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Flow of a heated ferrofluid over a stretching sheet in the presence of a magnetic dipole

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Summary. The flow of a viscous ferrofluid over a stretching sheet in the presence of a magnetic dipole is considered. The fluid momentum and thermal energy equations are formulated as a five-parameter problem, and the influence of the magneto-thermomechanical coupling is explored numerically. It is concluded that the primary effect of the magnetic field is to decelerate the fluid motion as compared to the hydrodynamic case, thereby increasing the skin friction and reducing the heat transfer rate at the sheet.

Nomenclature

a	distance
c	constant
c_p	specific heat at constant pressure
C_f	wall friction coefficient
e	2.71828 ...
f	dimensionless stream function
H	magnetic field
k	thermal conductivity
K	constant
M	magnetization
Nu_x	local Nusselt number
p	pressure
P	dimensionless pressure
Pr	Prandtl number, $\mu c_p/k$
Re_x	local Reynolds number, $\varrho c x^2/\mu$
T	temperature
u	velocity component along the sheet
v	velocity component normal to the sheet
x	coordinate along the sheet
y	coordinate normal to the sheet
α	dimensionless distance
β	ferrohydrodynamic interaction parameter
γ	constant
ε	dimensionless Curie temperature
η	dimensionless coordinate
θ	dimensionless temperature
λ	viscous dissipation parameter
μ	dynamic viscosity
μ_0	permeability
ξ	dimensionless coordinate
ϱ	density

τ shear stress
ϕ magnetic potential
ψ stream function

1 Introduction

Ferrofluids are artificially synthesized and consist of highly concentrated colloid suspensions of fine magnetic particles in a non-conducting carrier fluid. The resulting fluid behaves like a normal fluid except that it experiences a force due to the magnetization. A particularly attractive feature of the ferrofluids is the dependence of the magnetization upon the temperature, and this thermomagnetic coupling makes ferrofluids useful in various practical applications, see e.g. [1]–[4].

On the basis of electromagnetic theory, Neuringer and Rosensweig [1] demonstrated that the magnetic force per unit volume $\mu_0(\boldsymbol{M} \cdot \nabla)\boldsymbol{H}$ becomes $\mu_0 M \nabla H$ if the following assumptions are made: (i) the direction of the magnetization $\boldsymbol{M}$ of a fluid element is always in the direction of the local magnetic field $\boldsymbol{H}$, (ii) the fluid is electrically non-conducting and (iii) the displacement current is negligible. Thus, unlike magnetohydrodynamics, ferrohydrodynamics require the existence of a spatially varying field. The further assumption of a linear ferromagnetic equation of state enabled Neuringer [5] to treat the ferromagnetic extension of two classical problems in fluid mechanics, namely the Blasius boundary layer flow along a flat plate and the stagnation point flow.

The objective of the present study is to consider the ferrohydromagnetic analogue to another fundamental flow problem: the flow of a viscous fluid past a linearly stretching surface in otherwise quiescent surroundings. This problem was first considered by Crane [6] for a Newtonian fluid and subsequently extended to fluids obeying non-Newtonian constitutive equations like viscoelastic [7], micropolar [8] and inelastic power-law [9] fluids. Some of these cases were later extended to include the effect of a uniform transverse magnetic field on the motion of an electrically conducting fluid driven by the stretching sheet [10]–[12].

In this paper the influence of the magnetic field due to a magnetic dipole on the shear-driven motion of a viscous and non-conducting ferrofluid shall be explored. The focus of attention shall be on the magneto-thermomechanical interaction and its effect on skin friction and heat transfer at the stretching sheet.

