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On the boundary integral equations approach to a semi-infinite strip investigation

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Summary. An orthotropic semi-infinite strip under arbitrary boundary conditions is considered. By means of Fourier transforms, boundary integral relations of special type with moving and motionless singularities of the Cauchy type are obtained. These relations lead to a system of singular integral equations corresponding to the various mixed boundary value problem. The power of singularities at the corner points, stresses and stress intensity factors are calculated for different loads and various material properties.

1 Introduction

The problem of the isotropic semi-infinite strip has received considerable attention in the literature. Specifically, in [1]–[3] by means of singular integral equations an isotropic one has been studied and in [4] an orthotropic material was discussed. However, the method used in these papers allowed to consider only prescribed boundary conditions, particularly only first mode conditions on the longitudinal sides and homogeneous conditions on the strip end. In this paper boundary integral relations of a special type are derived. These relations give the possibility to formulate the system of singular integral equations for any boundary conditions on the strip edges.

2 Formulation of boundary integral relations

Consider the orthotropic semi-infinite strip shown in Fig. 1. Let in the points $x = \xi$, $y = \pm\eta$ concentrated forces $(\pm F_x, F_y)$ be symmetrically applied, and boundary conditions are arbitrary for a time.

Assuming a state of plane strain, the stress-displacement relations are

$$\sigma_x = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y}; \quad \sigma_y = A_{22}\frac{\partial v}{\partial y} + A_{12}\frac{\partial u}{\partial x};$$

$$\tau_{xy} = A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \quad \sigma_z = A_{13}\frac{\partial u}{\partial x} + A_{23}\frac{\partial v}{\partial y},$$

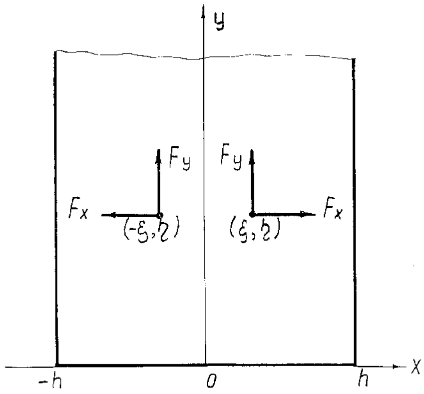

Fig. 1. The orthotropic semi-infinite strip

and the corresponding displacement equations of equilibrium become [5]

$$
\begin{aligned}
&A_{11}\frac{\partial^2 u}{\partial x^2} + A_{66}\frac{\partial^2 u}{\partial y^2} + (A_{12} + A_{66})\frac{\partial^2 v}{\partial x\,\partial y} = -\delta(x-\xi)\,\delta(y-\eta)\,F_x,\\
&A_{22}\frac{\partial^2 v}{\partial y^2} + A_{66}\frac{\partial^2 v}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 u}{\partial x\,\partial y} = -\delta(x-\xi)\,\delta(y-\eta)\,F_y,
\end{aligned}
\tag{1}
$$

where A_{ij} denote the stiffness coefficients for the orthotropic materials. Results for the corresponding problem of plane stress can be obtained from the present formulation by a simple rearrangement of the elastic constants, namely A_{11}, A_{12}, A_{22} should be replaced by $A_{11} - A_{13}^2/A_{33}$, $A_{12} - A_{13}A_{23}/A_{33}$, $A_{22} - A_{23}^2/A_{33}$ respectively.

We introduce the following functions:

$$
\begin{aligned}
&v(x,0) = Q(x), \qquad \tau_{xy}(x,0) = q_1(x);\\
&u(\pm h, y) = \pm F(y), \qquad \tau_{xy}(\pm h, y) = f_1(y).
\end{aligned}
\tag{2}
$$

In order to solve the differential Eq. (1) we introduce the finite Fourier transforms

$$
\bar{u}(p,y) = \int_0^h u(x,y)\sin(\omega p x)\,dx, \qquad \bar{v}(p,y) = \int_0^h v(x,y)\cos(\omega p x)\,dx, \qquad \omega = \pi/h
$$

in which case $\bar{u}$ and $\bar{v}$ are governed by the ordinary differential equations

$$
\begin{aligned}
&A_{66}\frac{d^2\bar{u}}{dy^2} - \omega p(A_{12} + A_{66})\frac{d\bar{v}}{dy} - \omega^2 p^2 A_{11}\bar{u} = -\delta(y-\eta)\sin(\omega p\xi)\,F_x + (-1)^p A_{11}\omega p F(y),\\
&A_{22}\frac{d^2\bar{v}}{dy^2} + \omega p(A_{12} + A_{66})\frac{d\bar{u}}{dy} - \omega^2 p^2 A_{66}\bar{v}\\
&= -\delta(y-\eta)\cos(\omega p\xi)\,F_y - (-1)^p f_1(y) - (-1)^p A_{12}\frac{du(h,y)}{dy}
\end{aligned}
\tag{3}
$$

