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Transient response of a piezothermoelastic circular disk under axisymmetric heating

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Summary. Transient temperature, displacement, stress and electric field intensities in a finite circular piezothermoelastic disk undergoing axisymmetric surface heating are examined. Exact solutions to the equations of equilibrium and electrostatics are obtained using a potential function approach based upon two piezothermoelastic potential functions, three piezoelastic potential functions and a piezoelectric potential function. The disk under consideration is assumed to exhibit hexagonal material symmetry of class 6 mm. The initial temperature of the disk is zero; thereafter one face is subjected to linear heat transfer from an adjacent medium (Newton's law of cooling), while the temperature of the other face remains constant. Both faces are taken to be free of traction. The cylindrical boundary of the disk is thermally insulated, electrically charge-free, and constrained against radial deformation. Numerical results are obtained for the stress and the electric potential distributions in a cadmium selenide disk.

1 Introduction

Piezoelectric materials have attracted considerable attention recently, due mainly to their potential for use in smart structural systems. Owing to the coupling that exists between the thermoelastic and electric fields in piezoelectric materials, thermomechanical disturbances can be determined from measurement of the induced electric potential, and the ensuing response can be controlled through application of an appropriate electric field. For successful and efficient utilization of piezoelectrics as sensors and actuators in intelligent systems, further research on piezothermoelastic behavior is needed.

Among the important earlier studies on piezothermoelastic response is the work of Tiersten [1] who derived nonlinear equations of thermoelectroelasticity, and Chandrasekharaiah [2] who presented a generalized linear piezothermoelastic formulation. Nowacki [3] considered steady-state problems involving thermopiezoelectric infinite and bounded bodies. Maysel's influence-function approach to thermoelasticity was generalized for static thermopiezoelectricity in [3], and was later generalized for piezoelectric vibration problems by Irschik and Ziegler [4]. Other investigations dealing with piezothermoelastic response of beams, plates and adaptive structural systems include the works of Dube et al. [5], Jonnalagadda et al. [6], Lee and Saravanos [7], Rao and Sunar [8], Shen and Weng [9], Tzou and Ye [10], and Xu and Noor [11].

The present authors proposed a general solution procedure for three-dimensional problems of piezothermoelastic solids of class 6mm in Cartesian coordinates [12], for a two-dimensional problem of an infinite plate of class mm2 [13], and for axisymmetric problems in cylindrical coordinates [14], [15]. More recently they examined the response of a finite circular disk, one face of which is in contact with a heated body [16]. The inverse problem was solved to determine the surface temperature when the electric potential difference between the faces of the disk is known, assuming a stationary temperature field. The possibility of utilizing piezoelectric elements as temperature sensors was thus demonstrated. In the present paper we extend the previous investigation [16] to include transient piezothermoelastic response of the disk.

2 Governing equations

Consider the axisymmetric response of a piezothermoelastic solid of crystal class 6mm. Constitutive equations for the elastic field are

$$\begin{aligned}
\sigma_{rr} &= c_{11}\varepsilon_{rr} + c_{12}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz} - e_1 E_z - \beta_1 T \\
\sigma_{\theta\theta} &= c_{12}\varepsilon_{rr} + c_{11}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz} - e_1 E_z - \beta_1 T \\
\sigma_{zz} &= c_{13}\varepsilon_{rr} + c_{13}\varepsilon_{\theta\theta} + c_{33}\varepsilon_{zz} - e_3 E_z - \beta_3 T \\
\sigma_{zr} &= c_{44}\varepsilon_{zr} - e_4 E_r
\end{aligned} \tag{1}$$

where σ_{ij} represent the stress components, ε_{ij} are the strain components, E_i are the electric field intensities, T denotes the temperature rise, c_{ij} are elastic stiffnesses, e_i are piezoelectric constants, and β_i are stress-temperature coefficients. The corresponding strains are related to the displacements u_i as

$$\varepsilon_{rr} = u_{r,,r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r} u_r, \qquad \varepsilon_{zz} = u_{z,z}, \qquad \varepsilon_{zr} = u_{z,r} + u_{r,z}. \tag{2}$$

For the electric field, the constitutive relations are

$$D_r = e_4 \varepsilon_{zr} + \eta_1 E_r, \qquad D_z = e_1 \varepsilon_{rr} + e_1 \varepsilon_{\theta\theta} + e_3 \varepsilon_{zz} + \eta_3 E_z + p_3 T \tag{3}$$

where D_i are the electric displacement components, η_i are dielectric permittivities and p_3 is the pyroelectric constant.

The temperature is assumed to satisfy the Fourier heat conduction equation

$$\Delta_1 T + \lambda^2 T_{,zz} = \frac{T_{,t}}{\varkappa} \tag{4}$$

where

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \qquad \lambda^2 = \frac{\lambda_z}{\lambda_r}, \qquad T_{,t} = \frac{\partial T}{\partial t} \tag{5}$$

in which λ_i are coefficients of heat conduction, and $\varkappa$ denotes thermal diffusivity. For the problem considered here it is convenient to represent the temperature as the sum of two functions, namely $T(r, z, t) = T_0(z, t) + T_1(r, z, t)$.

The equations of equilibrium are

$$\sigma_{rr,r} + \sigma_{rz,z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \qquad \sigma_{zr,r} + \sigma_{zz,z} + \frac{1}{r}\sigma_{zr} = 0, \tag{6}$$

and the equation of electrostatics is

$$D_{r,r} + D_{z,z} + \frac{1}{r} D_r = 0. \tag{7}$$

3 Solution procedure

Solutions to the equations of equilibrium (6) and electrostatics (7) are obtained using the potential function method introduced in [14] and [16]. In particular, the displacement and electric field intensities are expressed, respectively, in terms of potential functions as

$$u_r = (\Psi_1 + \Psi_2)_{,r}, \quad u_z = (\Psi_0 + k_1\Psi_1 + j\Psi_2)_{,z} \tag{8}$$