2 Formulation of the problem

Let us consider the steady two-dimensional flow past a flat and impermeable elastic sheet shown schematically in Fig. 1. By applying two equal and opposite forces along the x-axis, the sheet is being stretched with a speed proportional to the distance from the fixed origin $x = 0$. The resulting motion of the otherwise quiescent fluid is thus caused solely by the moving sheet. The incompressible, viscous and electrically non-conducting ferrofluid is confined to the half space $y > 0$ above the sheet, whereas a magnetic dipole is located some distance below the sheet. The dipole, whose center lies on the y-axis a distance a below the x-axis and whose magnetic field points in the positive x-direction, gives rise to a magnetic field of sufficient strength to saturate the ferrofluid. The stretching sheet is kept at a fixed temperature T_w below the Curie temperature T_c, while the fluid elements far away from the sheet are assumed to be at temperature $T = T_c$ and, hence, incapable of being magnetized until they begin to cool upon entering the thermal boundary layer adjacent to the sheet.

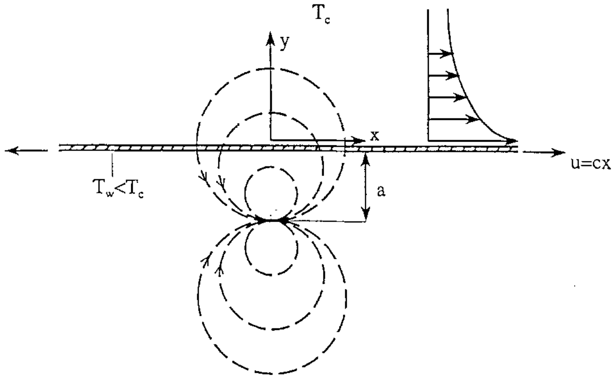

Fig. 1. Schematic representation of flow configuration. The broken lines represent the magnetic field

2.1 Ferrohydrodynamic and thermal energy equations

For the steady two-dimensional problem the equations governing mass conservation, fluid momentum and thermal energy are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\varrho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu_0 M\frac{\partial H}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$\varrho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu_0 M\frac{\partial H}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

$$\varrho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) + \mu_0 T\frac{\partial M}{\partial T}\left(u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}\right)$$

$$= k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right] \tag{4}$$

where u and v are the components of the velocity vector $\boldsymbol{v} = [u, v, 0]$, p denotes the pressure field, and the constant fluid properties ϱ, μ, μ_0, c_p and k are defined in the Nomenclature. Here, the second terms on the right-hand side of the ferrohydromagnetic momentum equations (2) and (3) represent the magnetic body force per unit volume $\mu_0 M \nabla H$, and the second term on the left-hand side of the thermal energy equation (4) accounts for heating due to adiabatic magnetization. The relevant boundary conditions at the sheet $y = 0$ are those of no-slip and prescribed temperature,

$$u = cx, \quad v = 0, \quad T = T_w, \tag{5}$$

whereas the boundary conditions infinitely far away from the sheet, i.e. as $y \to \infty$, are

$$u = 0, \quad T = T_c, \quad p + \frac{1}{2}\varrho\,(u^2 + v^2) = \text{constant} \tag{6}$$

where the latter requirement stems from the application of the Bernoulli equation outside the region affected by viscosity.

2.2 The magnetic field

The flow of the ferrofluid is affected by the magnetic field due to the magnetic dipole (cf. Fig. 1), whose magnetic scalar potential is given by

$$\phi = \frac{\alpha}{2\pi} \frac{x}{x^2 + (y + a)^2}, \tag{7}$$

and the corresponding magnetic field $\boldsymbol{H}$ has the components

$$H_x = -\frac{\partial \phi}{\partial x} = \frac{\gamma}{2\pi} \frac{x^2 - (y + a)^2}{[x^2 + (y + a)^2]^2} \tag{8}$$

$$H_y = -\frac{\partial \phi}{\partial y} = \frac{\gamma}{2\pi} \frac{2x(y + a)}{[x^2 + (y + a)^2]^2}. \tag{9}$$

Since the magnetic body force is proportional to the gradient of the magnitude of $\boldsymbol{H}$, cf. Eqs. (2), (3), we obtain from $H = [(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2]^{1/2}$ that

$$\frac{\partial H}{\partial x} = -\frac{\gamma}{2\pi} \frac{2x}{(y + a)^4} \tag{10}$$

$$\frac{\partial H}{\partial y} = \frac{\gamma}{2\pi} \left[-\frac{2}{(y + a)^3} + \frac{4x^2}{(y + a)^5} \right] \tag{11}$$

after having expanded in powers of x and retained terms up to order x^2.

