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Continuous accretion of a composite cylinder

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Summary. A new model is derived for winding of a composite thick-walled cylinder with finite strains. Continuous growth of a cylinder is treated as a limit of successive accretion of built-up portions (thin-walled shells) consisting of a fiber bundles and resin. Due to preload of fibers, a gradient of pressure arises in the cylinder which causes resin flow. Nonlinear partial differential equations are developed which permit stresses and displacements in a wound cylinder to be determined with account for the material accretion and resin flow. At infinitesimal strains, these equations are reduced to a linear Volterra integral equation for pressure on mandrel. This equation is solved numerically to analyze the effect of material and structural parameters on stresses in a wound cylinder.

1 Introduction

The paper is concerned with axi-symmetrical accretion of an elastic cylinder with finite strains. Continuous accretion is treated as a limit of the process when successive layers (thin-walled shells) are wound around the cylinder and immediately merge with it. Any built-up portion consists of fiber bundles and molten resin. When preloaded fibers are wound around the composite cylinder, some gradient of pressure arises in resin which causes its flow. The fluid flow leads to consolidation of fibers, which, in turn, reduces stress intensity. The objective of the present study is to derive a mathematical model for continuous accretion with account for the resin flow and to analyze the effect of material and structural parameters on stress distribution in a wound cylinder.

Estimation of residual stresses built-up in wound composite structures (composite pressure vessels and pipes) has been the focus of attention in the past three decades because of its applications in polymer engineering, see [1]. Filament winding consists of wrapping rovings of fibers over a mandrel with a given helical wind angle β (which serves to increase the load-bearing capacity of a pressure vessel) and with a suitable pretension (which controls the fiber position and provides a pressure necessary for compaction of the fiber network). The mandrel serves to resist sag due to the winding tension and the weight of a manufactured structure.

At the helical (chord) winding with $\beta \neq 0$, the filament is placed along straight lines on the lateral surface of a wound structure, and the slope angle β can change from layer to layer. The effect of the winding angle is accounted for by replacing the filament stress (in the direction of the fiber) σ_* by the circumferential component of the stress [2]

$$\Sigma = \sigma_* \sin^2 \beta .$$

The winding process consists of three main stages [3], [4], [5]:

(i) winding, when resin-impregnated fiber bundles are wounded around a rotating mandrel;

(ii) curing, when the assembly is placed into an oven, where it is subsequently heated, cured and cooled to consolidate the fiber network and to cure the resin;
(iii) removal from the mandrel.

Two types of the winding processes are distinguished: dry winding, when preimpregrated tows are wound consisting of fibers and partially cured resin, and wet winding, when the fiber is saturated with low-viscosity uncured resin during winding onto the mandrel.

Most of the works concerned with wound composite structures concentrate on the dry winding, when consolidation of wound layers and resin flow may be neglected, see [2]. For a survey of mathematical models for dry winding of composite structures, see [6], [7], [8].

The works [9], [10], [11] are concentrated on the measurement and calculation of residual stresses arisen in filament-wound composite structures when the material behavior obeys the constitutive equations of a linear orthotropic elastic medium.

A model of a nonlinear orthotropic elastic material with infinitesimal strains was used in [12], [13] to calculate stresses built-up in a wound cylinder under the action of fiber pretensioning and external pressure.

Recently, several new technological processes have been proposed, where the model of wet winding should be employed to adequately describe mechanical stresses, see [14], [15], [16] and the bibliography therein.

The first model accounting for changes in stresses due to preloading of fibers, thermal deformations of the wound structure, resin flow through the porous medium, consoldidation of fiber bundles, rheokinetics of curing, and shrinkage of resin was suggested in [17]. Similar models were derived recently in [2], [3], [7], [18].

In these models, winding is treated as successive accretion of thin-walled elements (curvilinear beams or shells) on a part of the boundary of a growing body. As a result, a complicated discrete system is derived for stresses and displacements which can be analyzed only numerically. An alternative approach consists in modeling winding as a continuous process when within an infinitesimal time interval $[t, t + dt]$, a thin layer with thickness proportional to dt merges with a growing solid [19], [20]. This method permits the accretion process to be described by partial differential or integro-differential equations, which may be solved explicitly.

A model for continuous accretion of linear elastic solids with application to mechanics of dams and embankments was derived in [21], [22], [23], [24], see also [25]. Similar approaches were developed later in [26], [27], [28].

An extension of the model of continuous accretion to linear viscoelastic solids subjected to aging was proposed in [29]. This model was extended to nonlinear viscoelasticity with finite strains in [30], [31], [32], and to viscoelasto-plasticity with finite strains in [33].

The above cited works on continuous accretion deal with dry winding, when resin flow and consolidation of fiber bundless are neglected. The objective of the present paper is to derive a new mathematical model for stresses arising in a wound composite cylinder with account for resin flow. We concentrate on continuous accretion with finite strains and analyze the effect of preloading and the resin viscosity on stresses and displacements built-up in a growing elastic body.

The exposition is as follows. Section 2 is concerned with kinematics of continuous accretion. In Section 3, we derive governing equations for stresses and displacements in a growing cylinder. These equations are solved explicitly under some additional assumptions in Section 4. Section 5 deals with continuous accretion with infinitesimal strains, where the governing equations are reduced to a linear Volterra integral equation. The latter equation is solved numerically to analyze the effect of material and structural parameters on the pressure on mandrel. Finally, some concluding remarks are formulated in Section 6.

2 Kinematics of deformation

This section is concerned with kinematics of the accretion process with finite strains. For convenience, we begin with some remarks regarding the main configurations for a growing medium, see [20].

For a body with a fixed mass, two basic configurations are employed: the reference configuration, where Lagrangian coordinates $\xi = \{\xi^i\}$ are introduced, and the actual configuration occupied by the medium under the action of external loads. As common practice, the reference configuration is assumed to coincide with the natural (stress-free) configuration.

For a growing medium, three basic configurations are introduced. The first is the reference configuration, where we fix Lagrangian coordinates and postulate a plan (schedule) of accretion. This plan determines which points on boundaries of built-up portions and on the accretion surface merge with each other at any instant t.

The second is the natural (stress-free) configuration of a built-up portion. For the initial body (existed at the instant $t = 0$), the reference configuration coincides with the natural configuration, whereas for built-up portions these configurations may differ from each other.

The third is the actual configuration occupied by a deformed medium at the current instant t. When the natural configuration of a built-up portion differs from the actual configuration of a growing body at instant $\tau^*(\xi)$ when they merge, the built-up portion should be deformed to join the growing body. Preload in an accreted layer is determined as the stress necessary to transform the built-up portion from its natural configuration to the actual configuration of the accretion surface.

We now apply this concept to describe winding of a composite elastic cylinder on a mandrel. The mandrel is modeled as an elastic cylinder with length l and external radius a_1. Winding of layers begins on its external surface at the instant $t = 0$ and occurs in the interval $[0, T]$. Accretion of material is treated as a continuous process. At instant $t \in [0, T]$, the wound composite cylinder occupies in the reference configuration the domain

$$\Omega(t) = \{a_1 \leqq r \leqq a(t), \quad 0 \leqq \theta < 2\pi, \quad 0 \leqq z \leqq l\},$$

where $\{r, \theta, z\}$ are cylindrical coordinates with unit vectors $\bar{e}_r$, $\bar{e}_\theta$, and $\bar{e}_z$. Within the interval $[t, t + dt]$, a cylindrical shell which occupies in the reference configuration the domain

$$d\Omega(t) = \{a(t) < r \leqq a(t + dt), \quad 0 \leqq \theta < 2\pi, \quad 0 \leqq z \leqq l\},$$

is wound around the accreted cylinder and immediately merges with it. The law of material supply

$$a = a(t), \quad a(0) = a_1, \quad a(T) = a_2,$$

is assumed to be given. The rate of accretion $v(t)$ is defined as volume of the material which joins the growing body per unit time

$$v(t) = 2\pi l a(t) \frac{da}{dt}(t). \tag{1}$$

Denote by $\tau^*(r)$ the instant when a built-up portion with polar radius r joins the growing cylinder. The function $\tau^*(r)$ is inverse to the function $a(t)$:

$$\tau^*\big(a(t)\big) = t, \quad a\big(\tau^*(r)\big) = r. \tag{2}$$

Differentiation of the first equality in Eq. (2) implies that

$$\frac{d\tau^*}{dr}(r) = \left[\frac{da}{dt}(\tau^*(r))\right]^{-1}. \tag{3}$$

Continuous growth of a cylinder is treated as a limit of the following process of successive accretion of thin shells (layers). We divide the interval $[0, T]$ by $N + 1$ points $t_n = n\Delta$, where $\Delta = T/N$ and $n = 0, 1, \dots, N$.