$$E_r = -(\Phi_1)_{,r}, \quad E_z = -(\Phi_0 + \Phi_1)_{,z} \tag{9}$$

where

$$\Psi_1 = \phi_1 + \psi_1 + \psi_2 + \psi_3 \tag{10}$$

$$\Psi_2 = \phi_2 + L_1\psi_1 + L_2\psi_2 + L_3\psi_3 \tag{11}$$

$$\Psi_{0,zz} = \bar{\gamma}_1 T_0 \tag{12}$$

$$\Phi_1 = \chi + (n_1\psi_1 + n_2\psi_2 + n_3\psi_3)_{,z} \tag{13}$$

$$\Phi_{0,z} = \bar{\gamma}_2 T_0. \tag{14}$$

As indicated in the following Section, the piezothermoelastic potential ϕ_1 and the three piezoelastic potentials ψ_i $(i = 1, 2, 3)$ are governed by uncoupled differential equations; the piezothermoelastic potential ϕ_2 is expressed in terms of ϕ_1 and T_1; and the piezoelectric function χ depends on ϕ_2 and T_1. The various coefficients appearing in Eqs. (8)–(14) are given by [14]

$$k_i = \frac{c_{11}v_i - c_{44}}{c_{13} + c_{44}}, \quad j = k_1 - \frac{k_1 v_2 c_{11}}{c_{44}} + \frac{c_{11}(e_3 - \eta e_4)}{(e_1 + e_4)(c_{44} + e_4{}^2/\eta_1)}, \quad \eta = \frac{\eta_3}{\eta_1}$$

$$\bar{\gamma}_1 = \frac{\beta_3\eta_3 - e_3 p_3}{c_{33}\eta_3 + e_3{}^2}, \quad \bar{\gamma}_2 = \frac{c_{33}p_3 + \beta_3 e_3}{c_{33}\eta_3 + e_3{}^2} \tag{15}$$

$$L_i = \ell_i - 1 = \frac{v_1 - \mu_i}{\xi_1} = \frac{m_i - k_1}{j}, \quad n_i = \frac{(v_1 - \mu_i)(v_2 - \mu_i)}{\xi_1\xi_2}$$

$$\xi_1 = (c_{13} + c_{44})\left[\frac{v_1 k_2}{c_{44}} - \frac{e_3 - \eta e_4}{(e_1 + e_4)(c_{44} + e_4{}^2/\eta_1)}\right], \quad \xi_2 = -\frac{e_1 + e_4}{c_{11}}$$

where v_i $(i = 1, 2)$ are roots of the equation

$$c_{11}\left(c_{44} + \frac{e_4{}^2}{\eta_1}\right)v^2 + \left[c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33} + \frac{(c_{13} + c_{44})(e_1 + e_4)e_4 - c_{11}e_3e_4 - c_{44}e_4{}^2}{\eta_1}\right]v$$
$$+ c_{44}\left(c_{33} + \frac{e_3e_4}{\eta_1}\right) = 0 \tag{16}$$

and μ_i $(i = 1, 2, 3)$ are roots of

$$\mu^3 - (v_1 + v_2 + \eta - a_2\xi_2)\mu^2 + [v_1v_2 + \eta(v_1 + v_2) - (v_1a_2 + b_2)\xi_1 - a_1\xi_1\xi_2]\mu$$
$$- \eta v_1 v_2 + v_1 b_2 \xi_2 + b_1\xi_1\xi_2 = 0 \tag{17}$$

in which

$$a_1 = \frac{e_1 + (1 + k_1)\, e_4}{\eta_1}, \quad b_1 = \frac{k_1 e_3}{\eta_1}, \quad a_2 = \frac{e_1 + (1 + j)\, e_4}{\eta_1}, \quad b_2 = \frac{j e_3}{\eta_1}. \tag{18}$$

The equations of equilibrium (6) and electrostatics (7) are then satisfied, provided the potential functions ϕ_i, χ and ψ_i satisfy the equations [14]

$$\left(\Delta_1 + \mu_1 \frac{\partial^2}{\partial z^2}\right)\left(\Delta_1 + \mu_2 \frac{\partial^2}{\partial z^2}\right)\left(\Delta_1 + \mu_3 \frac{\partial^2}{\partial z^2}\right)\phi_1 = d_2 \Delta_1 \Delta_1 T_1 + d_1 \Delta_1 T_{1,zz} + d_0 T_{1,zzzz} \tag{19}$$

$$\phi_{2,zz} = \frac{1}{\xi_1}(\Delta_1 \phi_1 + v_1 \phi_{1,zz} - \delta_1 T_1) \tag{20}$$

$$\chi_{,z} = \frac{1}{\xi_2}(\Delta_1 \phi_2 + v_2 \phi_{2,zz} - \delta_2 T_1) \tag{21}$$

$$\Delta_1 \psi_i + \mu_i \psi_{i,zz} = 0 \qquad (i = 1 \text{ to } 3) \tag{22}$$

in which

$$\delta_1 = \frac{\beta_1}{c_{11}} - \delta_2, \qquad \delta_2 = \frac{(e_1 + e_4)\,[k_1 v_2 (c_{44} + e_4{}^2/\eta_1)\, \beta_1 - c_{44}(\beta_3 - p_3 e_4/\eta_1)]}{c_{11}[(c_{44} + e_4{}^2/\eta_1)\,(e_1 + e_4)\, k_1 v_2 - c_{44}(e_3 - \eta e_4)]}$$

$$d_0 = \eta(v_2 \delta_1 + \xi_1 \delta_2) - b_2 \xi_2 \delta_1 + \frac{p_3 \xi_1 \xi_2}{\eta_1} \tag{23}$$

$$d_1 = v_2 \delta_1 + \xi_1 \delta_2 + \eta \delta_1 - a_2 \xi_2 \delta_1, \qquad d_2 = \delta_1.$$

4 Application

Consider a circular disk ($0 \leqq r \leqq a$, $0 \leqq z \leqq b$) of piezothermoelastic material, initially at zero temperature. One face ($z = b$) of the disk is subject to linear heat transfer from an adjacent medium at temperature $\Theta(r)$ (Fig. 1). The temperature of the other face remains constant, while the cylindrical boundary is thermally insulated. In this case the initial and boundary conditions are expressed as

$$T = 0 \quad \text{at} \quad t = 0 \tag{24}$$

$$T = 0 \quad \text{on} \quad z = 0 \tag{25}$$

$$T_{,z} + hT = h\Theta(r) \quad \text{on} \quad z = b \tag{26}$$

$$T_{,r} = 0 \quad \text{on} \quad r = a \tag{27}$$

where $h = H/\lambda_z$, in which H is the coefficient of surface heat transfer. It is convenient here to express the temperature of the contacting medium as

$$\Theta(r) = \bar{\Theta}_0 + \sum_{n=1}^{\infty} \bar{\Theta}_n J_0(\alpha_n r) \tag{28}$$