Assuming that the applied field H is sufficiently strong to saturate the ferrofluid and the variation of magnetization M with temperature can be approximated by a linear equation of state,

$$M = K(T_c - T), \tag{12}$$

the magneto-thermomechanical coupling is completely described. It is readily seen, however, that the manifestation of the ferrohydrodynamic interaction requires: (i) that the fluid is at a temperature T different from the Curie temperature T_c and (ii) the applied magnetic field is inhomogeneous.

3 Solution procedure

Let us now introduce the nondimensional variables

$$\psi(\xi, \eta) = \left(\frac{\mu}{\varrho}\right) \xi \cdot f(\eta) \tag{13}$$

$$P(\xi, \eta) \equiv \frac{p}{c\mu} = -P_1(\eta) - \xi^2 P_2(\eta) \tag{14}$$

$$\theta(\xi, \eta) \equiv \frac{T_c - T}{T_c - T_w} = \theta_1(\eta) + \xi^2 \theta_2(\eta) \tag{15}$$

and the dimensionless coordinates

$$\eta = (c\varrho/\mu)^{1/2}\, y \tag{16}$$

$$\xi = (c\varrho/\mu)^{1/2}\, x. \tag{17}$$

The velocity components u and v are related to the physical stream function ψ according to

$$u = \partial\psi/\partial y = cx \cdot f'(\eta) \tag{18}$$

$$v = -\partial\psi/\partial x = -(c\mu/\varrho)^{1/2} \cdot f(\eta) \tag{19}$$

where the prime signifies differentiation with respect to η. The mass conservation equation (1) is thus automatically satisfied, whereas the moment equations (2) and (3) and the thermal energy equation (4) transform into a set of ordinary differential equations (ODEs)

$$f''' + ff'' - (f')^2 + 2P_2 - \frac{2\beta\theta_1}{(\eta+\alpha)^4} = 0 \tag{20}$$

$$P_1' - f'' - ff' - \frac{2\beta\theta_1}{(\eta+\alpha)^3} = 0 \tag{21}$$

$$P_2' - \frac{2\beta\theta_2}{(\eta+\alpha)^3} + \frac{4\beta\theta_1}{(\eta+\alpha)^5} = 0 \tag{22}$$

$$\theta_1'' + \Pr f\theta_1' + \frac{2\lambda\beta(\theta_1 - \varepsilon) f}{(\eta+\alpha)^3} + 2\theta_2 - 4\lambda(f')^2 = 0 \tag{23}$$

$$\theta_2'' - \Pr(2f'\theta_2 - f\theta_2') + \frac{2\lambda\beta f\theta_2}{(\eta+\alpha)^3} - \lambda\beta(\theta_1 - \varepsilon)\left[\frac{2f'}{(\eta+\alpha)^4} + \frac{4f}{(\eta+\alpha)^5}\right] - \lambda(f'')^2 = 0 \tag{24}$$

where we have equated coefficients of equal powers of ξ up to ξ^2. The boundary conditions (5) and (6) become

$$f = 0, \quad f' = 1, \quad \theta_1 = 1, \quad \theta_2 = 0, \quad \text{at} \quad \eta = 0 \tag{25}$$

$$f' \to 0, \quad \theta_1 \to 0, \quad \theta_2 \to 0, \quad P_1 \to -P_\infty, \quad P_2 \to 0 \quad \text{as} \quad \eta \to \infty \tag{26}$$

at the stretching sheet and at infinity, respectively.