(we have taken into account that due to symmetry $u(0,y) = 0$ and $\partial v(0,y)/\partial x = 0$). Applying next Fourier transforms with respect to the y-coordinate,

$$
\bar{\bar{u}}(p,t) = \int_0^\infty \bar{u}(p,y)\cos(ty)\,dy, \qquad \bar{\bar{v}}(p,t) = \int_0^\infty \bar{v}(p,y)\sin(ty)\,dy,
$$

to Eq. (3), we arrive at the following algebraic system with respect to $\bar{\bar{u}}$ and $\bar{\bar{v}}$:

$$(A_{11}\omega^2p^2 + A_{66}t^2)\,\bar{\bar{u}} + (A_{12} + A_{66})\,\omega pt\bar{\bar{v}}$$
$$= \cos(t\eta)\sin(\omega p\xi)\,F_x + A_{12}\omega p\bar{Q}(p) - \bar{q}_1(p) - (-1)^pA_{11}\omega p\bar{F}(t),$$
$$(A_{12} + A_{66})\,\omega pt\bar{\bar{u}} + (A_{66}\omega^2p^2 + A_{22}t^2)\,\bar{\bar{v}} \tag{4}$$
$$= \sin(t\eta)\cos(\omega p\xi)\,F_y + A_{22}t\bar{Q}(p) + (-1)^p\bar{f}_1(t) - (-1)^pA_{12}t\bar{F}(t),$$

where

$$\bar{Q}(p) = \int_0^h Q(x)\cos(\omega px)\,dx, \qquad \bar{q}_1(p) = \int_0^h q_1(x)\sin(\omega px)\,dx,$$

$$\bar{F}(t) = \int_0^\infty F(y)\cos(ty)\,dy, \qquad \bar{f}_1(t) = \int_0^\infty f_1(y)\sin(ty)\,dy.$$

The determinant of the system (4) results in

$$D = A_{11}A_{66}(\omega^4p^4 + 2\alpha_1{}^2\omega^2p^2t^2 + \alpha_2{}^4t^4),$$

where

$$\alpha_1{}^2 = (A_{11}A_{22} - 2A_{12}A_{66} - A_{12}^2)/(2A_{11}A_{66}), \qquad \alpha_2{}^2 = A_{22}/A_{11}.$$

Next we assume that $\alpha_1{}^2 > \alpha_2{}^2$, i.e. we consider the most general type of orthotropic material classified as type I [6].

Solution of the system (4) and use of inverse Fourier transforms lead to the following expressions:

$$\frac{\partial u}{\partial x}(x, y) = \frac{1}{8h}\sum_{j=1}^{2}\left\{\int_{-h}^{h} S_j(x, y, t)\,q_j(t)\,dt + \int_0^\infty E_j(x, y, t)\,f_j(t)\,dt\right\} + \frac{\partial\tilde{u}}{\partial x}(x, y),$$
$$\frac{\partial v}{\partial y}(x, y) = \frac{1}{8h}\sum_{j=1}^{2}\left\{\int_{-h}^{h} L_j(x, y, t)\,q_j(t)\,dt + \int_0^\infty R_j(x, y, t)\,f_j(t)\,dt\right\} + \frac{\partial\tilde{v}}{\partial y}(x, y), \tag{5}$$

where

$$q_2(t) = Q'(t), \qquad f_2(t) = F'(t).$$

The formulas for S_j, E_j, L_j, R_j and other expressions needed for their calculation are given in the Appendix.

The following values are defined by the external load:

$$\frac{\partial\tilde{u}}{\partial x}(x, y) = -\frac{1}{16h}\sum_{j=1}^{2}\{F_xs_{j1}X_j(x, y; \xi, \eta) - F_ye_{j1}Y_j(x, y; \xi, \eta)\},$$

$$\frac{\partial\tilde{v}}{\partial y}(x, y) = -\frac{1}{16h}\sum_{j=1}^{2}\{F_xl_{j1}X_j(x, y; \xi, \eta) - F_yr_{j1}Y_j(x, y; \xi, \eta)\},$$

where X_j and Y_j are given in the Appendix.

