A LATTICE OF NORMAL MODAL LOGICS

L. L. Maksimova and V. V. Rybakov

UDC 517.11

In this paper we study a family of modal propositional logics that contain the Lewis logic S4. An essential role in this study is played by the mapping of an intuitionistic logic into the logic S4 constructed in [14]. The mapping T of the formulas of propositional calculus into the formulas of modal logic used in this case makes it also possible to reduce the study of superintuitionistic logics [5] to the study of corresponding modal logics. A number of results in this field were obtained in [10].

In the present article we obtain further relationships between a lattice $\mathcal{L}$ of superintuitionistic logics and a family $\mathcal{M}$ of normal modal logics containing S4 which is also a complete distributive lattice. First of all we shall prove that the mapping of $\mathcal{L}$ into $\mathcal{M}$ considered in [10] is a lattice isomorphism. On the other hand, it is possible to assign to any modal logic $M \in \mathcal{M}$ a superintuitionistic logic $\rho(M)$ consisting of all the formulas of propositional calculus whose mappings are deducible in M. The logic G constructed in [17] differs from S4 and is such that $\rho(G) = \rho(S4)$. Let us note that the family $\rho^{-1}(L)$ is infinite for any superintuitionistic logic L and that it has a minimal and a maximal element. Thus we have two monotonic mappings τ and σ from $\mathcal{L}$ into $\mathcal{M}$ and a monotonic mapping ρ from $\mathcal{M}$ into $\mathcal{L}$ such that τ is a lattice isomorphism, ρ is a homomorphism, and the following conditions are satisfied:

$$\rho\tau = \rho\sigma = id_{\mathcal{L}}, \quad \tau\rho \leq id_{\mu} \leq \sigma\rho.$$

The obtained relationships between $\mathcal{M}$ and $\mathcal{L}$ make it possible to apply the results relating to the lattice $\mathcal{L}$ in a study of the lattice $\mathcal{M}$ of normal modal logics. Let us note that the family $\mathcal{L}$ of superintuitionistic logics was studied by numerous investigators, for example, in [1, 2], [4–6], and [12, 13]. With regard to the family $\mathcal{M}$, it has been mainly studied up to now with respect to its individual representatives and some of its subfamilies. In the last section of this paper we shall consider tabular modal logics, i.e., sets of formulas that are valid in a certain finite topological Boolean algebra (TBA). In particular, we shall prove that all of them are finitely axiomatizable and that logics directly preceding tabular logics are tabular. As a corollary we obtain an algorithm that makes it possible to ascertain on the basis of a finite TBA $\mathcal{B}$ whether a given formula is an axiomatization of the TBA logic $\mathcal{B}$.

The methods used in this paper are primarily algebraic. We shall extensively use a theorem on completeness of modal logics with respect to appropriate varieties of topological Boolean algebras. The results of §§1, 2, and 4 were obtained by Rybakov, and those of §3 by Maksimova.

§0. Definitions and Preliminary Remarks

The set of formulas of propositional calculus Φ_J is a set of formulas constructed with the aid of the propositional variables $p_1, p_2, \ldots$ and the logical connectives $\&, \vee, \supset, \neg$,

$$A \equiv B \rightleftharpoons (A \supset B) \& (B \supset A).$$

The set of modal propositional formulas Φ_M is a set of formulas constructed with the aid of the propositional variables $p_1, p_2, \ldots$ and the logical connectives $\&, \vee, \rightarrow, \sim, \Box$ (where $\Box$ is a unary logical connective denoting "necessity").

$$\alpha \leftrightarrow \beta \rightleftharpoons (\alpha \rightarrow \beta) \& (\beta \rightarrow \alpha), \quad \Diamond\alpha \rightleftharpoons \sim\Box\sim\alpha.$$

Translated from Algebra i Logika, Vol. 13, No. 2, pp. 188-216, March-April, 1974. Original article submitted March 5, 1974.

In the following, the letters $\alpha, \beta, \gamma, \delta, \ldots$ will denote formulas belonging to $\mathcal{F}_M$, whereas the letters A, B, C denote formulas belonging to $\mathcal{F}_J$.