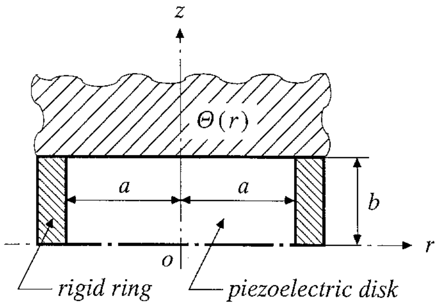

Fig. 1. Circular piezoelectric disk

where

$$\bar{\Theta}_0 = \frac{2}{a^2}\int_0^a r\Theta(r)\,dr, \qquad \bar{\Theta}_n = \frac{2}{a^2 J_0^2(\alpha_n a)}\int_0^a r\Theta(r)\,J_0(\alpha_n r)\,dr. \tag{29}$$

The solution to the heat conduction equation (4) satisfying conditions (24)–(27) has been obtained using the Laplace transform technique. The resulting temperature $T = T_0 + T_1$ is given by

$$T_0 = A_{00}\frac{z}{b} + \sum_{m=1}^{\infty} A_{0m}\sin\left(\frac{\gamma_m z}{\lambda}\right) \tag{30}$$

$$T_1 = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left[A_{n0}\frac{\sinh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty} A_{nm}\sin\left(\frac{\gamma_m z}{\lambda}\right)\right] \tag{31}$$

where

$$A_{00} = \frac{hb\bar{\Theta}_0}{1+hb}, \qquad A_{0m} = \frac{2\lambda h\bar{\Theta}_0 e^{-\varkappa\gamma_m^2 t}}{\gamma_m[(1+hb)\cos(\gamma_m b/\lambda) - (\gamma_m b/\lambda)\sin(\gamma_m b/\lambda)]} \tag{32}$$

$$A_{n0} = \frac{\lambda h\bar{\Theta}_n}{\lambda h + \alpha_n\coth(\alpha_n b/\lambda)}, \qquad A_{nm} = \frac{2\lambda h\gamma_m\bar{\Theta}_n e^{-\varkappa(\alpha_n^2+\gamma_m^2)t}}{(\alpha_n^2+\gamma_m^2)[(1+hb)\cos(\gamma_m b/\lambda) - (\gamma_m b/\lambda)\sin(\gamma_m b/\lambda)]}$$

and α_n and γ_m are, respectively, the roots of the equations

$$J_1(\alpha_n a) = 0 \tag{33}$$

$$\gamma_m\cos(\gamma_m b/\lambda) + \lambda h\sin(\gamma_m b/\lambda) = 0. \tag{34}$$

We next examine the induced elastic and electric fields, assuming the faces of the disk are free of both traction and electric charge, while the cylindrical edge is constrained against radial deformation (e.g., by a rigid ring as shown in Fig. 1) and is charge free; i.e.,

$$\sigma_{rz} = \sigma_{zz} = D_z = 0 \quad \text{on} \quad z = 0, b \tag{35}$$

$$u_r = D_r = 0 \quad \text{on} \quad r = a. \tag{36}$$

The piezothermoelastic potential function ϕ_1 in Eq. (19) is represented in a form similar to the expression for temperature T_1 in Eq. (31), namely

$$\phi_1 = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left[B_{n0}\frac{\sinh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty} B_{nm}\sin\left(\frac{\gamma_m z}{\lambda}\right)\right]. \tag{37}$$

By substituting Eqs. (31) and (37) into (19), the coefficients B_{n0} and B_{nm} are found to be

$$B_{n0} = \frac{\lambda^2(d_2\lambda^4 - d_1\lambda^2 + d_0)\, A_{n0}}{\alpha_n{}^2(\mu_1 - \lambda^2)\,(\mu_2 - \lambda^2)\,(\mu_3 - \lambda^2)}$$

$$B_{nm} = -\frac{\lambda^2(d_2\lambda^4\alpha_n{}^4 + d_1\lambda^2\alpha_n{}^2\gamma_m^2 + d_0\gamma_m^4)\, A_{nm}}{(\lambda^2\alpha_n{}^2 + \mu_1\gamma_m^2)\,(\lambda^2\alpha_n{}^2 + \mu_2\gamma_m^2)\,(\lambda^2\alpha_n{}^2 + \mu_3\gamma_m^2)}. \tag{38}$$

The piezothermoelastic potential ϕ_2 is then found by integrating Eq. (20), with the result

$$\phi_2 = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left[C_{n0}\frac{\sinh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty} C_{nm}\sin\left(\frac{\gamma_m z}{\lambda}\right)\right] \tag{39}$$

where

$$C_{n0} = \frac{(\nu_1 - \lambda^2)\,\alpha_n{}^2 B_{n0} - \delta_1\lambda^2 A_{n0}}{\xi_1\alpha_n{}^2}, \qquad C_{nm} = \frac{(\nu_1\gamma_m^2 + \lambda^2\alpha_n{}^2)\,B_{nm} + \delta_1\lambda^2 A_{nm}}{\xi_1\gamma_m^2}. \tag{40}$$

Likewise, integration of Eq. (21) for the piezoelectric potential χ gives

$$\chi = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left[D_{n0}\frac{\alpha_n\cosh(\alpha_n z/\lambda)}{\lambda\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty} D_{nm}\frac{\gamma_m}{\lambda}\cos\left(\frac{\gamma_m z}{\lambda}\right)\right] \tag{41}$$

with

$$D_{n0} = \frac{(\nu_2 - \lambda^2)\,\alpha_n{}^2 C_{n0} - \delta_2\lambda^2 A_{n0}}{\xi_2\alpha_n{}^2}, \qquad D_{nm} = \frac{(\nu_2\gamma_m^2 + \lambda^2\alpha_n{}^2)\,C_{nm} + \delta_2\lambda^2 A_{nm}}{\xi_2\gamma_m^2}. \tag{42}$$