Extracting in (5) singular components of the integrands we obtain the following formulas for strains and stresses on the strip edges:

$$u'(x, 0) = \Omega_1(x) + \tilde{u}'(x, 0), \tag{6}$$

$$\sigma_y(x,0) = \Omega_2(x) + \tilde{\sigma}_y(x,0), \tag{7}$$

$$v'(h,y) = \Omega_3(y) + \tilde{v}'(h,y), \tag{8}$$

$$\sigma_x(h,y) = \Omega_4(y) + \tilde{\sigma}_x(h,y), \tag{9}$$

where

$$\tilde{\sigma}_y(x,0) = A_{22}\tilde{v}'(x,0) + A_{12}\tilde{u}'(x,0),$$

$$\tilde{\sigma}_x(h,y) = A_{11}\tilde{u}'(h,y) + A_{12}\tilde{v}'(h,y),$$

$$\Omega_i(x) = \frac{1}{4\pi}\sum_{j=1}^{2}\left\{\gamma_{ij}\int_{-h}^{h}[B(x,t) + \varkappa M_{ij}(x,t)]\, q_j(t)\,dt + 2\int_0^\infty [C_{ij}(x,t) + \varkappa M_{i,j+2}(x,t)]\, f_j(t)\,dt\right\}$$

$$\Omega_{i+2}(y) = \frac{1}{4\pi}\sum_{j=1}^{2}\left\{\int_{-h}^{h}[D_{ij}(y,t) + \varkappa M_{i+2,j}(y,t)]\, q_j(t)\,dt \right.$$

$$\left. + \int_0^\infty [\mu_{i+2,j}G(y,t) + \varkappa M_{i+2,j+2}(y,t)]\, f_j(t)\,dt\right\}, \qquad i,j = 1, 2.$$

Singular components of the kernels in the expressions (6)—(9) are the following:

$$B(x,t) = \frac{1}{2h - x - t} - \frac{1}{t - x} - \frac{1}{2h + x + t};$$

$$G(y,t) = \frac{1}{t - y} + \frac{1}{t + y};$$

$$C_{ij}(x,t) = \mu_{ij}\left[\frac{t}{t^2 + [(h-x)/k_1]^2} + \frac{t}{t^2 + [(h+x)/k_1]^2}\right]$$

$$+ \mu_{i,j+2}\left[\frac{t}{t^2 + [(h-x)/k_2]^2} + \frac{t}{t^2 + [(h+x)/k_2]^2}\right];$$

$$D_{ij}(y,t) = \gamma_{i+2,j}\left[\frac{h-t}{(h-t)^2 + (k_1y)^2} - \frac{h+t}{(h+t)^2 + (k_1y)^2}\right]$$

$$+ \gamma_{i+2,j+2}\left[\frac{h-t}{(h-t)^2 + (k_2y)^2} - \frac{h+t}{(h+t)^2 + (k_2y)^2}\right],$$

and the regular ones can be written in the form

$$M_{ij}(x,t) = -\operatorname{ctg}\left(\frac{\varrho}{2}\right) - \frac{2}{2\pi - \varrho} + \frac{2}{2\pi + \varrho} + \frac{2}{\varrho}, \qquad \varrho = \omega(t - x),$$

$$M_{i,j+2}(x,t) = \mu_{ij}k_1H_1(x,t) + \mu_{i,j+2}k_2H_2(x,t),$$

$$M_{i+2,j}(y,t) = \gamma_{i+2,j}N_1(y,t) + \gamma_{i+2,j+2}N_2(y,t),$$

$$M_{3,j+2}(y,t) = r_{1j}\Gamma_1(y,t) + r_{2j}\Gamma_2(y,t),$$

$$M_{4,j+2}(y,t) = (A_{11}e_{1j} + A_{12}r_{1j})\,\Gamma_1(y,t) + (A_{11}e_{2j} + A_{12}r_{2j})\,\Gamma_2(y,t), \qquad i,j = 1, 2$$

where expressions for H_i, N_i, Γ_i and values needed for their calculations are given in the Appendix. In the last expressions functions $M_{ij}(x, t) \in H$ $(i, j = \overline{1, 4})$, and H is the class of Hölder functions [7].

Formulas (6)–(9) can be called boundary integral relations (BIR) for any orthotropic semi-infinite strip. They are very convenient for systems of singular integral equations formulation corresponding to prescribed, especially mixed boundary conditions at the strip edges.