At instant t_n, the accreted cylinder occupies in the reference configuration the domain

$$\Omega(t_n) = \{a_1 \leqq r \leqq a(t_n), \quad 0 \leqq \theta < 2\pi, \quad 0 \leqq z \leqq l\}.$$

At instant t_{n+1}, a thin-walled cylindrical shell which occupies in the reference configuration the domain

$$\Delta\Omega(t_n) = \{a(t_n) \leqq r \leqq a(t_{n+1}), \quad 0 \leqq \theta < 2\pi, \quad 0 \leqq z \leqq l\},$$

merges with the cylinder $\Omega(t_n)$.

Any built-up portion consists of a fiber bundle impregnated by a molten resin. For definiteness, we assume that in the layer $\Delta\Omega(t_n)$ fibers occupy the domain

$$\Delta_f\Omega(\tau_n) = \{a(t_n) \leqq r \leqq b(t_n), \quad 0 \leqq \theta < 2\pi, \quad 0 \leqq z \leqq l\},$$

and resin occupies the domain

$$\Delta_r\Omega(t_n) = \{b(t_n) \leqq r \leqq a(t_{n+1}), \quad 0 \leqq \theta < 2\pi, \quad 0 \leqq z \leqq l\}.$$

Points of the wound cylinder in the actual configuration refer to cylindrical coordinates $\{R, \Theta, Z\}$ with unit vectors $\bar{e}_R$, $\bar{e}_\Theta$, and $\bar{e}_Z$. We suppose that deformation of successive layers for transition from the reference to actual configuration is described by the formulas

$$R = \Phi_n(t, r), \quad \Theta = \theta, \quad Z = z, \quad a(t_n) \leqq r \leqq b(t_n), \tag{4}$$

where $\Phi_n(t, r)$ is a function to be found.

The radius-vectors in the reference and actual configurations equal

$$\bar{r}_0 = r\bar{e}_r + z\bar{e}_z, \quad \bar{r} = R\bar{e}_R + Z\bar{e}_Z. \tag{5}$$

Differentiating Eqs. (5), we obtain tangent vectors in the reference and actual configurations

$$\bar{g}_{01} = \bar{e}_r, \quad \bar{g}_{02} = r\bar{e}_\theta, \quad \bar{g}_{03} = \bar{e}_z, \quad \bar{g}_1 = \frac{\partial \Phi_n}{\partial r}(t, r)\,\bar{e}_R, \quad \bar{g}_2 = \Phi_n(t, r)\,\bar{e}_\Theta, \quad \bar{g}_3 = \bar{e}_Z. \tag{6}$$

It follows from Eq. (6) that dual vectors are calculated as

$$\bar{g}_0^{\,1} = \bar{e}_r, \quad \bar{g}_0^{\,2} = \frac{1}{r}\bar{e}_\theta, \quad \bar{g}_0^{\,3} = \bar{e}_z, \quad \bar{g}^1 = \left[\frac{\partial \Phi_n}{\partial r}(t, r)\right]^{-1}\bar{e}_R, \quad \bar{g}^2 = \frac{1}{\Phi_n(t, r)}\bar{e}_\Theta, \quad \bar{g}^3 = \bar{e}_Z. \tag{7}$$

Substituting expressions (6) and (7) into the formula for the deformation gradient for transition from the reference to actual configuration

$$\bar{\nabla}_0\bar{r} = \bar{g}_0^{\,i}\bar{g}_i,$$

we find that

$$\bar{\nabla}_0\bar{r}(t)=\frac{\partial\Phi_n}{\partial r}(t,r)\,\bar{e}_r\bar{e}_R+\frac{\Phi_n(t,r)}{r}\,\bar{e}_\theta\bar{e}_\Theta+\bar{e}_z\bar{e}_Z,\qquad a(t_n)\le r\le b(t_n). \tag{8}$$

The Finger tensor $\hat{F}$ for transition from the reference to actual configuration equals

$$\hat{F}=(\bar{\nabla}_0\bar{r})^T\cdot\bar{\nabla}_0\bar{r}, \tag{9}$$

where T stands for transpose. Combining Eqs. (8) and (9), we find that

$$\hat{F}(t,r)=\left[\frac{\partial\Phi_n}{\partial r}(t,r)\right]^2\bar{e}_R\bar{e}_R+\left[\frac{\Phi_n(t,r)}{r}\right]^2\bar{e}_\Theta\bar{e}_\Theta+\bar{e}_Z\bar{e}_Z,\qquad a(t_n)\le r\le b(t_n). \tag{10}$$

We assume that both fibers and resin are incompressible

$$I_3(\hat{F})=1.$$

It follows from this condition and Eq. (10) that the funciton $\Phi_n(t,r)$ satisfies the equation

$$\frac{\partial\Phi_n}{\partial r}(t,r)=\frac{r}{\Phi_n(t,r)},\qquad a(t_n)\le r\le b(t_n). \tag{11}$$

Integration of Eq. (11) implies that

$$\Phi_n^2(t,r)=r^2+C(t,t_n),\qquad a(t_n)\le r\le b(t_n), \tag{12}$$

where $C(t,t_n)$ is an arbitrary function of time t. Bearing in mind Eq. (11), we present Eq. (8) as follows:

$$\bar{\nabla}_0\bar{r}(t)=\frac{r}{\Phi_n(t,r)}\,\bar{e}_r\bar{e}_R+\frac{\Phi_n(t,r)}{r}\,\bar{e}_\theta\bar{e}_\Theta+\bar{e}_z\bar{e}_Z,\qquad a(t_n)\le r\le b(t_n). \tag{13}$$

We suppose that transition of a built-up portion from its reference to natural (stress-free) configuration is determined by the formulas similar to Eq. (4)

$$r^*=\phi_n(r),\qquad \theta^*=\theta,\qquad z^*=z,\qquad a(t_n)\le r\le b(t_n). \tag{14}$$

Here $\{r^*,\theta^*,z^*\}$ are cylindrical coordinates in the natural configuration with unit vectors $\bar{e}_r{}^*$, $\bar{e}_\theta{}^*$, $\bar{e}_z{}^*$, and $\phi_n(r)$ is a function to be determined. Using the incompressibility condition, it is easy to show that the function $\phi_n(r)$ satisfies Eq. (11), and, therefore, can be presented in the form

$$\phi_n^2(r)=r^2+c(t_n),\qquad a(t_n)\le r\le b(t_n), \tag{15}$$

where $c(t_n)$ is an arbitrary constant. According to Eq. (15), the deformation gradient for transition from the reference to natural configuration equals

$$\bar{\nabla}_0\bar{r}^*=\frac{r}{\phi_n(r)}\,\bar{e}_r\bar{e}_r{}^*+\frac{\phi_n(r)}{r}\,\bar{e}_\theta\bar{e}_\theta{}^*+\bar{e}_z\bar{e}_z{}^*,\qquad a(t_n)\le r\le b(t_n).$$