The piezoelastic potentials ψ_i, which represent solutions to Eqs. (22), are

$$\psi_i = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left[E_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right] \quad (i = 1 \text{ to } 3) \tag{43}$$

where E_{in} and F_{in} are unknown coefficients. The displacement function Ψ_1, obtained by substituting Eqs. (37) and (43) into (10), then becomes

$$\Psi_1 = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left\{B_{n0}\frac{\sinh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty} B_{nm}\sin\left(\frac{\gamma_m z}{\lambda}\right)\right.$$
$$\left. + \sum_{i=1}^{3}\left[E_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right]\right\} \tag{44}$$

while substitution of Eqs. (39) and (43) into (11) gives

$$\Psi_2 = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left\{C_{n0}\frac{\sinh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty} C_{nm}\sin\left(\frac{\gamma_m z}{\lambda}\right)\right.$$
$$\left. + \sum_{i=1}^{3} L_i\left[E_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right]\right\}, \tag{45}$$

and substitution of T_0 given by Eq. (30) into (12) leads to

$$\Psi_0 = \bar{\gamma}_1\left[A_{00}\frac{z^3}{6b} - \sum_{m=1}^{\infty} A_{0m}\frac{\lambda^2}{\gamma_m^2}\sin\left(\frac{\gamma_m z}{\lambda}\right)\right]. \tag{46}$$

The electric potential Φ_1, found by introducing Eqs. (41) and (43) into (13), is

$$\Phi_1 = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left\{\frac{\alpha_n}{\lambda} D_{n0} \frac{\cosh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty} \frac{\gamma_m}{\lambda} D_{nm} \cos\left(\frac{\gamma_m z}{\lambda}\right)\right.$$
$$\left. + \sum_{i=1}^{3} n_i \frac{\alpha_n}{\sqrt{\mu_i}}\left[E_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right]\right\}, \tag{47}$$

and substitution of T_0 into Eq. (14) gives

$$\Phi_0 = \bar{\gamma}_2\left[A_{00}\frac{z^2}{2b} - \sum_{m=1}^{\infty} \frac{\lambda}{\gamma_m} A_{0m} \cos\left(\frac{\gamma_m z}{\lambda}\right)\right]. \tag{48}$$

Finally, inserting the expressions for Ψ_0, Ψ_1, Ψ_2, Φ_0 and Φ_1 into (8), (2), (1), (9) and (3) yields, respectively, the following relations for the displacements, stresses and electric displacements:

$$u_r = -\sum_{n=1}^{\infty} \alpha_n J_1(\alpha_n r)\left\{(B_{n0} + C_{n0})\frac{\sinh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty} (B_{nm} + C_{nm}) \sin\left(\frac{\gamma_m z}{\lambda}\right)\right.$$
$$\left. + \sum_{i=1}^{3} \ell_i \left[E_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right]\right\} \tag{49}$$

$$u_z = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left\{\frac{\alpha_n}{\lambda}(k_1 B_{n0} + jC_{n0})\frac{\cosh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty} \frac{\gamma_m}{\lambda}(k_1 B_{nm} + jC_{nm}) \cos\left(\frac{\gamma_m z}{\lambda}\right)\right.$$
$$\left. + \sum_{i=1}^{3} m_i \frac{\alpha_n}{\sqrt{\mu_i}}\left[E_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right]\right\}$$
$$+ \bar{\gamma}_1\left[A_{00}\frac{z^2}{2b} - \sum_{m=1}^{\infty} \frac{\lambda}{\gamma_m} A_{0m} \cos\left(\frac{\gamma_m z}{\lambda}\right)\right] \tag{50}$$

$$\sigma_{rr} = -\sum_{n=1}^{\infty} \alpha_n^{\,2}\left[\left\{J_0(\alpha_n r)\left[c_{11}(B_{n0} + C_{n0}) - \frac{c_{13}}{\lambda^2}(k_1 B_{n0} + jC_{n0}) - \frac{e_1}{\lambda^2} D_{n0} + \frac{\beta_1}{\alpha_n^{\,2}} A_{n0}\right]\right.\right.$$
$$\left. - (c_{11} - c_{12})\frac{J_1(\alpha_n r)}{\alpha_n r}(B_{n0} + C_{n0})\right\}\frac{\sinh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)}$$
$$+ \sum_{m=1}^{\infty}\left\{J_0(\alpha_n r)\left[c_{11}(B_{nm} + C_{nm}) + \frac{c_{13}\gamma_m^2}{\lambda^2\alpha_n^{\,2}}(k_1 B_{nm} + jC_{nm}) + \frac{e_1\gamma_m^2}{\lambda^2\alpha_n^{\,2}} D_{nm} + \frac{\beta_1}{\alpha_n^{\,2}} A_{nm}\right]\right.$$
$$\left. - (c_{11} - c_{12})\frac{J_1(\alpha_n r)}{\alpha_n r}(B_{nm} + C_{nm})\right\}\sin(\gamma_m z/\lambda)$$
$$+ \sum_{i=1}^{3}\left\{\left(c_{11}\ell_i - c_{13}\frac{m_i}{\mu_i} - e_1\frac{n_i}{\mu_i}\right) J_0(\alpha_n r) - (c_{11} - c_{12})\,\ell_i\frac{J_1(\alpha_n r)}{\alpha_n r}\right\}$$
$$\left.\times\left\{E_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right\}\right]$$
$$+ (c_{13}\bar{\gamma}_1 + e_1\bar{\gamma}_2 - \beta_1)\left\{A_{00}\frac{z}{b} + \sum_{m=1}^{\infty} A_{0m}\sin(\gamma_m z/\lambda)\right\} \tag{51}$$

$$\sigma_{\theta\theta} = -\sum_{n=1}^{\infty} \alpha_n{}^2 \Bigg[\Bigg\{J_0(\alpha_n r)\left[c_{12}(B_{n0}+C_{n0}) - \frac{c_{13}}{\lambda^2}(k_1 B_{n0} + jC_{n0}) - \frac{e_1}{\lambda^2} D_{n0} + \frac{\beta_1}{\alpha_n{}^2} A_{n0}\right]$$

$$+ (c_{11} - c_{12}) \frac{J_1(\alpha_n r)}{\alpha_n r}(B_{n0} + C_{n0})\Bigg\} \frac{\sinh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)}$$

$$+ \sum_{m=1}^{\infty} \Bigg\{J_0(\alpha_n r)\left[c_{12}(B_{nm} + C_{nm}) + \frac{c_{13}\gamma_m^2}{\lambda^2 \alpha_n{}^2}(k_1 B_{nm} + jC_{nm}) + \frac{e_1 \gamma_m^2}{\lambda^2 \alpha_n{}^2} D_{nm} + \frac{\beta_1}{\alpha_n{}^2} A_{nm}\right]$$