3 Solution for fixed end conditions

To illustrate the correctness of BIR, we consider the problem in question under the previously reported boundary conditions [1], [2], [4]:

$$v(x, 0) = 0, \quad u(x, 0) = 0, \quad |x| < h, \tag{10}$$

$$\sigma_x(\pm h, y) = 0, \quad \tau_{xy}(\pm h, y) = 0, \quad y > 0, \tag{11.1, 2}$$

i.e. the end of the strip is fixed and the longitudinal sides are traction free.

It follows from (10), (11) that $q_2(x) \equiv 0$, $f_1(y) \equiv 0$, and from relation $u'(x, 0) = 0$ and Eq. (11.1) we obtain the following system of singular integral equations:

$$\begin{gathered}
\gamma_{11} \int_{-h}^{h} [B(x, t) + \varkappa M_{11}(x, t)]\, q_1(t)\, dt + 2 \int_0^\infty [C_{12}(x, t) + \varkappa M_{14}(x, t)]\, f_2(t)\, dt + 4\pi \tilde{u}'(x, 0) = 0, \\
x \in (-h, h), \\
\int_{-h}^{h} [D_{21}(y, t) + \varkappa M_{41}(y, t)]\, q_1(t)\, dt + \int_0^\infty [\mu_{42} G(y, t) + \varkappa M_{44}(y, t)]\, f_2(t)\, dt + 4\pi \tilde{\sigma}_x(h, y) = 0, \\
y \in (0, \infty).
\end{gathered} \tag{12}$$

The solution of (12) has the form

$$q_1(t) = \frac{q_1^*(t)}{(h^2 - t^2)^\alpha}, \quad f_2(t) = \frac{f_2^*(t)}{t^\alpha}, \tag{13.1, 2}$$

where $q_1^*(t)$, $f_2^*(t) \in H$ and $0 \leqq \mathrm{Re}(\alpha) < 1$.

Next we consider Eq. (13.1) for $x \to h - 0$ and Eq. (13.2) for $y \to +0$ and use the method based upon Cauchy type integrals [7]. This yields the following relations:

$$\begin{gathered}
\frac{\gamma_{11}[1 + \cos(\pi\alpha)]}{(2h)^\alpha}\, q_1^*(h) + (\mu_{12} k_1^{\alpha} + \mu_{14} k_2^{\alpha})\, (e^{(3/2)\pi\alpha i} + e^{(\pi/2)\alpha i})\, f_2^*(0) = \Phi_1^\circ(x)\, (h - x)^\alpha|_{x \to h - 0}, \\
\frac{\gamma_{41} k_1^{-\alpha} + \gamma_{43} k_2^{-\alpha}}{(2h)^\alpha}\, (e^{-(3/2)\pi\alpha i} + e^{-(\pi/2)\alpha i})\, q_1^*(h) + 2\mu_{42}[1 + \cos(\pi\alpha)]\, f_2^*(0) \\
= \Phi_2^\circ(y)\, y^\alpha|_{y \to +0}
\end{gathered} \tag{14}$$

where Φ_i° $(i = 1, 2)$ can have weaker singularities than the functions (13). It follows from (14) that the non-zero solution of this system appears if α satisfies the following equation:

$$d_1[\cos(\pi\alpha) + 1] - d_2\theta^\alpha - d_3\theta^{-\alpha} - d_4 = 0, \tag{15}$$

where

$$\theta = \frac{k_1}{k_2}, \quad d_1 = \gamma_{11}\mu_{42}, \quad d_2 = \gamma_{43}\mu_{12}, \quad d_3 = \gamma_{41}\mu_{14}, \quad d_4 = \gamma_{41}\mu_{12} + \gamma_{43}\mu_{14}.$$

It is important to note that for any values of A_{ij} the solution of (15) coincides with the roots of the corresponding equation (47) of [4], which were obtained by using another approach.

The numerical solution of the system (12) has been found under additional conditions,

$$\int_0^h \sigma_y(x, 0)\, dx = P_y,$$

$$\int_{-h}^{h} q_1(t)\, dt = 0,$$

and for various positions of applied forces. It turned out that for previously reported cases in [2], [4] the values of $\sigma_y(x, 0)$ and $\tau_{xy}(x, 0)$ are in good agreement to each other.