The inverse tensor is calculated as

$$(\bar{\nabla}_0\bar{r}^*)^{-1}=\frac{\phi_n(r)}{r}\,\bar{e}_r{}^*\bar{e}_r+\frac{r}{\phi_n(r)}\,\bar{e}_\theta{}^*\bar{e}_\theta+\bar{e}_z{}^*\bar{e}_z,\qquad a(t_n)\le r\le b(t_n). \tag{16}$$

We substitute expressions (13) and (16) into the formula for the deformation gradient $\bar{\nabla}^*\bar{r}$ at transition from the natural to actual configuration

$$\bar{\nabla}^*\bar{r} = (\bar{\nabla}_0\bar{r}^*)^{-1} \cdot \bar{\nabla}_0\bar{r}$$

and obtain

$$\bar{\nabla}^*\bar{r}(t) = \frac{\phi_n(r)}{\Phi_n(t,r)}\,\bar{e}_r{}^*\bar{e}_R + \frac{\Phi_n(t,r)}{\phi_n(r)}\,\bar{e}_\theta{}^*\bar{e}_\Theta + \bar{e}_z{}^*\bar{e}_Z, \qquad a(t_n) \le r \le b(t_n). \tag{17}$$

According to Eq. (17), the Finger tensor at transition from the natural to actual configuration

$$\hat{F}^0 = (\bar{\nabla}^*\bar{r})^T \cdot \bar{\nabla}^*\bar{r}$$

is calculated as follows:

$$\hat{F}^0(t,r) = \left[\frac{\phi_n(r)}{\Phi_n(t,r)}\right]^2 \bar{e}_R\bar{e}_R + \left[\frac{\Phi_n(t,r)}{\phi_n(r)}\right]^2 \bar{e}_\Theta\bar{e}_\Theta + \bar{e}_Z\bar{e}_Z, \qquad a(t_n) \le r \le b(t_n). \tag{18}$$

3 Governing equations

Let us suppose that the response of fibers is governed by the constitutive equation of an isotropic elastic medium with strain energy density $W(I_1{}^0, I_2{}^0)$, where $I_k{}^0$ is the kth principal invariant of the Finger tensor $\hat{F}^0$. We employ the Finger formula for the Cauchy stress tensor, see e.g. [34],

$$\hat{\sigma} = -p\hat{I} + 2[\psi_1\hat{F}^0 + \psi_2(\hat{F}^0)^2], \tag{19}$$

where p is pressure, $\hat{I}$ is the unit tensor, and the scalar functions ψ_1 and ψ_2 equal

$$\psi_1 = \frac{\partial W}{\partial I_1{}^0} + I_1{}^0\frac{\partial W}{\partial I_2{}^0}, \qquad \psi_2 = -\frac{\partial W}{\partial I_2{}^0}. \tag{20}$$

Substituting expression (18) into Eq. (19), we find the following non-zero components of the stress tensor:

$$\sigma_{RR}(t,r) = -p(t,r) + 2\left[\frac{\phi_n(r)}{\Phi_n(t,r)}\right]^2 \left\{\tilde{\psi}_1(t,r) + \tilde{\psi}_2(t,r)\left[\frac{\phi_n(r)}{\Phi_n(t,r)}\right]^2\right\},$$

$$\sigma_{\Theta\Theta}(t,r) = -p(t,r) + 2\left[\frac{\Phi_n(t,r)}{\phi_n(r)}\right]^2 \left\{\tilde{\psi}_1(t,r) + \tilde{\psi}_2(t,r)\left[\frac{\Phi_n(t,r)}{\phi_n(r)}\right]^2\right\}, \tag{21}$$

$$\sigma_{ZZ}(t,r) = -p(t,r) + 2[\tilde{\psi}_1(t,r) + \tilde{\psi}_2(t,r)], \qquad a(t_n) \le r \le b(t_n),$$

where

$$\tilde{\psi}_k(t,r) = \psi_k\big(I^0(t,r), I^0(t,r)\big), \qquad I^0(t,r) = \left[\frac{\Phi_n(t,r)}{\phi_n(r)}\right]^2 + \left[\frac{\phi_n(r)}{\Phi_n(t,r)}\right]^2 + 1. \tag{22}$$

Let $P(t, t_n)$ denote pressure in resin in the domain $\Delta_r\Omega(t_n)$. The equilibrium equation for the layer $\Delta_f\Omega(t_n)$ reads

$$\frac{\partial\sigma_{RR}}{\partial R} + \frac{1}{R}(\sigma_{RR} - \sigma_{\Theta\Theta}) = 0. \tag{23}$$

Integration of Eq. (23) with the boundary conditions

$$\sigma_{RR}|_{r=a(t_n)} = -P(t, t_{n-1}), \qquad \sigma_{RR}|_{r=b(t_n)} = -P(t, t_n),$$

implies that

$$P(t, t_{n-1}) - P(t, t_n) + \int_{\Phi_n(a(t_n))}^{\Phi_n(b(t_n))} (\sigma_{RR} - \sigma_{\Theta\Theta}) R^{-1}\, dR = 0. \tag{24}$$

According to Eqs. (4) and (11),

$$\frac{dR}{R} = \frac{\partial \Phi_n}{\partial r}(t, r) \frac{dr}{\Phi_n(t, r)} = \frac{r\, dr}{\Phi_n^2(t, r)}.$$

Substitution of this expression and Eqs. (21) into Eq. (24) yields

$$P(t, t_n) - P(t, t_{n-1}) = 2 \int_{a(t_n)}^{b(t_n)} \left\{ \tilde{\psi}_1(t, r) \left[\left(\frac{\phi_n(r)}{\Phi_n(t, r)} \right)^2 - \left(\frac{\Phi_n(t, r)}{\phi_n(r)} \right)^2 \right] \right.$$
$$\left. + \tilde{\psi}_2(t, r) \left[\left(\frac{\phi_n(r)}{\Phi_n(t, r)} \right)^4 - \left(\frac{\Phi_n(t, r)}{\phi_n(r)} \right)^4 \right] \right\} \frac{r\, dr}{\Phi_n^2(t, r)}. \tag{25}$$

Denote by $\eta(t_n)$ concentration of resin in a built-up portion which merges with the wound cylinder at instant t_n

$$\eta(t_n) = \frac{a^2(t_{n+1}) - b^2(t_n)}{a^2(t_{n+1}) - a^2(t_n)}.$$

It follows from this equality that

$$b(t_n) = a(t_n) \sqrt{1 + [1 - \eta(t_n)] \left[\left(\frac{a(t_{n+1})}{a(t_n)} \right)^2 - 1 \right]}. \tag{26}$$

Up to the second order terms compared with Δ,

$$a(t_{n+1}) = a(t_n) + \frac{da}{dt}(t_n)\, \Delta. \tag{27}$$

Substituting expression (27) into Eq. (26) and neglecting terms of the second order compared with Δ, we obtain

$$b(t_n) = a(t_n) + [1 - \eta(t_n)] \frac{da}{dt}(t_n)\, \Delta. \tag{28}$$

Combining Eqs. (25) and (28), we arrive at the formula

$$\frac{1}{\Delta}[P(t, t_n) - P(t, t_{n-1})]$$
$$= \frac{2[1 - \eta(t_n)]\, a(t_n)}{\Phi_n^2(t, a(t_n))} \frac{da}{dt}(t_n) \left\{ \tilde{\psi}_1(t, a(t_n)) \left[\left(\frac{\phi_n(a(t_n))}{\Phi_n(t, a(t_n))} \right)^2 - \left(\frac{\Phi_n(t, a(t_n))}{\phi_n(a(t_n))} \right)^2 \right] \right.$$
$$\left. + \tilde{\psi}_2(t, a(t_n)) \left[\left(\frac{\phi_n(a(t_n))}{\Phi_n(t, a(t_n))} \right)^4 - \left(\frac{\Phi_n(t, a(t_n))}{\phi_n(a(t_n))} \right)^4 \right] \right\}. \tag{29}$$