$$+ (c_{11} - c_{12}) \frac{J_1(\alpha_n r)}{\alpha_n r}(B_{nm} + C_{nm})\Bigg\} \sin(\gamma_m z/\lambda)$$

$$+ \sum_{i=1}^{3} \left\{\left(c_{12}\ell_i - c_{13}\frac{m_i}{\mu_i} - e_1 \frac{n_i}{\mu_i}\right) J_0(\alpha_n r) - (c_{11} - c_{12})\,\ell_i \frac{J_1(\alpha_n r)}{\alpha_n r}\right\}$$

$$\times \left\{E_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right\}\Bigg]$$

$$+ (c_{13}\bar{\gamma}_1 + e_1 \bar{\gamma}_2 - \beta_1)\left\{A_{00}\frac{z}{b} + \sum_{m=1}^{\infty} A_{0m} \sin(\gamma_m z/\lambda)\right\} \tag{52}$$

$$\sigma_{zz} = -\sum_{n=1}^{\infty} \alpha_n{}^2 J_0(\alpha_n r)\Bigg\{\left[c_{13}(B_{n0} + C_{n0}) - \frac{c_{33}}{\lambda^2}(k_1 B_{n0} + jC_{n0}) - \frac{e_3}{\lambda^2} D_{n0} + \frac{\beta_3}{\alpha_n{}^2} A_{n0}\right]$$

$$\times \frac{\sinh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty}\Bigg[c_{13}(B_{nm} + C_{nm}) + \frac{c_{33}\gamma_m^2}{\lambda^2\alpha_n{}^2}(k_1 B_{nm} + jC_{nm})$$

$$+ \frac{e_3\gamma_m^2}{\lambda^2\alpha_n{}^2} D_{nm} + \frac{\beta_3}{\alpha_n{}^2} A_{nm}\Bigg] \sin(\gamma_m z/\lambda)$$

$$+ \sum_{i=1}^{3}\left(c_{13}\ell_i - c_{33}\frac{m_i}{\mu_i} - e_3\frac{n_i}{\mu_i}\right)\left[E_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right]\Bigg\} \tag{53}$$

$$\sigma_{rz} = -\sum_{n=1}^{\infty}\alpha_n{}^2 J_1(\alpha_n r)\Bigg\{\left[\frac{c_{44}\{(1+k_1)B_{n0} + (1+j)C_{n0}}{\lambda} + \frac{e_4}{\lambda}D_{n0}\right]\frac{\cosh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)}$$

$$+ \sum_{m=1}^{\infty}\frac{\gamma_m}{\lambda\alpha_n}[c_{44}\{(1+k_1)B_{nm} + (1+j)C_{nm}\} + e_4 D_{nm}]\cos(\gamma_m z/\lambda)$$

$$+ \sum_{i=1}^{3}\frac{c_{44}(\ell_i + m_i) + e_4 n_i}{\sqrt{\mu_i}}\left[E_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right]\Bigg\} \tag{54}$$

$$D_r = -\sum_{n=1}^{\infty} a_n{}^2 J_1(\alpha_n r)\Bigg\{\left[\frac{e_4\{(1+k_1)B_{n0} + (1+j)C_{n0}}{\lambda} - \frac{\eta_1}{\lambda}D_{n0}\right]\frac{\cosh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)}$$

$$+ \sum_{m=1}^{\infty}\frac{\gamma_m}{\lambda\alpha_n}[e_4\{(1+k_1)B_{nm} + (1+j)C_{nm}\} - \eta_1 D_{nm}]\cos(\gamma_m z/\lambda)$$

$$+ \sum_{i=1}^{3}\frac{e_4(\ell_i + m_i) - \eta_1 n_i}{\sqrt{\mu_i}}\left[E_{in}\frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in}\frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})}\right]\Bigg\} \tag{55}$$

$$D_z = -\sum_{n=1}^{\infty} \alpha_n^2 J_0(\alpha_n r) \left\{ \left[e_1(B_{n0} + C_{n0}) - \frac{e_3}{\lambda^2}(k_1 B_{n0} + jC_{n0}) + \frac{\eta_3}{\lambda^2} D_{n0} - \frac{p_3}{\alpha_n^2} A_{n0} \right] \right.$$

$$\times \frac{\sinh(\alpha_n z/\lambda)}{\sinh(\alpha_n b/\lambda)} + \sum_{m=1}^{\infty} \left[e_1(B_{nm} + C_{nm}) + \frac{e_3 \gamma_m^2}{\lambda^2 \alpha_n^2}(k_1 B_{nm} + jC_{nm}) - \frac{\eta_3 \gamma_m^2}{\lambda^2 \alpha_n^2} D_{nm} - \frac{p_3}{\alpha_n^2} A_{nm} \right]$$

$$\left. \times \sin(\gamma_m z/\lambda) + \sum_{i=1}^{3} \left(e_1 \ell_i - e_3 \frac{m_i}{\mu_i} + \eta_3 \frac{n_i}{\mu_i} \right) \left[E_{in} \frac{\sinh(\alpha_n z/\sqrt{\mu_i})}{\sinh(\alpha_n b/\sqrt{\mu_i})} + F_{in} \frac{\cosh(\alpha_n z/\sqrt{\mu_i})}{\cosh(\alpha_n b/\sqrt{\mu_i})} \right] \right\}. \tag{56}$$

The $6n$ unknown coefficients E_{in} and F_{in} ($i = 1, 2, 3$; $n = 1$ to ∞) in these equations are determined through application of the boundary conditions. Conditions (36) are satisfied identically by expressions (49) and (55). Conditions (35) governing the stresses σ_{rz} and σ_{zz} and electric displacement D_z imply, respectively,

$$\sum_{i=1}^{3} \frac{c_{44}(\ell_i + m_i) + e_4 n_i}{\sqrt{\mu_i} \sinh(\alpha_n b/\sqrt{\mu_i})} E_{in} + \frac{c_{44}[(1 + k_1) B_{n0} + (1 + j) C_{n0}] + e_4 D_{n0}}{\lambda \sinh(\alpha_n b/\lambda)} \Bigg]$$

$$+ \sum_{m=1}^{\infty} \frac{\gamma_m}{\lambda \alpha_n} \{c_{44}[(1 + k_1) B_{nm} + (1 + j) C_{nm}] + e_4 D_{nm}\} = 0 \tag{57}$$