4 Mixed boundary conditions at the strip end

We consider now the problem shown in Fig. 2, where a rigid stamp of width 2a is acted without friction to the strip end. Boundary conditions are the following:

$$v'(x, 0) = 0, \qquad \tau_{xy}(x, 0) = 0, \qquad |x| < a; \tag{16}$$

$$\sigma_y(x, 0) = 0, \qquad \tau_{xy}(x, 0) = 0, \qquad a < |x| < h; \tag{17.1, 2}$$

$$u(\pm h, y) = 0, \qquad v(\pm h, y) = 0, \qquad y > 0. \tag{18}$$

Taking into account that in this case $q_1(x) \equiv 0$, $f_2(y) \equiv 0$ and $q_2(x) = 0$ for $|x| < a$ and using relations (6)–(9) we obtain from Eq. (17.1) and $v'(h, y) = 0$ $(y > 0)$ the following system of singular integral equations:

$$\gamma_{22} \int_a^h \left[B(x, t) - B(x, -t) + \varkappa\left(M_{22}(x, t) - M_{22}(x, -t)\right)\right] q_2(t)\, dt$$

$$+ 2 \int_0^c \left[C_{21}(x, t) + \varkappa M_{23}(x, t)\right] f_1(t)\, dt = 0, \qquad x \in (a, h); \tag{19}$$

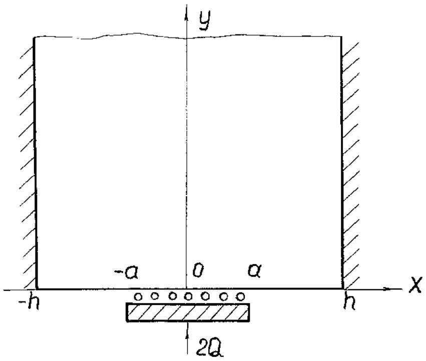

Fig. 2. Orthotropic semi-infinite strip under frictionless action of a rigid stamp

$$\int_a^h [D_{12}(y,t) - D_{12}(y,-t) + \varkappa(M_{32}(y,t) - M_{32}(y,-t))]\, q_2(t)\, dt$$

$$+ \int_0^c [\mu_{31} G(y,t) + \varkappa M_{33}(y,t)]\, f_1(t)\, dt = 0, \qquad y \in (0,c). \tag{19}$$

In the last relations, due to St. Venant's principle, we use $c \gg h$ instead of ∞.

The solution of the system (19) we present in the form

$$q_2(t) = \frac{q_2{}^*(t)}{\sqrt{t-a}\,(h-t)^\alpha}, \qquad f_1(t) = \frac{f_1{}^*(t)}{t^\alpha \sqrt{c-t}}, \qquad q_2{}^*(t),\ f_1{}^*(t) \in H. \tag{20}$$

The power of singularity α in this case has the same value as in (13).

The additional conditions for the system (19) yield the equations of equilibrium and can be written in the form

$$\int_0^c f_1(t)\, dt = -Q, \qquad \int_0^a \sigma_y(x,0)\, dx = -Q. \tag{21}$$

For the numerical solution of the system (19), (21), a numerical method based upon the Gauss-Jacobi integration formula [8]

$$\int_{-1}^{1} f(\tau)\, w(\tau)\, d\tau = \sum_{k=1}^{n} A_k f(\tau_k) \tag{22}$$

was used. In this formula $w(\tau) = (1-\tau)^{-\alpha}(1+\tau)^{-\beta}$; A_k are weight coefficients; τ_k are the zeros of Jacobi polynomials $P_n^{(-\alpha,-\beta)}(\tau)$. To apply formula (22) to the singular integral evaluations we used the zeros z_m $(m = 1, 2, \ldots, n-1)$ of the second kind Jacobi functions [9]

$$Q_N^{(-\alpha,-\beta)}(z) = -\int_{-1}^{1} \frac{P_N^{(-\alpha,-\beta)}(\tau)\, w(\tau)\, d\tau}{\tau - z}$$

as the points of collocations. Application of this method to the system (19), (21) yields the following algebraic equations:

$$\zeta_2 \gamma_{22} \sum_{k=1}^{n} A_{1k}[B(x_{1m}, t_{1k}) - B(x_{1m}, -t_{1k}) + \varkappa(M_{22}(x_{1m}, t_{1k}) - M_{22}(x_{1m}, -t_{1k}))]\, q_2{}^*(t_{1k})$$

$$+ 2\chi \sum_{k=1}^{n} A_{2k}[C_{21}(x_{1m}, t_{2k}) + \varkappa M_{23}(x_{1m}, t_{2k})]\, f_1{}^*(t_{2k}) = 0$$