Continuous accretion is determined by a function of two variables $P(t, \tau^*)$ which equals the pressure at instant t in resin in a built-up portion manufactured at instant τ^*, and a function of one variable $\eta(\tau^*(r))$ which equals the concentration of the liquid phase at the instant of accretion at point with polar radius r:

$$P(t, \tau^*(r)) = P(t, t_n), \quad \eta(\tau^*(r)) = \eta(t_n), \quad a(t_n) \le r \le a(t_{n+1}).$$

Approaching the limit as $N \to \infty$ in Eq. (29), we find that

$$\frac{\partial P}{\partial \tau^*}(t, \tau^*(r)) = \frac{2r[1-\eta(\tau^*(r))]}{\Phi^2(t, r)} \frac{da}{dt}(\tau^*(r)) \left\{ \tilde{\psi}_1(t, r) \left[\left(\frac{\phi(r)}{\Phi(t, r)} \right)^2 - \left(\frac{\Phi(t, r)}{\phi(r)} \right)^2 \right] + \tilde{\psi}_2(t, r) \left[\left(\frac{\phi(r)}{\Phi(t, r)} \right)^4 - \left(\frac{\Phi(t, r)}{\phi(r)} \right)^4 \right] \right\}. \tag{30}$$

The functions $\phi(r)$ and $\Phi(t, r)$ are determined by analogy with Eqs. (12) and (15)

$$\phi^2(r) = r^2 + c(\tau^*(r)), \quad \Phi^2(t, r) = r^2 + C(t, \tau^*(r)), \tag{31}$$

where $C(t, \tau^*)$ and $c(\tau^*)$ are limits of the step-wise functions $C(t, t_n)$ and $c(t_n)$. To transform Eq. (30), we set

$$\Pi(t, r) = P(t, \tau^*(r)).$$

It follows from Eq. (3) that

$$\frac{\partial \Pi}{\partial r}(t, r) = \frac{\partial P}{\partial \tau^*}(t, \tau^*(r)) \left[\frac{da}{dt}(\tau^*(r)) \right]^{-1}.$$

Combining this equality with Eq. (30), we find that

$$\frac{\partial \Pi}{\partial r}(t, r) = \frac{2r[1-\eta(\tau^*(r))]}{\Phi^2(t, r)} \left\{ \tilde{\psi}_1(t, r) \left[\left(\frac{\phi(r)}{\Phi(t, r)} \right)^2 - \left(\frac{\Phi(t, r)}{\phi(r)} \right)^2 \right] + \tilde{\psi}_2(t, r) \left[\left(\frac{\phi(r)}{\Phi(t, r)} \right)^4 - \left(\frac{\Phi(t, r)}{\phi(r)} \right)^4 \right] \right\}. \tag{32}$$

Let us now return to successive accretion of thin layers and calculate volume $\Delta V(t, t_n)$ of a part of the built-up portion $\Delta\Omega(t_n)$ occupied by resin at instant t

$$\Delta V(t, t_n) = \pi l [\Phi_{n+1}^2(t, a(t_{n+1})) - \Phi_n{}^2(t, b(t_n))].$$

Substitution of expression (12) into this equality implies that

$$\Delta V(t, t_n) = \pi l \{[a^2(t_{n+1}) + C(t, t_{n+1})] - [b^2(t_n) + C(t, t_n)]\}.$$

Up to the second order terms compared with δt, the increment $\delta V(t, t_n)$ of the volume $\Delta V(t, t_n)$ in the interval $[t, t + \delta t]$ equals

$$\delta V(t, t_n) = \pi l \frac{\partial}{\partial t} [C(t, t_{n+1}) - C(t, t_n)] \, \delta t. \tag{33}$$

The volume $\delta V(t, t_n)$ equals the difference between the amount of resin $\delta Q_n(t)$ which enters the portion $\Delta\Omega(t_n)$ through the surface $r = a(t_n)$ and the amount of resin $\delta Q_{n+1}(t)$ which leaves this portion through the surface $r = b(t_n)$

$$\delta V(t, t_n) = \delta Q_n(t) - \delta Q_{n+1}(t). \tag{34}$$

The resin flow is assumed to obey the Darcy law, which states that volume of a fluid which passes through a porous wall (per unit area and unit time) is proportional to the difference of pressure and is inversely proportional to the thickness of the wall

$$\begin{aligned} q_n(t) &= \varkappa(t, t_n)\frac{P(t, t_{n-1}) - P(t, t_n)}{\Phi_n\big(t, b(t_n)\big) - \Phi_n\big(t, a(t_n)\big)}, \\ q_{n+1}(t) &= \varkappa(t, t_{n+1})\frac{P(t, t_n) - P(t, t_{n+1})}{\Phi_{n+1}\big(t, b(t_{n+1})\big) - \Phi_{n+1}\big(t, a(t_{n+1})\big)}, \end{aligned} \tag{35}$$

where $\varkappa(t, t_n)$ is a coefficient, which depends on permeability of the fiber bundle manufactured at instant t_n and on the fluid viscosity at instant t. It follows from Eqs. (35) that

$$\begin{aligned} \delta Q_n(t) &= 2\pi\varkappa(t, t_n)\, l\delta t\,\frac{[P(t, t_{n-1}) - P(t, t_n)]\,\Phi_n\big(t, a(t_n)\big)}{\Phi_n\big(t, b(t_n)\big) - \Phi_n\big(t, a(t_n)\big)}, \\ \delta Q_{n+1}(t) &= 2\pi\varkappa(t, t_{n+1})\, l\delta t\,\frac{[P(t, t_n) - P(t, t_{n+1})]\,\Phi_{n+1}\big(t, a(t_{n+1})\big)}{\Phi_{n+1}\big(t, b(t_{n+1})\big) - \Phi_{n+1}\big(t, a(t_{n+1})\big)}. \end{aligned} \tag{36}$$

Substitution of expression (33) and (36) into Eq. (34) implies tat

$$\begin{aligned} \frac{\partial}{\partial t}[C(t, t_{n+1}) - C(t, t_n)] = 2\Bigg\{&\varkappa(t, t_n)\,[P(t, t_{n-1}) - P(t, t_n)]\left[\frac{\Phi_n\big(t, b(t_n)\big)}{\Phi_n\big(t, a(t_n)\big)} - 1\right]^{-1} \\ &- \varkappa(t, t_{n+1})\,[P(t, t_n) - P(t, t_{n+1})]\left[\frac{\Phi_{n+1}\big(t, b(t_{n+1})\big)}{\Phi_{n+1}\big(t, a(t_{n+1})\big)} - 1\right]^{-1}\Bigg\}. \end{aligned} \tag{37}$$

Up to the second order terms compared with Δ, the left-hand side of Eq. (37) is presented as

$$\frac{\partial}{\partial t}\Big[C(t, t_{n+1}) - C(t, t_n)\Big] = \frac{\partial^2 C}{\partial t\,\partial\tau^*}(t, t_n)\,\Delta. \tag{38}$$