$$\sum_{i=1}^{3} \frac{c_{44}(\ell_i + m_i) + e_4 n_i}{\sqrt{\mu_i}} [E_{in} \coth(\alpha_n b/\sqrt{\mu_i}) + F_{in} \tanh(\alpha_n b/\sqrt{\mu_i})]$$

$$+ \frac{c_{44}[(1 + k_1) B_{n0} + (1 + j) C_{n0}] + e_4 D_{n0}}{\lambda} \coth(\alpha_n b/\lambda)$$

$$+ \sum_{m=1}^{\infty} \frac{\gamma_m}{\lambda \alpha_n} \{c_{44}[(1 + k_1) B_{nm} + (1 + j) C_{nm}] + e_4 D_{nm}\} \cos(\gamma_m b/\lambda) = 0 \tag{58}$$

$$\sum_{i=1}^{3} \left[c_{13} \ell_i - c_{33} \frac{m_i}{\mu_i} - e_3 \frac{n_i}{\mu_i} \right] \frac{1}{\cosh(\alpha_n b/\sqrt{\mu_i})} F_{in} = 0 \tag{59}$$

$$\sum_{i=1}^{3} \left[c_{13} \ell_i - c_{33} \frac{m_i}{\mu_i} - e_3 \frac{n_i}{\mu_i} \right] (E_{in} + F_{in})$$

$$+ c_{13}(B_{n0} + C_{n0}) - \frac{c_{33}}{\lambda^2}(k_1 B_{n0} + jC_{n0}) - \frac{e_3}{\lambda^2} D_{n0} + \frac{\beta_3}{\alpha_n^2} A_{n0} = 0 \tag{60}$$

$$\sum_{i=1}^{3} \left[e_1 \ell_i - e_3 \frac{m_i}{\mu_i} + \eta_3 \frac{n_i}{\mu_i} \right] \frac{1}{\cosh(\alpha_n b/\sqrt{\mu_i})} F_{in} = 0 \tag{61}$$

$$\sum_{i=1}^{3} \left[e_1 \ell_i - e_3 \frac{m_i}{\mu_i} + \eta_3 \frac{n_i}{\mu_i} \right] (E_{in} + F_{in})$$

$$+ e_1(B_{n0} + C_{n0}) - \frac{e_3}{\lambda^2}(k_1 B_{n0} + jC_{n0}) + \frac{\eta_3}{\lambda^2} D_{n0} - \frac{p_3}{\alpha_n^2} A_{n0} = 0. \tag{62}$$

Equations (57)–(62) are sufficient for the determination of E_{in} and F_{in}.

5 Numerical example

As an illustrative example, the temperature of the contacting medium is presumed to have a distribution described by

$$\Theta(r) = T_M\left(1 - 2f\,\frac{r^2}{a^2} + f\,\frac{r^4}{a^4}\right) \tag{63}$$

in which T_M is a constant and f is a specified parameter.

The disk material is considered to be cadmium selenide, with the following properties [17]:

$$\begin{aligned}
&c_{11} = 74.1\times 10^9 Nm^{-2}, \quad c_{12} = 45.2\times 10^9 Nm^{-2}, \quad c_{13} = 39.3\times 10^9 Nm^{-2},\\
&c_{33} = 83.6\times 10^9 Nm^{-2}, \quad c_{44} = 13.2\times 10^9 Nm^{-2},\\
&\beta_1 = 0.621\times 10^6 NK^{-1}m^{-2}, \quad \beta_3 = 0.551\times 10^6 NK^{-1}m^{-2},\\
&e_1 = -0.160 Cm^{-2}, \quad e_3 = 0.347 Cm^{-2}, \quad e_4 = -0.138 Cm^{-2},\\
&\eta_1 = 82.6\times 10^{-12} C^2N^{-1}m^{-2}, \quad \eta_3 = 90.3\times 10^{-12} C^2N^{-1}m^{-2},\\
&p_3 = -2.94\times 10^{-6} CK^{-1}m^{-2}, \quad Y_r = 42.8\times 10^9 Nm^{-2},\\
&\alpha_r = 4.4\times 10^{-6} K^{-1}, \quad d_1 = -3.92\times 10^{-12} CN^{-1}
\end{aligned} \tag{64}$$

where Y_r is Young's modulus, α_r is the coefficient of linear thermal expansion, and d_1 is the piezoelectric coefficient. Since the values of the coefficients of heat conduction for cadmium selenide could not be found in the literature, the value $\lambda^2 = \lambda_z/\lambda_r = 1.5$ is assumed.

The following dimensionless quantities are introduced for convenience in the presentation of the numerical results:

$$\begin{aligned}
&\bar{b} = \frac{b}{a}, \quad \bar{r} = \frac{r}{a}, \quad \bar{z} = \frac{z}{a}, \quad \bar{t} = \frac{\varkappa t}{a^2}, \quad B_i = ah, \quad \bar{T} = \frac{T}{T_M}\\
&\bar{u}_i = \frac{u_i}{a\alpha_r T_M}, \quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\alpha_r Y_r T_M}, \quad \bar{\Phi} = \frac{|d_1|\,\Phi}{a\alpha_r T_M}, \quad \bar{D}_i = \frac{D_i}{|d_1|\,\alpha_r Y_r T_M}.
\end{aligned} \tag{65}$$

Response quantities were calculated by retaining the first 50 terms in each of the corresponding infinite series. Figure 2 shows the nondimensional surface temperature distribution at various times $\bar{t}$ for a disk of thickness $\bar{b} = 0.2$, in the case of a Biot number $B_i = 1$ and parameter $f = 1$. The resulting radial distributions of electric potential difference $[\bar{\Phi}]_{\bar{z}=\bar{b}} - [\bar{\Phi}]_{\bar{z}=0}$, stresses $[\bar{\sigma}_{rr}, \bar{\sigma}_{\theta\theta}]_{\bar{z}=\bar{b}}$ and radial electric displacement $[\bar{D}_r]_{\bar{z}=0,\bar{b}}$ are given in Figs. 3 – 5, respectively. The maximum (absolute) value of each response quantity is seen to occur when the temperature reaches its steady-state value (at $\bar{t} = \infty$).