$$\zeta_2 \sum_{k=1}^{n} A_{1k}[D_{12}(x_{2m}, t_{1k}) - D_{12}(x_{2m}, -t_{1k}) + \varkappa(M_{32}(x_{2m}, t_{1k}) - M_{32}(x_{2m}, -t_{1k}))]\, q_2{}^*(t_{1k})$$

$$+ \chi \sum_{k=1}^{n} A_{2k}[\mu_{31} G(x_{2m}, t_{2k}) + \varkappa M_{33}(x_{2m}, t_{2k})]\, f_1{}^*(t_{2k}) = 0$$

where

$$t_{1k} = \zeta_1 + \zeta_2 \tau_{1k}, \qquad t_{2k} = \chi(1 + \tau_{2k}), \qquad x_{1m} = \zeta_1 + \zeta_2 z_{1m}, \qquad x_{2m} = \chi(1 + z_{2m}),$$

$$\zeta_1 = \frac{h+a}{2}, \qquad \zeta_2 = \frac{h-a}{2}, \qquad \chi = \frac{c}{2}.$$

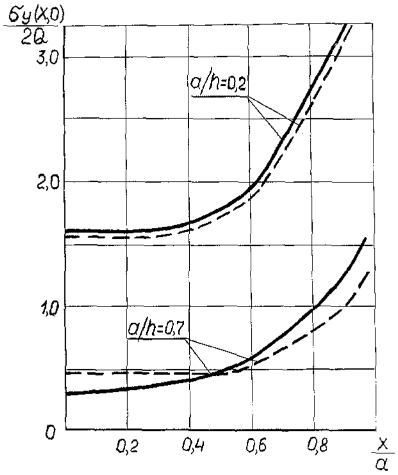

Fig. 3. Normal stresses under rigid stamp acting to the semi-infinite strip

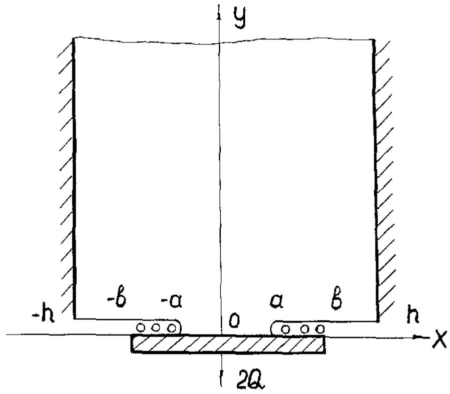

Fig. 4. Delamination of rigid stamp from semi-infinite strip

The solid line in Fig. 3 shows the variation of the normal stress acting under the stamp for nearly isotropic material with $v = 0.3$ and various relative stamp widths a/h. Dashed lines relate to the exact solution for a semi-infinite plane [10]. It is clear from Fig. 3 that for relatively small stamp widths these results almost coincide, but for stamp widths nearly equal to strip widths their differences are essential.

Next we consider the bonded rigid stamp pull off the end of the strip (Fig. 4). To avoid the oscillating singularity we introduce two frictionless contact zones $[-b, -a]$ and $[a, b]$ at the stamp corners. In this case the boundary conditions are the following:

$$u(x, 0) = 0, \qquad v(x, 0) = 0, \qquad |x| < a; \tag{23}$$

$$v(x, 0) = 0, \qquad \tau_{xy}(x, 0) = 0, \qquad a < |x| < b; \tag{24}$$

$$\tau_{xy}(x, 0) = 0, \qquad \sigma_y(x, 0) = 0, \qquad b < |x| < h; \tag{25}$$

$$\sigma_x(\pm h, y) = 0, \qquad \tau_{xy}(\pm h, y) = 0, \qquad y > 0. \tag{26}$$

It follows from the last equations that $q_1(x) = 0$ for $|x| > a$, $q_2(x) = 0$ for $|x| < b$ and $f_2(y) \equiv 0$. By means of the remaining boundary conditions satisfaction we arrive at the following system of singular integral equations (similarly to (19) we use here $c \gg h$ instead of ∞):