The five dimensionless parameters, which appear explicitly in the transformed problem, are the Prandtl number Pr, the viscous dissipation parameter λ, the dimensionless Curie temperature ε, the ferrohydrodynamic interaction parameter β, and the dimensionless distance α from the origin to the center of the magnetic dipole, defined respectively as

$$\Pr = \mu c_p/k \tag{27}$$

$$\lambda = c\mu^2/\varrho k(T_c - T_w) \tag{28}$$

$$\varepsilon = T_c/(T_c - T_w) \tag{29}$$

$$\beta = (\gamma/2\pi)\, \mu_0 K(T_c - T_w)\, \varrho/\mu^2 \tag{30}$$

$$\alpha = (c\varrho/\mu)^{1/2}\, a. \tag{31}$$

Here, the product of λ and the local Reynolds number $\mathrm{Re}_x = \varrho c x^2/\mu$ can be recognized as the Brinkman number (i.e. the product of the Prandtl and Eckert numbers). Thus, λ is a measure of the relative importance of the viscous dissipation.

The five coupled differential equations (20)–(24) subject to the boundary conditions (25), (26) constitute a non-linear two-point boundary-value problem, which can be solved by means of a standard shooting method. The higher-order ODEs are formulated as first-order equations, and the resulting set of nine first-order equations can be integrated as an initial value problem using a fourth-order Runge-Kutta scheme. Since P_1 appears only in (20), this equation was not solved in the present study. Trial values of $f''(0)$, $\theta_1'(0)$, $\theta_2'(0)$ and $P_2(0)$ were adjusted iteratively by Newton's method to assure a quadratic convergence of the iterations required in order to fulfil the outer boundary conditions (26).

4 Results and discussion

It is interesting to notice that when the ferrohydromagnetic interaction parameter $\beta = 0$, P_2 becomes a constant equal to its value zero at infinity. The flow problem is thereby decoupled from the thermal energy problem, and the streamwise momentum equation (20) reduces to the hydrodynamic case, for which Crane [6] deduced the analytical solution

$$f(\eta) = 1 - \exp(-\eta). \tag{32}$$

Since Crane's analysis was based on conventional boundary layer theory, it was implicitly assumed that the pressure is constant across the boundary layer. Within the present framework, i.e. the Navier-Stokes equations, the pressure distribution can readily be obtained from the momentum balance (21) normal to the sheet as

$$P_1(\eta) = \frac{1}{2}\exp(-2\eta) - P_\infty. \tag{33}$$

Since $P_2 = 0$ in this particular case, the explicit solution (33) shows that the pressure P increases monotonically with the distance η from the sheet. Furthermore, if viscous energy dissipation can be neglected so that $\lambda = 0$, Eq. (24) gives the trivial solution $\theta_2 = 0$ and Eq. (23) reduces to the more familiar $\theta_1'' + \mathrm{Pr}\, f\theta_1' = 0$, for which Crane [6] found the explicit solution

$$\theta_1(\eta) = \{1 - \exp[-\exp(-\eta)]\}\, e/(e-1) \tag{34}$$

for the particular parameter value $\mathrm{Pr} = 1$.

In the general case $\beta \neq 0$ the five-parameter problem defined in Eqs. (20)–(24) was solved numerically for interaction parameters β in the range from 0 to 5 and Prandtl number $\mathrm{Pr} = 1$ and $\mathrm{Pr} = 7$. The three remaining parameters were fixed as $\lambda = 0.01$, $\varepsilon = 2.0$ and $\alpha = 1.0$, i.e. in accordance with the parameter values adopted by Neuringer [5] for the stagnation point flow. As pointed out in [5], $\lambda = 0.01$ corresponds to a situation in which the source terms due to viscous dissipation in the energy equations (23) and (24) are of marginal importance compared to the conduction and convection terms.

In order to give an impression of the effect of the magneto-thermomechanical interaction, some characteristic profiles for velocity $f'(\eta)$, pressure $P_2(\eta)$, and temperature $\theta_1(\eta)$ are presented in Figs. 2–4, respectively. The numerical results for $\beta = 0$ are also compared with the exact