To transform the right-hand side of Eq. (37), we use the equalities

$$\begin{aligned} P(t, t_{n-1}) - P(t, t_n) &= -\frac{\partial P}{\partial\tau^*}(t, t_n)\,\Delta + \frac{1}{2}\frac{\partial P}{\partial(\tau^*)^2}(t, t_n)\,\Delta^2, \\ P(t, t_n) - P(t, t_{n+1}) &= -\frac{\partial P}{\partial\tau^*}(t, t_n)\,\Delta - \frac{1}{2}\frac{\partial P}{\partial(\tau^*)^2}(t, t_n)\,\Delta^2, \end{aligned} \tag{39}$$

which are satisfied up to the third order terms compared with Δ. Eqs. (12) and (28) imply that up to the second order terms compared with Δ

$$\begin{aligned} \frac{\Phi_n\big(t, b(t_n)\big)}{\Phi_n\big(t, a(t_n)\big)} - 1 &= \sqrt{\frac{b^2(t_n) + C(t, t_n)}{a^2(t_n) + C(t, t_n)}} - 1 \\ &= \sqrt{1 + \frac{2a(t_n)\,[1 - \eta(t_n)]}{a^2(t_n) + C(t, t_n)}\frac{da}{dt}(t_n)\,\Delta} - 1 = \frac{a(t_n)\,[1 - \eta(t_n)]}{a^2(t_n) + C(t, t_n)}\frac{da}{dt}(t_n)\,\Delta. \end{aligned} \tag{40}$$

It follows from Eqs. (39) and (40) that up to the second order terms compared with Δ the right-hand side of Eq. (37) equals

$$2\frac{\partial P}{\partial \tau^*}(t, t_n)\left\{\frac{\varkappa(t, t_{n+1})[a^2(t_{n+1}) + C(t, t_{n+1})]}{a(t_{n+1})[1 - \eta(t_{n+1})]}\left[\frac{da}{dt}(t_{n+1})\right]^{-1}\right.$$

$$\left. - \frac{\varkappa(t, t_n)[a^2(t_n) + C(t, t_n)]}{a(t_n)[1 - \eta(t_n)]}\left[\frac{da}{dt}(t_n)\right]^{-1}\right\}$$

$$+ \frac{\partial^2 P}{\partial(\tau^*)^2}(t, t_n)\,\Delta\left\{\frac{\varkappa(t, t_{n+1})[a^2(t_{n+1}) + C(t, t_{n+1})]}{a(t_{n+1})[1 - \eta(t_{n+1})]}\left[\frac{da}{dt}(t_{n+1})\right]^{-1}\right.$$

$$\left. + \frac{\varkappa(t, t_n)[a^2(t_n) + C(t, t_n)]}{a(t_n)[1 - \eta(t_n)]}\left[\frac{da}{dt}(t_n)\right]^{-1}\right\}.$$

With the above level of accuracy, this expression reads

$$2\Delta\left\{\frac{\partial P}{\partial \tau^*}(t, \tau^*)\frac{\partial}{\partial \tau^*}\left[\frac{\varkappa(t, \tau^*)\left(a^2(\tau^*) + C(t, \tau^*)\right)}{a(\tau^*)\left(1 - \eta(\tau^*)\right)}\left(\frac{da}{dt}(\tau^*)\right)^{-1}\right]\right.$$

$$\left. + \frac{\partial^2 P}{\partial(\tau^*)^2}(t, \tau^*)\left[\frac{\varkappa(t, \tau^*)\left(a^2(\tau^*) + C(t, \tau^*)\right)}{a(\tau^*)\left(1 - \eta(\tau^*)\right)}\left(\frac{da}{dt}(\tau^*)\right)^{-1}\right]\right\}_{\tau^* = t_n}$$

$$= 2\Delta\frac{\partial}{\partial \tau^*}\left[\frac{\partial P}{\partial \tau^*}(t, \tau^*)\frac{\varkappa(t, \tau^*)\left(a^2(\tau^*) + C(t, \tau^*)\right)}{a(\tau^*)\left(1 - \eta(\tau^*)\right)}\left(\frac{da}{dt}(\tau^*)\right)^{-1}\right]_{\tau^* = t_n}. \tag{41}$$

Substituting expressions (38) and (41) into Eq. (37), we obtain

$$\frac{\partial^2 C}{\partial t\,\partial \tau^*}(t, \tau^*) = 2\frac{\partial}{\partial \tau^*}\left[\frac{\partial P}{\partial \tau^*}(t, \tau^*)\frac{\varkappa(t, \tau^*)\left(a^2(\tau^*) + C(t, \tau^*)\right)}{a(\tau^*)\left(1 - \eta(\tau^*)\right)}\cdot\left(\frac{da}{dt}(\tau^*)\right)^{-1}\right]. \tag{42}$$

Combining Eqs. (30) and (42) and using Eq. (31), we find that

$$\frac{\partial^2 C}{\partial t\,\partial \tau^*}(t, \tau^*) = 4\frac{\partial}{\partial \tau^*}\left\{\varkappa(t, \tau^*)\left[\tilde{\psi}_1(t, r)\left(\left(\frac{\phi(r)}{\Phi(t, r)}\right)^2 - \left(\frac{\Phi(t, r)}{\phi(r)}\right)^2\right)\right.\right.$$

$$\left.\left. + \tilde{\psi}_2(t, r)\left(\left(\frac{\phi(r)}{\Phi(t, r)}\right)^4 - \left(\frac{\Phi(t, r)}{\phi(r)}\right)^4\right)\right]\right\}.$$

Finally, employing Eq. (31), we present this equation as follows:

$$\frac{\partial^2 C}{\partial t\,\partial \tau^*} = 4\frac{\partial}{\partial \tau^*}\left\{\varkappa(t, \tau^*)\left[\tilde{\psi}_1\left(t, a(\tau^*)\right)\left(\frac{a^2(\tau^*) + c(\tau^*)}{a^2(\tau^*) + C(t, \tau^*)} - \frac{a^2(\tau^*) + C(t, \tau^*)}{a^2(\tau^*) + c(\tau^*)}\right)\right.\right.$$

$$\left.\left. + \tilde{\psi}_2\left(t, a(\tau^*)\right)\left(\left(\frac{a^2(\tau^*) + c(\tau^*)}{a^2(\tau^*) + C(t, \tau^*)}\right)^2 - \left(\frac{a^2(\tau^*) + C(t, \tau^*)}{a^2(\tau^*) + c(\tau^*)}\right)^2\right)\right]\right\}. \tag{43}$$

For a given function $a(t)$, which determines the accretion process, and for a given function $c(\tau^*)$, which characterizes preloading, Eq. (43) is a nonlinear partial differential equation for the function of two variables $C(t, \tau^*)$.

Denote by $\Sigma(t_n)$ preloading in a built-up portion $\Delta\Omega(t_n)$ which is wound around the growing cylinder at instant t_n. The stress $\Sigma(t_n)$ equals tangential stress $\sigma_{\Theta\Theta}$ in the built-up portion at instant t_n when this portion merges with the cylinder, i.e. when its configuration coincides with the actual configuration of the growing cylinder. Neglecting radial stress σ_{RR} (which is proportional to thickness of a built-up portion), and equating $\sigma_{\Theta\Theta}$ to $\Sigma(t_n)$, we obtain from Eq. (21) a nonlinear algebraic equation for the function $c(t_n)$

$$2\left\{\tilde{\psi}_1(t_n, a(t_n))\left[\frac{a^2(t_n)+C(t_n, t_n)}{a^2(t_n)+c(t_n)}-\frac{a^2(t_n)+c(t_n)}{a^2(t_n)+C(t_n, t_n)}\right.\right.$$
$$\left.\left.+\tilde{\psi}_2(t_n, a(t_n))\left[\left(\frac{a^2(t_n)+C(t_n, t_n)}{a^2(t_n)+c(t_n)}\right)^2-\left(\frac{a^2(t_n)+c(t_n)}{a^2(t_n)+C(t_n, t_n)}\right)^2\right]\right\}=\Sigma(t_n).$$

Calculating the limit as $\Delta\to 0$, we find that

$$\tilde{\psi}_1(\tau^*, a(\tau^*))\left[\frac{a^2(\tau^*)+C(\tau^*, \tau^*)}{a^2(\tau^*)+c(\tau^*)}-\frac{a^2(\tau^*)+c(\tau^*)}{a^2(\tau^*)+C(\tau^*, \tau^*)}\right]$$
$$+\tilde{\psi}_2(\tau^*, a(\tau^*))\left[\left(\frac{a^2(\tau^*)+C(\tau^*, \tau^*)}{a^2(\tau^*)+c(\tau^*)}\right)^2-\left(\frac{a^2(\tau^*)+c(\tau^*)}{a^2(\tau^*)+C(\tau^*, \tau^*)}\right)^2\right]=\frac{1}{2}\Sigma(\tau^*). \quad (44)$$