Figures 6 – 9 show the influence of the parameter f upon the distributions of temperature, electric potential difference and stresses, again for the case of $\bar{b} = 0.2$ and $B_i = 1$. Note that in the case of a uniform temperature $\Theta = T_M$ of the contacting medium (described by the curve $f = 0$ in Fig. 6), the induced electric potential difference and stresses are independent of $\bar{r}$ for all times $\bar{t}$.

The influence of disk thickness $\bar{b}$ on the temperature, potential difference and stresses at $\bar{r} = 0$ is illustrated in Figs. 10 – 12 for the case $f = 1$ and $B_i = 1$. For very small time $\bar{t}$, the thickness $\bar{b}$ has little effect on the response quantities; however for larger time $\bar{t}$, the greater the disk thickness, the greater the (absolute) value of the response quantities.

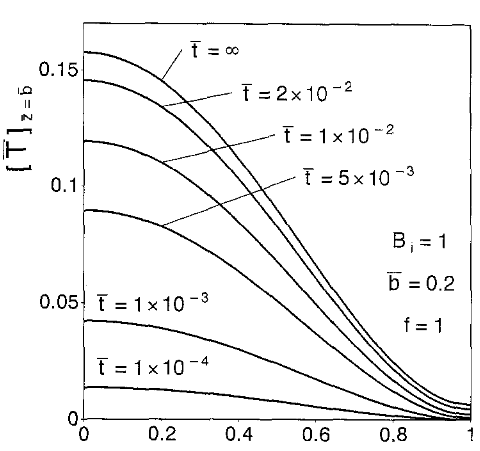

Fig. 2

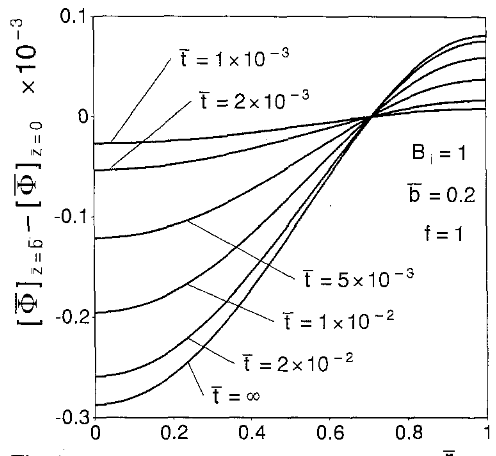

Fig. 3

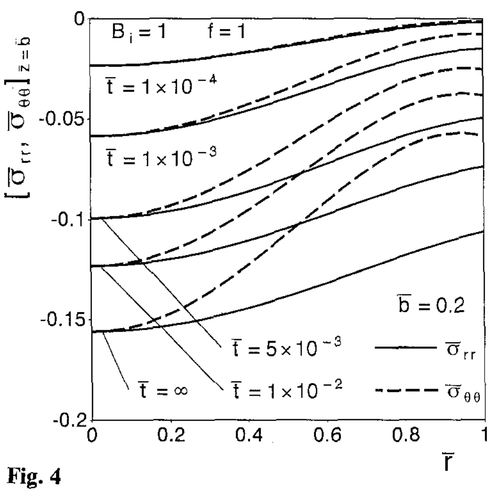

Fig. 4

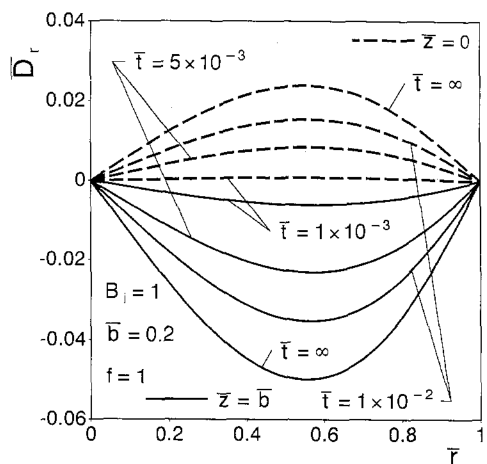

Fig. 5

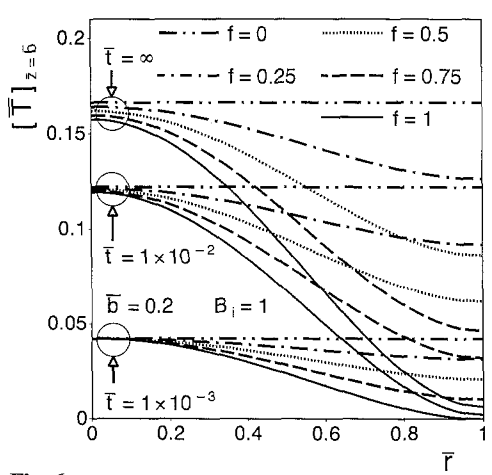

Fig. 6

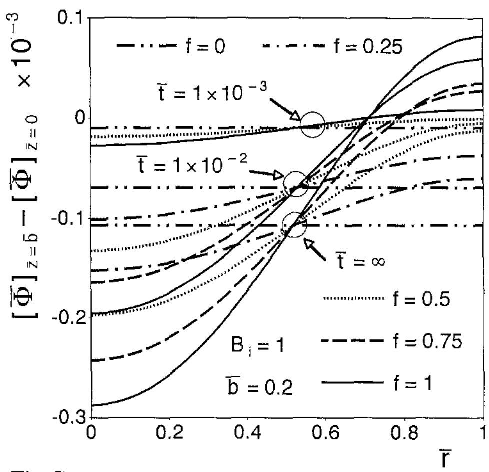

Fig. 7

Fig. 2. Distributions of surface temperature at various times. **Fig. 3.** Distributions of electric potential difference at various times. **Fig. 4.** Distributions of radial and circumferential surface stresses at various times. **Fig. 5.** Distributions of radial electric displacements at various times. **Fig. 6.** Influence of parameter f on surface temperature. **Fig. 7.** Influence of parameter f on electric potential difference