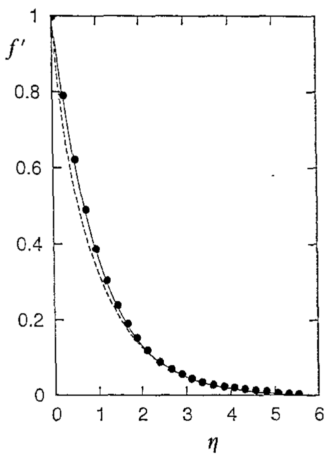

Fig. 2

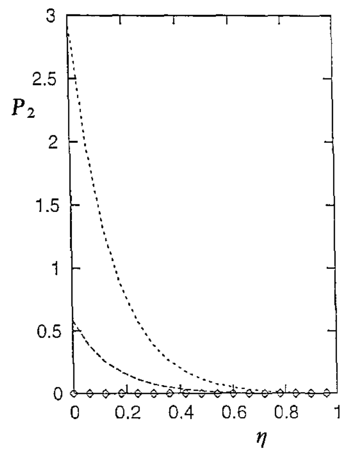

Fig. 3

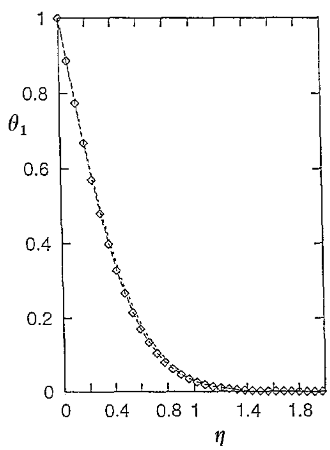

Fig. 4

Fig. 2. Dimensionless velocity profiles for $\Pr = 7.0$. Solid and broken lines represent $\beta = 0$ and $\beta = 5$, respectively. The symbols denote the exact analytical solution due to Crane [6]
Fig. 3. Dimensionless pressure profiles for $\Pr = 7.0$. The symbols denote the exact solution $P_2 = 0$ for $\beta = 0$. Broken and dotted lines represent $\beta = 1$ and $\beta = 5$, respectively
Fig. 4. Dimensionless temperature profiles for $\Pr = 7.0$. The symbols denote the results for $\beta = 0$ and the broken and dotted lines represent $\beta = 1$ and $\beta = 5$, respectively

analytical solution $f'(\eta) = \exp(-\eta)$ in Fig. 2 and $P_2(\eta) = 0$ in Fig. 3 to demonstrate the accuracy of the numerical integration. The ferrohydromagnetic interaction tends to reduce f' and increase θ_1 and P_2 in the vicinity of the stretching sheet, thereby implying a reduction not only in the velocity component u along the sheet but also in the pressure p and the temperature T.

The variation of some important flow and heat transfer characteristics with β is shown in Figs. 5–7 for $\Pr = 1$ and $\Pr = 7$. Here, the local skin friction coefficient C_f, which is a dimensionless form of the shear stress τ at the sheet, is defined as

$$C_f \equiv \frac{-2\tau_w}{\varrho(cx)^2} = -2f''(0)\,\mathrm{Re}_x^{-1/2} \tag{35}$$

whereas the local heat transfer rate is expressed in dimensionless form as the local Nusselt number

$$Nu_x \equiv \frac{x}{T_c - T_w}\cdot\left.\frac{\partial T}{\partial y}\right|_{y=0} = -[\theta_1'(0) + \xi^2\theta_2'(0)]\cdot \mathrm{Re}_x^{1/2} \tag{36}$$

where $\theta'(0)$ is the dimensionless temperature gradient evaluated at the sheet. It is readily observed that, for a given Prandtl number, the skin friction $-f''(0)$ and the wall pressure $P_2(0)$ both