4 Filament winding on a rigid mandrel

In applications, rigidity of a mandrel essentially exceeds rigidity of polymer fibers, which implies that the mandrel may be treated as a rigid body. It follows from this assumption and the incompressibility condition for polymer fibers and resin that displacements for transition from the reference to actual configuration equal zero on the interface between the mandrel and the composite cylinder and on the outer boundary of the cylinder

$$C(t, 0)=0, \qquad C(\tau^*, \tau^*)=0. \quad (45)$$

For definiteness, we confine ourselves to a neo-Hookean elastic medium with

$$W=\frac{\mu}{2}(I_1{}^0-3), \quad (46)$$

where μ is the generalized shear modulus. Substitution of expression (46) into Eq. (20) yields

$$\psi_1=\frac{\mu}{2}, \qquad \psi_2=0. \quad (47)$$

Combining equalities (45) and (47) with Eq. (44), we find that

$$\frac{1}{1+\tilde{c}(\tau^*)}-[1+\tilde{c}(\tau^*)]=\Sigma_0(\tau^*), \quad (48)$$

where

$$\tilde{c}(\tau^*)=\frac{c(\tau^*)}{a^2(\tau^*)}, \qquad \Sigma_0(\tau^*)=\frac{\Sigma(\tau^*)}{\mu}. \quad (49)$$

Introducing the notation $\zeta=1+\tilde{c}$, we transform Eq. (48) into the quadratic equation

$$\zeta^2+\Sigma_0(\tau^*)\,\zeta-1=0,$$

which has the only positive root

$$\zeta(\tau^*)=\frac{1}{2}\left[\sqrt{4+\Sigma_0{}^2(\tau^*)}-\Sigma_0(\tau^*)\right]. \quad (50)$$

Returning to the initial notation, we obtain from Eqs. (49) and (50)

$$c(\tau^*) = a^2(\tau^*) \left\{ \frac{1}{2} \left[\sqrt{4 + \left(\frac{\Sigma(\tau^*)}{\mu} \right)^2} - \frac{\Sigma(\tau^*)}{\mu} \right] - 1 \right\}.$$

We substitute expressions (47) and (49) into Eq. (43) and integrate the obtained equality from 0 to τ^*. Bearing in mind Eqs. (45), we arrive at the formula

$$\frac{\partial C}{\partial t}(t, \tau^*) = 2\mu \left\{ \varkappa(t, \tau^*) \left[\frac{1 + \tilde{c}(\tau^*)}{1 + \tilde{C}(t, \tau^*)} - \frac{1 + \tilde{C}(t, \tau^*)}{1 + \tilde{c}(\tau^*)} \right] - \varkappa(t, 0) \left[1 + \tilde{c}(0) - \frac{1}{1 + \tilde{c}(0)} \right] \right\}, \tag{51}$$

where

$$\tilde{C}(t, \tau^*) = \frac{C(t, \tau^*)}{a^2(\tau^*)}. \tag{52}$$

Eq. (51) together with Eq. (48) implies that

$$a^2(\tau^*) \frac{\partial \tilde{C}}{\partial t}(t, \tau^*) = 2\mu\varkappa(t, \tau^*) \left[\frac{\zeta(\tau^*)}{1 + \tilde{C}(t, \tau^*)} - \frac{1 + \tilde{C}(t, \tau^*)}{\zeta(\tau^*)} \right] + 2\Sigma(0)\, \varkappa(t, 0). \tag{53}$$

As common practice, temperature in the winding chamber is not sufficient for curing of resin. Neglecting changes in the resin viscosity caused by the polymerization process, we set

$$\varkappa(t, \tau^*) = \varkappa_0, \tag{54}$$

where $\varkappa_0$ is a constant. It follows from Eqs. (53) and (54) that

$$\frac{\partial \tilde{C}}{\partial t}(t, \tau^*) = \frac{2\mu\varkappa_0}{a^2(\tau^*)} \left\{ \Sigma_0(0) - \left[\frac{1 + \tilde{C}(t, \tau^*)}{\zeta(\tau^*)} - \frac{\zeta(\tau^*)}{1 + \tilde{C}(t, \tau^*)} \right] \right\}. \tag{55}$$

Eqs. (45) and (50) imply that at instant τ^* the right-hand side of Eq. (55) equals

$$\frac{2\mu\varkappa_0}{a^2(\tau^*)} [\Sigma_0(0) - \Sigma_0(\tau^*)].$$

Therefore, differential equation (55) with the second boundary condition in Eq. (45) has a non-zero solution if and only if the dimensionless preload intensity $\Sigma_0(\tau^*)$ is not constant. For a constant preload, $\Sigma(\tau^*) = \Sigma(0)$, fiber bundles do not move with respect to resin. This rather surprising result is obtained under the assumption that the mandrel is rigid.

For a monotonically increasing preload intensity, we integrate Eq. (55) from τ^* to t, utilize the second boundary condition in Eq. (45) and find that

$$\int_0^{\tilde{C}(t,\tau^*)} \left\{ \Sigma_0(0) - \left[\frac{1 + C}{\zeta(\tau^*)} - \frac{\zeta(\tau^*)}{1 + C} \right] \right\}^{-1} dC = \frac{2\mu\varkappa_0}{a^2(\tau^*)} (t - \tau^*). \tag{56}$$

The integral in the left-hand side of Eq. (56) is calculated in elementary functions, but we do not dwell on this question.

5 Accretion with infinitesimal strains

We now concentrate on infinitesimal strains when the dimensionless ratios (49) and (52) are small compared with unity. Bearing in mind Eq. (47), we find from Eq. (44) that

$$\tilde{C}(\tau^*, \tau^*) - \tilde{c}(\tau^*) = \frac{1}{2}\Sigma_0(\tau^*). \tag{57}$$

Linearization of Eq. (43) with the use of Eq. (57) implies that

$$\frac{\partial}{\partial \tau^*}\left[a^2(\tau^*)\frac{\partial \tilde{C}}{\partial t}(t, \tau^*)\right] = -4\mu\frac{\partial}{\partial \tau^*}\left\{\varkappa(t, \tau^*)\left[\tilde{C}(t, \tau^*) - \tilde{C}(\tau^*, \tau^*) + \frac{1}{2}\Sigma_0(\tau^*)\right]\right\}. \tag{58}$$

Integrating Eq. (58) from 0 to τ^*, we obtain

$$a^2(\tau^*)\frac{\partial \tilde{C}}{\partial t}(t, \tau^*) + 4\mu\varkappa(t, \tau^*)\,[\tilde{C}(t, \tau^*) - \tilde{C}(\tau^*, \tau^*)]$$

$$= a_1{}^2\frac{\partial \tilde{C}}{\partial t}(t, 0) + 4\mu\varkappa(t, 0)\,[\tilde{C}(t, 0) - \tilde{C}(0, 0)] + 2\mu[\varkappa(t, 0)\,\Sigma_0(0) - \varkappa(t, \tau^*)\,\Sigma_0(\tau^*)]. \tag{59}$$

Substitution of expressions (49), (52), and (57) into Eq. (30) yields

$$\frac{\partial P}{\partial \tau^*}(t, \tau^*) = -\frac{2\mu[1 - \eta(\tau^*)]}{a(\tau^*)}\frac{da}{dt}(\tau^*)\left[\tilde{C}(t, \tau^*) - \tilde{C}(\tau^*, \tau^*) + \frac{1}{2}\Sigma_0(\tau^*)\right]. \tag{60}$$