A normal modal logic is defined as a set $M \subseteq \mathcal{F}_M$ such that M contains all the axioms of the logic S4 and is closed under the following deduction rules: 1) substitution, 2) $\frac{\alpha, \alpha \to \beta}{\beta}$, and 3) $\frac{\alpha}{\Box\alpha}$.

If $\Gamma \subseteq \mathcal{F}_M$, $\alpha \in \mathcal{F}_M$, then $\Gamma \models \alpha$ signifies that from Γ we can deduce α with the aid of the deduction rules 1, 2, 3 and the axioms of S4, whereas $\Gamma \vdash \alpha$ signifies that from Γ we can deduce α with the aid of the deduction rules 2 and 3 and the axioms of S4;

$$[\Gamma] \rightleftharpoons \{\alpha / \alpha \in \mathcal{F}_M, \Gamma \models \alpha\}.$$

If $\Gamma \subseteq \mathcal{F}_J$, $A \in \mathcal{F}_J$, then $\Gamma \models_J A$ will signify that from Γ we can deduce A with the aid of the substitution rule, modus ponens, and the axioms of the intuitionistic propositional calculus J, whereas $\Gamma \vdash_J A$ signifies that from Γ we can deduce A with the aid of modus ponens and the axioms of intuitionistic propositional calculus;

$$[\Gamma]_J \rightleftharpoons \{A / A \in \mathcal{F}_J, \Gamma \models_J A\}.$$

The set of all normal modal logics $\mathcal{M}$ is ordered by inclusion and it forms a complete lattice with the operations

$$\bigvee_{i \in J} M_i = [\bigcup_{i \in J} M_i], \quad \bigwedge_{i \in J} M_i = \bigcap M_i \quad (\{M_i\}_{i \in J} \subseteq \mathcal{M}).$$

The definitions relating to superintuitionistic logics and their semantics, i.e., to pseudo-Boolean algebras (PBA), can be found in [1, 2, 5, 7].

In the following we shall denote a PBA by the letter $\mathfrak{A}$ (if necessary, with subscripts), superintuitionistic logics will be denoted by the letter L (if necessary, with subscripts), a lattice of superintuitionistic logics will be denoted by $\mathcal{L}$, and the PBA logic $\mathfrak{A}$ will be denoted by $L\mathfrak{A}$. We shall stipulate that the fact that formula A takes the value 1 for all the interpretations on the PBA $\mathfrak{A}$ will be expressed by the relation $\mathfrak{A} \models A$.

A PBA $\mathfrak{A} = \langle \mathcal{A}; \&, \vee, \supset, \neg; 1 \rangle$ is said to be completely connected if from $a \in \mathcal{A}$, $b \in \mathcal{A}$, and $a \vee b = 1$ it follows that either $a = 1$ or $b = 1$.

A topological Boolean algebra (TBA) is an algebra $\mathfrak{B} = \langle \mathcal{B}; \&, \vee, \to, \sim, \Box; 1 \rangle$ such that $\langle \mathcal{B}; \&, \vee, \to, \sim; 1 \rangle$ is a Boolean algebra, and the operation $\Box$ (the interior) satisfies the following condition: for any $a, b \in \mathcal{B}$,

$$\Box a \leq a, \Box\Box a = \Box a, \quad \Box(a \& b) = \Box a \& \Box b, \quad \Box 1 = 1.$$

We shall denote a TBA by the symbol $\mathfrak{B}$ (with subscripts, if necessary). $\mathfrak{B}$ is said to be completely connected if from $a, b \in \mathcal{B}$ and $\Box a \vee \Box b = 1$ it follows that $a = 1$ or $b = 1$. The power of the TBA $\mathfrak{B}$ will be denoted by $\bar{\mathfrak{B}}$.

A formula $\alpha(p_1, \ldots, p_n)$ is said to be true on $\mathfrak{B}$ ($\mathfrak{B} \models \alpha$) if on $\mathfrak{B}$ we have a true identity $(\forall x_1) \ldots (\forall x_n)\ \alpha(x_1, \ldots, x_n) = 1$.