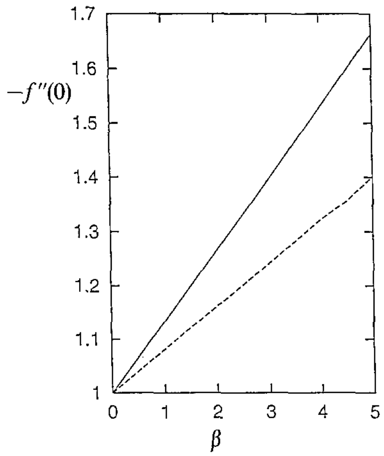

Fig. 5

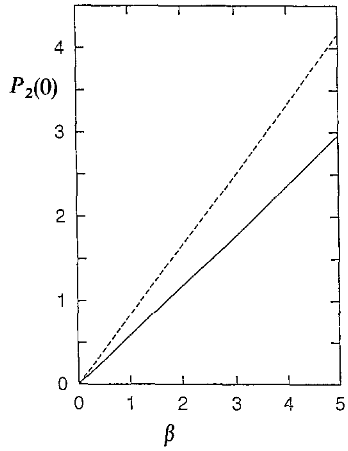

Fig. 6

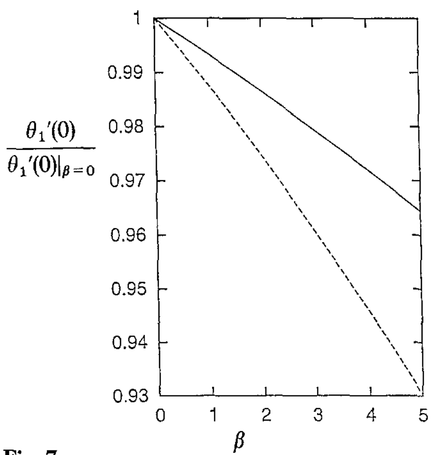

Fig. 7

Fig. 5. Wall shear stress for Pr = 1.0 (---) and Pr = 7.0 (—)
Fig. 6. Wall pressure for Pr = 1.0 (---) and Pr = 7.0 (—)
Fig. 7. Heat transfer rate at the sheet for Pr = 1.0 (---) and Pr = 7.0 (—)

increase approximately linearly with β, whereas the heat transfer rate $-\theta_1'(0)$ decreases linearly with increasing ferromagnetic interaction. Since for the small parameter value $\lambda = 0.01$ considered here, θ_2 is typically small compared to θ_1, and the dominance of the last term in Eq. (22) makes $P_2' < 0$ and thus $P_2(0) > 0$, cf. Fig. 3. The effect of increasing the Prandtl number is to reduce the thickness of the thermal boundary layer, across which the temperature adjusts from T_c to T_w, thereby reducing the effect of the last term in Eq. (22). This explains why the influence of the ferromagnetic interaction is reduced at the higher Prandtl number in Fig. 6.

The flow field is affected by β through the two last terms in Eq. (20), which are responsible for any departure from the analytical solution (32) for $\beta = 0$. Of these two terms the last one due to the magnetic body force is the greatest and tends to reduce the velocity components $f'(\eta)$ and $f(\eta)$, cf. Fig. 2, thereby increasing the wall friction $-f''(0)$, as shown in Fig. 5. As the Prandtl number is increased from 1 to 7, however, the reduced effect of the magnetic body force is more than outweighed by the reduction in the pressure term P_2. The net effect of increasing Pr is therefore to enhance the influence of β on the wall friction, as observed from Fig. 5.

The thermal energy budget (23) is essentially a balance between diffusion of thermal energy towards the stretching sheet (θ_1'') and convective transport $-\Pr f\theta_1'$ away from the sheet, while the other terms are relatively small for the present set of parameter values. The ferromagnetic interaction has thus essentially an indirect effect on the heat transfer through its influence on the velocity field. The reduction in $f(\eta)$ for increasing β-values tends to reduce the convective heat transport and consequently also the heat transfer rate at the sheet, cf. Fig. 7. However, the greater reduction in f at higher Pr is more than compensated by the proportionality of the convective term $\Pr f\theta_1'$ with the Prandtl number. This is why the ferromagnetic interaction has receding impact on the heat transfer rate as Pr increases.

It can be concluded that the primary effect of the magneto-thermomechanical interaction, which couples the ferrohydromagnetic momentum equations and the thermal energy equation, is to decelerate the flow along the stretching sheet, thereby increasing the skin friction C_f and reducing the heat transfer rate Nu_x at the sheet. It is noteworthy that the imposition of the magnetic field has receding influence on Nu_x as the Prandtl number is increased, whereas the impact on C_f is enhanced.

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