$$\begin{aligned}
&\gamma_{11} \int_{-a}^{a} [B(x, t) + \varkappa M_{11}(x, t)]\, q_1(t)\, dt + \gamma_{12} \int_{b}^{h} [B(x, t) - B(x, -t) + \varkappa(M_{12}(x, t) - M_{12}(x, -t))] \\
&\times q_2(t)\, dt + 2 \int_{0}^{c} [C_{11}(x, t) + \varkappa M_{13}(x, t)]\, f_1(t)\, dt = 0, \qquad x \in (-a, a); \\
&\gamma_{21} \int_{-a}^{a} [B(x, t) + \varkappa M_{21}(x, t)]\, q_1(t)\, dt + \gamma_{22} \int_{b}^{h} [B(x, t) - B(x, -t) + \varkappa(M_{22}(x, t) - M_{22}(x, -t))] \\
&\times q_2(t)\, dt + 2 \int_{0}^{c} [C_{21}(x, t) + \varkappa M_{23}(x, t)]\, f_1(t)\, dt = 0, \qquad x \in (b, h); \\
&\int_{-a}^{a} [D_{11}(y, t) + \varkappa M_{31}(y, t)]\, q_1(t)\, dt + \int_{b}^{h} [D_{12}(y, t) - D_{12}(y, -t) + \varkappa(M_{32}(y, t) - M_{32}(y, -t))] \\
&\times q_2(t)\, dt + \int_{0}^{c} [\mu_{31} G(y, t) + \varkappa M_{33}(y, t)]\, f_1(t)\, dt = 0, \qquad y \in (0, c).
\end{aligned} \tag{27}$$

Table 1. Stress intensity factors for interface crack between rigid stamp and semi-infinite strip

a/hr	λ	$\tilde{K}_1$	$\tilde{K}_2$	$\tilde{K}$
0,2	10^{-2}	0,483	0,594	0,812
	10^{-3}	0,276	0,759	0,823
	10^{-4}	0,180	0,802	0,828
0,5	10^{-2}	0279	0,421	0,526
	10^{-3}	0,132	0,492	0,513
	10^{-4}	0,065	0,498	0,510
0,7	10^{-2}	0,235	0,382	0,462
	10^{-3}	0,085	0,440	0,450
	10^{-4}	0,016	0,443	0,449

Unknown functions we present in the form

$$q_1(t) = \frac{q_1{}^*(t)}{\sqrt{a^2 - t^2}}, \quad q_2(t) = \frac{q_2{}^*(t)}{\sqrt{t - b}\,(h - t)^\alpha}, \quad f_1(t) = \frac{f_1{}^*(t)}{t^\alpha \sqrt{c - t}}, \quad q_1{}^*(t), q_2{}^*(t), f_1{}^*(t) \in H \tag{28}$$

and α is the same as earlier. Additional conditions are the following:

$$\int_0^c f_1(t)\,dt = Q, \quad \int_0^a \sigma_y(x, 0)\,dx = Q, \quad \int_{-a}^a q_1(t)\,dt = 0. \tag{29}$$

Next we introduce the following stress intensity factors at the points a and b:

$$K_1 = \lim_{x \to b-0} \sqrt{2(b - x)}\,\sigma_y(x, 0) \qquad K_2 = \lim_{x \to a-0} \sqrt{2(a - x)}\,\tau_{xy}(x, 0). \tag{30}$$

It follows from (28) and (7) that

$$K_1 = -\frac{\sqrt{2}\,\gamma_{22} q_2{}^*(b)}{4(h - b)^\alpha}, \qquad K_2 = \frac{q_1{}^*(a)}{\sqrt{a}}.$$

A numerical solution of the system (27), (29) was obtained by the method outlined above. The values of $\tilde{K}_1 = K_1/(2Q\sqrt{h})$, $\tilde{K}_2 = K_2/(2Q\sqrt{h})$ and

$$\tilde{K} = \frac{1}{2Q\sqrt{h}} \sqrt{\omega K_1{}^2 + K_2{}^2}, \qquad \omega = \frac{4(1 - \nu)^2}{3 - 4\nu}$$

for various a/h and $\lambda = (b - a)/(2b)$ are given in the Table. A nearly isotropic material with $\nu = 0.02$ has been used. It can be easily seen from the results that in spite of $\tilde{K}_1$ and $\tilde{K}_2$ essentially depending on λ, parameter $\tilde{K}$ similarly to infinite domains [11] is quasi-invariant with respect to λ.

5 Conclusions

The method of orthotropic semi-infinite strip investigation under arbitrary boundary conditions is described. It is based upon boundary integral relations of special type. Good agreement with previously reported results was found for a particular case of boundary conditions and external load. New results corresponding to the interaction of strip end with a rigid stamp have been obtained by approximate solution of the boundary integral equations.