The TBA $\mathfrak{B}$ is defined by the set

$$M(\mathfrak{B}) \rightleftharpoons \{\alpha / \alpha \in \mathcal{F}_M, \mathfrak{B} \models \alpha\}.$$

It is evident that $M(\mathfrak{B}) \in \mathcal{M}$. It is well known ([14], Theorem 3.6) that for any $M \in \mathcal{M}$ there exists a $\mathfrak{B}$ such that $M = M(\mathfrak{B})$. M is said to be tabular if there exists a finite $\mathfrak{B}$ such that $M = M(\mathfrak{B})$, and nontabular otherwise. Logics that are maximal in the class of nontabular logics are said to be pretabular. A logic M is said to be finitely approximable if it can be represented by an intersection of tabular logics.

Between the lattice $\mathcal{M}$ of modal logics and the lattice of all TBA varieties there exists a dual isomorphism, i.e., to any $M \in \mathcal{M}$ we assign a variety defined by the identities $\{\alpha = 1 / \alpha \in M\}$.

SECTION 1

Our next aim is to prove that $\mathcal{M}$ is a distributive lattice. But at first we have to prove a number of special results that are, however, also of intrinsic interest.

LEMMA 1.1. (Deduction theorem for S4). If $\Gamma \subseteq \mathcal{F}_M$, $\Gamma, \alpha \vdash \beta$, then $\Gamma \vdash \Box\alpha \to \beta$.

The proof is based on induction on the length of the deduction. The following lemma is easy to prove.

LEMMA 1.2. If $\alpha_1, \dots, \alpha_m \models \gamma$, then there exist $\alpha_{11} \dots \alpha_{1K_1}, \dots, \alpha_{m1} \dots \alpha_{mK_m}$ such that any α_{ji} can be obtained from α_j $(1 \le j \le m)$ by a substitution and $\alpha_{11}, \dots, \alpha_{1K_1}, \dots, \alpha_{m1}, \dots, \alpha_{mK_m} \vdash \gamma$.

LEMMA 1.3. If $\alpha_1, \dots, \alpha_n \models \gamma$ and $\alpha_1, \dots, \alpha_n$ do not have common propositional variables with β, then

$$\Box\alpha_1 \vee \beta, \dots, \Box\alpha_n \vee \beta \models \gamma \vee \beta.$$

Proof. At first we shall consider the case:

1. $\alpha_1, \dots, \alpha_n \vdash \gamma$, i.e., γ is deducible from $\alpha_1, \dots, \alpha_n$ without the substitution rule.

It then follows from Lemma 1.1 that $\vdash (\Box\alpha_1 \to (\Box\alpha_2 \to \dots (\Box\alpha_n \to \gamma) \dots))$. Moreover, $\vdash (\delta_1 \to (\delta_2 \to \dots (\delta_n \to \gamma) \dots)) \to (\delta_1 \vee \beta \to \dots \to (\delta_n \vee \beta \to \gamma \vee \beta))$ (which is a formula deducible in classical calculus). Then $\vdash (\Box\alpha_1 \vee \beta \to \dots (\Box\alpha_n \vee \beta \to \gamma \vee \beta) \dots)$, and hence

$$\Box\alpha_1 \vee \beta, \dots, \Box\alpha_n \vee \beta \vdash \gamma \vee \beta.$$

Now let us consider the case:

2. $\alpha_1, \dots, \alpha_n \models \gamma$, and use in the deduction the substitution rule. It then follows from Lemma 1.2 that there exist $\alpha_{11} \dots \alpha_{1K_1}, \dots, \alpha_{n1} \dots \alpha_{nK_n}$ such that α_{ji} can be obtained from α_j by substitutions and $\alpha_{11}, \dots, \alpha_{1K_1}, \dots, \alpha_{n1}, \dots, \alpha_{nK_n} \vdash \gamma$. By using the proof for case 1, we have

$$\Box\alpha_{11} \vee \beta, \dots, \Box\alpha_{1K_1} \vee \beta, \dots, \Box\alpha_{n1} \vee \beta, \dots, \Box\alpha_{nK_n} \vee \beta \vdash \gamma \vee \beta.$$

But $\Box\alpha_j \vee \beta \models \Box\alpha_{ji} \vee \beta$ (since β does not have common propositional variables with α_j); then $\Box\alpha_1 \vee \beta, \dots, \Box\alpha_n \vee \beta \models \gamma \vee \beta$. This completes the proof of the lemma.