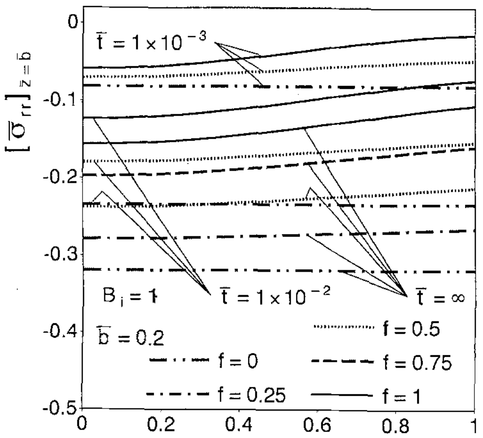

Fig. 8

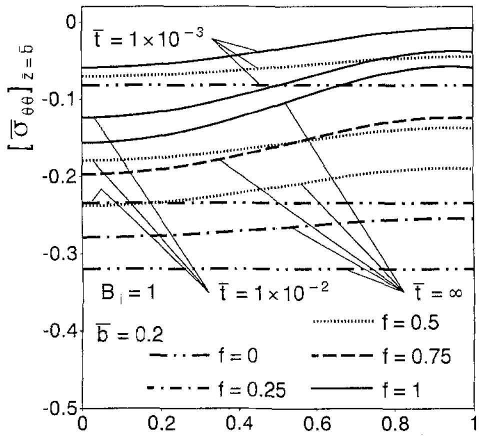

Fig. 9

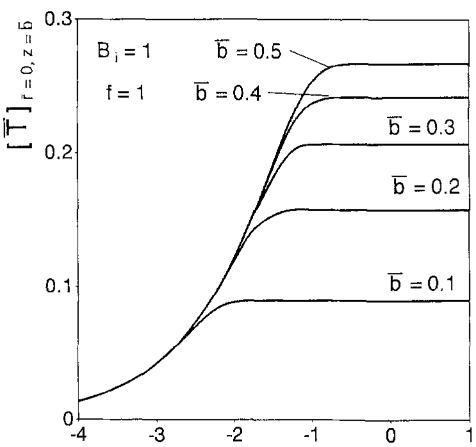

Fig. 10

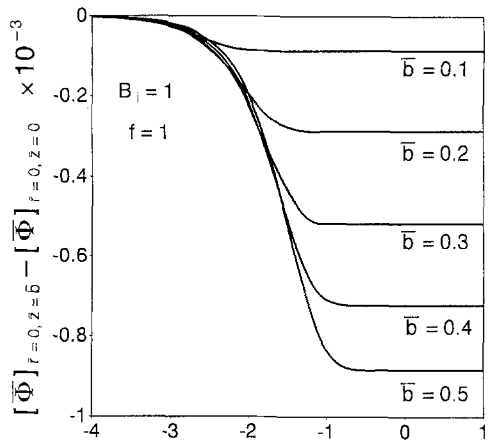

Fig. 11

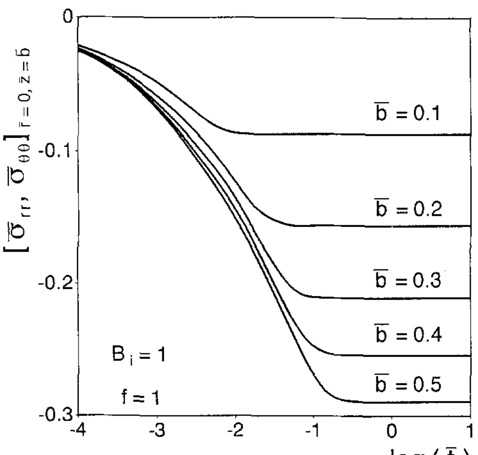

Fig. 12

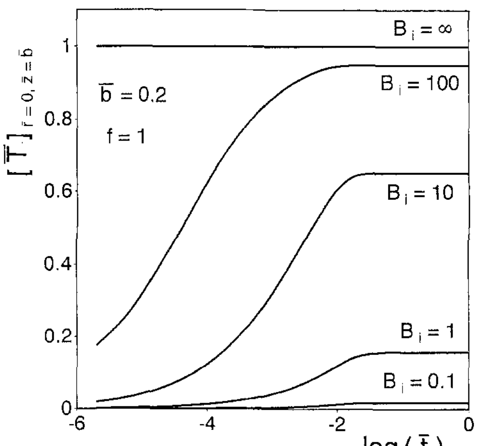

Fig. 13

Fig. 8. Influence of parameter f on radial stress. **Fig. 9.** Influence of parameter f on circumferential stress. **Fig. 10.** Influence of disk thickness on temperature. **Fig. 11.** Influence of disk thickness on electric potential difference. **Fig. 12.** Influence of disk thickness on radial and circumferential stresses. **Fig. 13.** Influence of Biot number on temperature

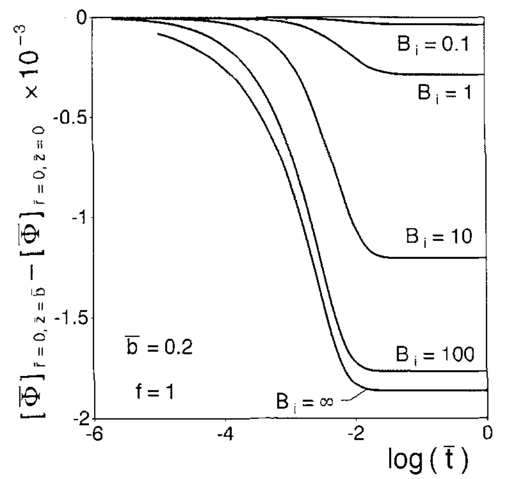

Fig. 14

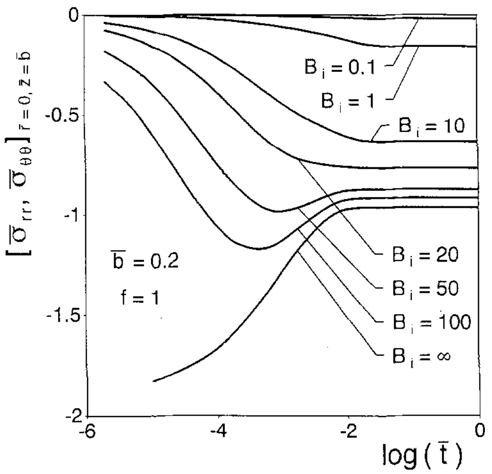

Fig. 15

Fig. 14. Influence of Biot number on electric potential difference. **Fig. 15.** Influence of Biot number on radial and circumferential stresses

Figures 13–15 show the effect of the Biot number B_i upon the surface temperature, electric potential and stresses at $\bar{r} = 0$. The special case $B_i = \infty$ corresponds to the case where the temperature is prescribed on the top surface of the disk; $B_i = 0$ implies zero heat flux into the disk, in which case all response quantities are zero.

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