We integrate Eq. (60) from 0 to t and employ the boundary condition

$$P(t, t) = 0.$$

As a result, we find the pressure on mandrel $P_0(t) = P(t, 0)$

$$P_0(t) = 2\mu\int_0^t \frac{1 - \eta(\tau^*)}{a(\tau^*)}\frac{da}{dt}(\tau^*)\left[\tilde{C}(t, \tau^*) - \tilde{C}(\tau^*, \tau^*) + \frac{1}{2}\Sigma_0(\tau^*)\right]d\tau^*$$

$$= 2\mu\int_0^t \frac{1 - \eta(\tau^*)}{a(\tau^*)}\frac{da}{dt}(\tau^*)\left[M(t, \tau^*) + \frac{1}{2}\Sigma_0(\tau^*)\right]d\tau^*, \tag{61}$$

where

$$M(t, \tau^*) = \tilde{C}(t, \tau^*) - \tilde{C}(\tau^*, \tau^*). \tag{62}$$

Equation (62) implies that

$$M(\tau^*, \tau^*) = 0. \tag{63}$$

Let K be rigidity of the mandrel and

$$\tilde{C}(t, 0) = -\frac{P_0(t)}{K}. \tag{64}$$

The constant K may be easily expressed in terms of elastic moduli and internal and external radii of the mandrel, provided that it is modeled as a linear elastic cylinder. To avoid tedious calculations, we assume K to be given.

It follows from Eqs. (61) and (64) that $C(0, 0) = 0$. Accounting for this equality and substituting expressions (62) and (64) into Eq. (59), we find that

$$\frac{\partial M}{\partial t}(t, \tau^*) + \frac{4\mu\varkappa(t, \tau^*)}{a^2(\tau^*)} M(t, \tau^*) = F(t, \tau^*), \tag{65}$$

where

$$F(t, \tau^*) = -\frac{1}{K}\left[\left(\frac{a(0)}{a(\tau^*)}\right)^2 \frac{dP_0}{dt}(t) + \frac{4\mu\varkappa(t, 0)}{a^2(\tau^*)} P_0(t)\right] + \frac{2}{a^2(\tau^*)}[\varkappa(t, 0)\, \Sigma(0) - \varkappa(t, \tau^*)\, \Sigma(\tau^*)]. \tag{66}$$

Neglecting curing of resin, see Eq. (54), we integrate Eq. (65) with the boundary condition (63) and obtain

$$M(t, \tau^*) = \int_{\tau^*}^{t} F(s, \tau^*) \exp\left[-\frac{4\mu\varkappa_0}{a^2(\tau^*)}(t - s)\right] ds. \tag{67}$$

Equations (66) and (67) imply that

$$\begin{aligned} M(t, \tau^*) = &-\frac{1}{K}\left(\frac{a(0)}{a(\tau^*)}\right)^2 \int_{\tau^*}^{t} \frac{dP_0}{dt}(s) \exp\left[-\frac{4\mu\varkappa_0}{a^2(\tau^*)}(t - s)\right] ds \\ &- \frac{4\mu\varkappa_0}{a^2(\tau^*)\, K} \int_{\tau^*}^{t} P_0(s) \exp\left[-\frac{4\mu\varkappa_0}{a^2(\tau^*)}(t - s)\right] ds \\ &+ \frac{2\varkappa_0}{a^2(\tau^*)}[\Sigma(0) - \Sigma(\tau^*)] \int_{\tau^*}^{t} \exp\left[-\frac{4\mu\varkappa_0}{a^2(\tau^*)}(t - s)\right] ds. \end{aligned}$$

We integrate by parts the first term in the right-hand side of this equality and calculate the third term. As a result, we arrive at the formula

$$\begin{aligned} M(t, \tau^*) = &-\frac{1}{K}\left(\frac{a(0)}{a(\tau^*)}\right)^2 \left\{P_0(t) - P_0(\tau^*) \exp\left[-\frac{4\mu\varkappa_0}{a^2(\tau^*)}(t - \tau^*)\right]\right\} \\ &- \frac{4\mu\varkappa_0}{a^2(\tau^*)\, K}\left[1 - \left(\frac{a(0)}{a(\tau^*)}\right)^2\right] \int_{\tau^*}^{t} P_0(s) \exp\left[-\frac{4\mu\varkappa_0}{a^2(\tau^*)}(t - s)\right] ds \\ &+ \frac{1}{2}[\Sigma_0(0) - \Sigma_0(\tau^*)]\left\{1 - \exp\left[-\frac{4\mu\varkappa_0}{a^2(\tau^*)}(t - \tau^*)\right]\right\}. \end{aligned} \tag{68}$$

Substitution of expression (68) into Eq. (61) leads to the linear Volterra integral equation

$$H(t)\, P_0(t) + \int_0^t L(t, s)\, P_0(s)\, ds = G(t) \tag{69}$$

with

$$H(t) = 1 + \frac{2\mu}{K} \int_0^t \frac{1-\eta(\tau)}{a(\tau)} \left(\frac{\alpha(0)}{\alpha(\tau)}\right)^2 \frac{da}{dt}(\tau)\, d\tau,$$

$$G(t) = \int_0^t \frac{1-\eta(\tau)}{a(\tau)} \frac{da}{dt}(\tau) \left\{ \Sigma(\tau) + [\Sigma(0) - \Sigma(\tau)] \left[1 - \exp\left(-\frac{4\mu\varkappa_0}{a^2(\tau)}(t-\tau) \right) \right] \right\} d\tau,$$

$$L(t,s) = L_1(t,s) - L_2(t,s),$$

$$L_1(t,s) = \frac{2\mu}{K} \frac{1-\eta(\tau)}{a(\tau)} \left(\frac{a(0)}{a(\tau)}\right)^2 \exp\left[-\frac{4\mu\varkappa_0}{a^2(s)}(t-s) \right] \frac{da}{dt}(s),$$

$$L_2(t,s) = \frac{2\mu}{K} \frac{4\mu\varkappa_0}{a^2(s)} \int_0^s \frac{1-\eta(s)}{a(s)} \left(\frac{a(0)}{a(s)}\right)^2 \left[1 - \left(\frac{a(0)}{a(\tau)}\right)^2 \right] \frac{da}{dt}(\tau) \exp\left[-\frac{4\mu\varkappa_0}{a^2(\tau)}(t-s) \right] d\tau.$$

Assuming the parameters η and Σ to be independent of τ^*, we introduce the dimensionless variables and parameters

$$t_* = \frac{t}{T}, \quad a_* = \frac{a}{a(0)}, \quad P_{0*} = \frac{P_0}{\Sigma(1-\eta)}, \quad v_* = \frac{vT}{\pi a^2(0)\, l}, \quad \tilde{\mu} = \frac{\mu(1-\eta)}{K}, \quad \tilde{\varkappa} = \frac{4\mu\varkappa_0 T}{a^2(0)}.$$

In the new notation, Eqs. (1) and (69) are presented as follows (asterisks are omitted):

$$\frac{dA}{dt}(t) = v(t),$$

$$H(t)\, P_0(t) + \tilde{\mu} \int_0^t [L_1(t,s) - L_2(t,s)]\, P_0(s)\, ds = G(t), \tag{70}$$

where $A(t) = a^2(t)$ and

$$H(t) = 1 + \tilde{\mu} \left[1 - \frac{1}{A(t)} \right],$$

$$G(t) = \frac{1}{2} \ln A(t),$$

$$L_1(t,s) = \frac{v(s)}{A^2(s)} \exp\left(-\tilde{\varkappa} \frac{t-s}{A(s)} \right), \tag{71}$$

$$L_2(t,s) = \frac{1}{t-s} \left\{ \left[1 - \frac{1}{A(s)} \right] \exp\left(-\tilde{\varkappa} \frac{t-s}{A(s)} \right) + \frac{1}{\tilde{\varkappa}(t-s)} \left[\exp\left(-\tilde{\varkappa}(t-s)\right) - \exp\left(-\tilde{\varkappa} \frac{t-s}{A(s)} \right) \right] \right\}.$$

To study the effect of material and structural parameters on the pressure on mandrel P_0, we solve numerically Eqs. (70) for accretion with a constant rate of material supply v. Results of numerical simulation are presented in Figs. 1–4.

