'$,

$$\nu_p^t(m_0) = \nu_p^{t'}(m_0) = a_{n_0} \leftrightarrows \nu_p(m_0)$$

and

$$\nu_q^t(m_1) = \nu_q^t(m_1) = a_{n_1} = \nu_q(m_1).$$

5°. For any i, we can find step t_i on which all the marks $\boxed{0}, \ldots, \boxed{i-1}, \boxed{i}$ are stabilized.

6°. Let mark i correspond to a) pair $\langle p,\tau\rangle$, b) triple $\langle p,q,\tau\rangle$, and let f_τ be general recursive. Then, on step t_i mark $\boxed{i}$ will stand. We assume that, on step t_i, mark $\boxed{i}$ nowhere appears, and let $\bar{m}_0$ be the $\mu_p^{t_i}$ ordinal number labeled on step t_i by marks less than $\boxed{i}$. It is understandable that, when $t > t_i$, it is true that $\nu_p^t(\bar{m}_0) = \nu_p(\bar{m}_0) = \bar{a}_0$. By hypothesis, we can find a formula $\varphi(\bar{a}_0, \bar{x}_1)$, compatible with $T(\bar{a}_0)$, and a sequence $\bar{a}_1$ of elements of model $\mathfrak{A}$ such that $\varphi(\bar{a}_0,\bar{x}_1)$ is true on $\bar{a}_1$ and does not follow from any of the $\exists$-formulas which are true on $\bar{a}_1$. For some sufficiently remote step $t'(> t_i)$ and sequence of natural numbers $\bar{m}_1$, we have

$$\nu_p^t : \bar{m}_1 \to \bar{a}_1 ;$$
$$\langle t' \rangle_1^4 = p ;$$
$$\varphi_{\langle t \rangle_3^4} = \varphi(\bar{m}_0, \bar{n}_1)$$

and

$$\langle t \rangle_3^4 = i \quad .$$

By virtue of the general recursiveness of function f_τ we can also assume that, in case a), $f_\tau^{t'}$ is defined on $\langle t \rangle_3^4$, and, in case b), that f_τ^t is so defined; $\bar{m}_0 \wedge \bar{m}_1 \to \bar{n}_0 \wedge \bar{n}_1$ and all the numbers of $\bar{n}_0 \wedge \bar{n}_1$ are ν_φ^t ordinal numbers. By the inductive definition of step $t+1$, mark $\boxed{i}$ must stand on step $t'+1$. We have arrived at a contradiction.

Finally, it follows from property 4° that, for any p, a ν_p-enumeration of model $\mathfrak{A}$ is constructive by virtue of 1°. It follows from property 6° that all the models $(\mathfrak{A}, \nu_0), \ldots, (\mathfrak{A}, \nu_p), \ldots$ are weakly constructive and pairwise are elementarily constructively nonembeddable in one another. Theorem 2 is proven.

COROLLARY 2. Weakly constructivizable model $\mathfrak{A}$ does not have weak constructivizations if and only if there exists a finite sequence $\bar{a}_0$ of elements of $\mathfrak{A}$ such that, for any $\bar{a}_1$ and $\varphi(\bar{a}_0, \bar{x}_1)$, if $\mathfrak{A} \models \varphi(\bar{a}_0, \bar{a}_1)$ then there exists an $\exists$-formula $\psi(\bar{a}_0, \bar{x}_1)$ such that $\mathfrak{A} \models \psi(\bar{a}_0, \bar{a}_1)$ and $T(\mathfrak{A}, \bar{a}_0) \vdash \psi(\bar{a}_0, \bar{x}_1) \to \varphi(\bar{a}_0, \bar{x}_1)$.

Example 3. If, in the foregoing corollary, $Th(\mathfrak{A}, \bar{a}_0)$ is $\aleph_0$-categorial, it then turns out to be model complete.

Example 4. In [5] there was constructed a model whose theory is categorial in all cardinalities. It can be verified that it is autostable with respect to strong constructivizations but, by the previous example, has infinitely many weak ones.

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