Appendix

$$H_i(x, t) = \frac{sh(\omega k_i t)}{ch(\omega k_i t) + \cos(\omega x)} - 2 \sum_{n=1}^{2} \frac{\omega k_i t}{[\omega k_i t]^2 + [\pi + (-1)^n \omega x]^2};$$

$$N_i(y, t) = \frac{\sin(\omega t)}{\cos(\omega t) + ch(\omega k_i y)} + 2 \sum_{n=1}^{2} (-1)^n \frac{\pi + (-1)^n \omega t}{[\pi + (-1)^n \omega t]^2 + [\omega k_i y]^2};$$

$$\Gamma_i(y, t) = \sum_{n=1}^{2} \left[\frac{sh[\omega k_i(t + (-1)^n y)]}{ch[\omega k_i(t + (-1)^n y)] - 1} - \frac{2}{\omega k_i[t + (-1)^n y]} \right];$$

$$S_j(x, y, t) = \sum_{i=1}^{2} s_{ij} Z_i(x, y, t), \quad E_j(x, y, t) = \sum_{i=1}^{2} e_{ij} T_i(x, y, t),$$

$$L_j(x, y, t) = \sum_{i=1}^{2} l_{ij} Z_i(x, y, t), \quad R_j(x, y, t) = \sum_{i=1}^{2} r_{ij} T_i(x, y, t),$$

$$Z_i(x, y, t) = \frac{\sin[\omega(t - x)]}{\cos[\omega(t - x)] - ch(\omega k_i y)},$$

$$T_i(x, y, t) = \sum_{n=1}^{2} \frac{sh[\omega k_i(t + (-1)^n y)]}{ch[\omega k_i(t + (-1)^n y)] + \cos(\omega x)},$$

$$X_j(x, y; \xi, \eta) = \sum_{n=1}^{4} \frac{\sin[\omega(\xi + c_n x)]}{\cos[\omega(\xi + c_n x)] - ch[\omega k_j(\eta + (-1)^n y)]},$$

$$Y_j(x, y; \xi, \eta) = \sum_{n=1}^{4} \frac{sh[\omega k_j(\eta + c_n y)]}{ch[\omega k_j(\eta + c_n y)] - \cos[\omega(\xi + (-1)^n x)]},$$

$\gamma_{1i} = s_{1i} + s_{2i}, \quad \gamma_{2i} = A_{12}(s_{1i} + s_{2i}) + A_{22}(l_{1i} + l_{2i}), \quad \gamma_{3i} = l_{1i}, \quad \gamma_{3,i+2} = l_{2i},$

$\gamma_{4i} = A_{11}s_{1i} + A_{12}l_{1i}, \quad \gamma_{4,i+2} = A_{11}s_{2i} + A_{12}l_{2i}, \quad \mu_{1i} = e_{1i}/k_1, \quad \mu_{1,i+2} = e_{2i}/k_2,$

$\mu_{2i} = (A_{12}e_{1i} + A_{22}r_{1i})/k_1, \quad \mu_{2,i+2} = (A_{12}e_{2i} + A_{22}r_{2i})/k_2, \quad \mu_{3i} = r_{1i}/k_1 + r_{2i}/k_2,$

$\mu_{4i} = (A_{11}e_{1i} + A_{12}r_{1i})/k_1 + (A_{11}e_{2i} + A_{12}r_{2i})/k_2, \quad s_{ij} = \alpha_2{}^2 m_{ij}, \quad e_{ij} = m_{i+2,j+2},$

$l_{ij} = k_i m_{5-i,j+2}, \quad r_{ij} = k_i m_{3-i,j}, \quad m_{ij} = g_{jj} + (-1)^i g_{j,j+1}, \quad g_{11} = [(A_{11}\alpha_2{}^2)^{-1} + A_{66}^{-1}]/\alpha_4,$

$g_{12} = [(A_{11}\alpha_2{}^2)^{-1} - A_{66}^{-1}]/\alpha_5, \quad g_{22}[A_{12}/(A_{11}\alpha_2{}^2) - 1]/\alpha_4, \quad g_{33} = 0,$

$g_{34} = -(A_{12} + A_{66})/(\alpha_4\alpha_5 A_{11}A_{66}), \quad g_{44} = -2, \quad g_{45} = (A_{11}A_{22} - A_{12}^2)/(2\alpha_4\alpha_5 A_{11}A_{66}),$

$k_1 = (\alpha_4 + \alpha_5)/\alpha_2{}^2, \quad k_2 = (\alpha_4 - \alpha_5)/\alpha_2{}^2, \quad c_1 = c_2 = 1,$

$c_3 = c_4 = -1, \quad \varkappa = \omega/2, \quad j = 1, 2.$

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