Figure 1 demonstrates that the pressure P_0 decreases with the growth of $\tilde{\mu}$. This result is quite natural, since $\tilde{\mu}$ is proportional to the shear modulus of fibers μ. With an increase in μ, any built-up layer effectively resists its deformation, and the influence of mandrel decreases.

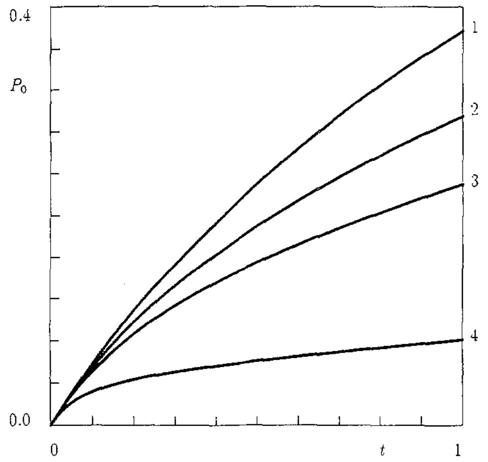

Fig. 1. The dimensionless pressure on mandrel P_0 *versus* the dimensionless time t for a growing cylinder with $a_2 = 1.5a_1$, $v(t) = 1.25t$, and $\tilde{\varkappa} = 0.5$. Curve 1: $\tilde{\mu} = 0.1$, curve 2: $\tilde{\mu} = 0.5$, curve 3: $\tilde{\mu} = 1.0$, curve 4: $\tilde{\mu} = 5.0$

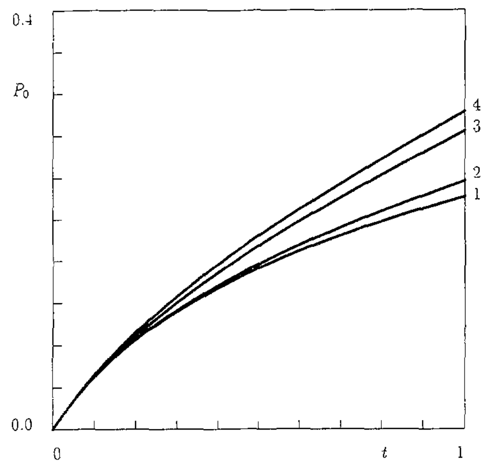

Fig. 2. The dimensionless pressure on mandrel P_0 *versus* the dimensionless time t for a growing cylinder with $a_2 = 1.5a_1$, $v(t) = 1.25t$, and $\tilde{\mu} = 1.0$. Curve 1: $\tilde{\varkappa} = 0.01$, curve 2: $\tilde{\varkappa} = 1.0$, curve 3: $\tilde{\varkappa} = 10.0$, curve 4: $\tilde{\varkappa} = 50.0$

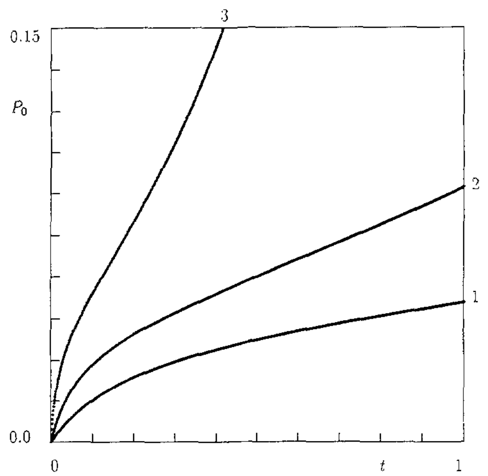

Fig. 3. The dimensionless pressure on mandrel P_0 *versus* the dimensionless time t for a growing cylinder with $\tilde{\mu} = 10.0$, $\tilde{\varkappa} = 500.0$ and a time-independent rate of accretion v. Curve 1: $a_2 = 1.2a_1$, curve 2: $a_2 = 1.5a_1$, curve 3: $a_2 = 2.5a_1$

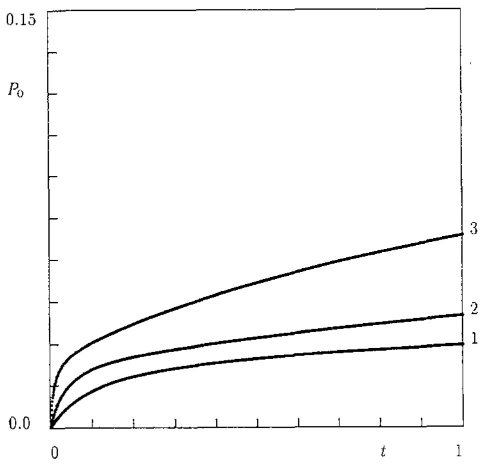

Fig. 4. The dimensionless pressure on mandrel P_0 *versus* the dimensionless time t for a growing cylinder with $\tilde{\mu} = 10.0$, $\tilde{\varkappa} = 0.1$ and a time-independent rate of accretion v. Curve 1: $a_2 = 1.2a_1$, curve 2: $a_2 = 1.5a_1$, curve 3: $a_2 = 2.5a_1$

Figure 2 shows that the pressure P_0 increases with the growth of $\tilde{\varkappa}$. Since $\tilde{\varkappa}$ is inversely proportional to the fluid viscosity, the latter means that an increase in the resin viscosity (e.g. due to its polymerization) leads to a decrease in the pressure on mandrel. This effect follows from Eqs. (71), since for small $\tilde{\varkappa}$

$$L_1(t, s) - L_2(t, s) = \frac{v(s)}{A^2(s)} - \tilde{\varkappa}\left\{\frac{v(s)\,(t-s)}{A^3(s)} + \frac{1}{2}\left[1 + \frac{1}{A^2(s)}\right]\right\}.$$

Combining this equality with Eq. (70), we find that the function $P_0(t)$ increases in $\tilde{\varkappa}$, at least for sufficiently small $\tilde{\varkappa}$ values.

Figures 3 and 4 demonstrate that the pressure P_0 increases with the growth in the accretion rate v. This effect is rather strong for large $\tilde{\varkappa}$ values, when the resin viscosity is small, and it is essentially less pronounced for small $\tilde{\varkappa}$ values.

6 Concluding remarks

A new model is derived for accretion of a composite cylinder with finite strains when resin flow is taken into account. Continuous material supply is treated as a limit of the process of successive merging thin-walled shells (built-up portions) with a growing body, when thicknesses of the shells tend to zero. Preloading of fibers leads to some stresses in built-up layers and to a gradient of pressure in resin. This gradient implies resin flow through the fiber bundles, which is governed by the Darcy law.

The nonlinear partial differential equation (43) is derived for finite radial displacements of fibers when the accretion rate and preload intensity are given. A solution of this equation is developed under the assumption that the mandrel rigidity essentially exceeds rigidity of fibers.

At infinitesimal strains, two linear integro-differential equations (58) and (60) are derived for radial displacements of fibers and pressure on the mandrel. These equations are reduced to a linear Volterra equation (69), which is solved numerically. At any instant t, the pressure on mandrel P_0 is determined by three dimensionless parameters $\tilde{\mu}$, $\tilde{\varkappa}$, and v. It is shown that P_0 decreases with the growth of the shear modulus of fibers $\tilde{\mu}$ and with the growth of the resin viscosity $\tilde{\varkappa}^{-1}$, and it increases with an increase in the rate of accretion v.

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