Let us prove a theorem on the structure of axioms of intersection of modal logics,

Similarly to [1] and [2] we shall define the concept of unrepeated disjunction $\vee'$. For any formulas F_1, F_2 ($F_1, F_2 \in \mathcal{F}_M$ or $F_1, F_2 \in \mathcal{F}_J$) we have $F_1 \vee' F_2 = F_1 \vee F_2'$, where F_2' is obtained from F_2 by increasing the numbers of propositional variables that have an occurrence in F_2 by the maximal number of propositional variables that have an occurrence in F_1.

THEOREM 1. If $M_1, M_2 \in \mathcal{M}$ and $M_1 = [\Gamma_1]$, $M_2 = [\Gamma_2]$, then $M_1 \wedge M_2 = [\{\Box\alpha \vee' \Box\beta / \alpha \in \Gamma_1, \beta \in \Gamma_2\}]$.

Proof. It is evident that $M_1 \wedge M_2 \supseteq [\{\Box\alpha \vee' \Box\beta / \alpha \in \Gamma_1, \beta \in \Gamma_2\}]$.

Let us prove the inverse inclusion. We shall assume that $\gamma \in M_1 \wedge M_2$. Then there exist $\alpha_1, \dots, \alpha_n \in \Gamma_1$, $\beta_1, \dots, \beta_m \in \Gamma_2$ such that $\alpha_1, \dots, \alpha_n \models \gamma$, $\beta_1, \dots, \beta_m \models \gamma$. Let N be the maximal number of propositional variables that have occurrences in at least one of the formulas $\alpha_1, \dots, \alpha_n, \gamma$. For any K $(1 \le K \le m)$ let us denote by β_K' the formula obtained from β_K by renaming of variables, more precisely, by replacing any variables p_i by p_{i+N}.

Then $\beta_1', \dots, \beta_m' \models \gamma$ (since $\beta_K' \models \beta_K$). It follows from Lemma 1.3 that $\gamma \vee \Box\beta_1', \dots, \gamma \vee \Box\beta_m' \models \gamma \vee \gamma$. It also follows from Lemma 1.3 that:

$$\Box\alpha_1 \vee \Box\beta_1', \ldots, \Box\alpha_n \vee \Box\beta_1' \models \gamma \vee \Box\beta_1'; \ldots; \Box\alpha_1 \vee \Box\beta_m', \ldots, \Box\alpha_n \vee \Box\beta_m' \models \gamma \vee \Box\beta_m'.$$

Hence

$$\Box\alpha_1 \vee \Box\beta_1', \ldots, \Box\alpha_n \vee \Box\beta_1', \ldots, \Box\alpha_1 \vee \Box\beta_m', \ldots, \Box\alpha_n \vee \Box\beta_m' \models \gamma,$$

since $\gamma \vee \gamma \models \gamma$. But $\Box\alpha_j \vee' \Box\beta_i \models \Box\alpha_j \vee \Box\beta_i'$; hence

$$\Box\alpha_1 \vee' \Box\beta_1, \ldots, \Box\alpha_n \vee' \Box\beta_1, \ldots, \Box\alpha_1 \vee' \Box\beta_m, \ldots, \Box\alpha_n \vee' \Box\beta_m \models \gamma,$$

i.e., $\gamma \in [\{\Box\alpha \vee' \Box\beta / \alpha \in \Gamma_1, \beta \in \Gamma_2\}]$. This completes the proof of the theorem.

COROLLARY. A family of finitely axiomatizable logics is a sublattice of the lattice $\mathcal{M}$.

It is evident that a union of finitely axiomatizable logics is finitely axiomatizable. The intersection of finitely axiomatizable logics is finitely axiomatizable by virtue of Theorem 1.

Let us prove the principal result of this section.

THEOREM 2. The lattice $\mathcal{M}$ is a pseudo-Boolean algebra.

Proof. It is evident that we must prove only the existence of a relative pseudocomplement. Let $M_1, M_2 \in \mathcal{M}$ and $M_1 = [\Gamma]$. Let us define a set $\mathcal{D} \subseteq \Phi_M$:

$$\mathcal{D} \rightleftharpoons \{\beta / \beta \in \Phi_M, (\forall \alpha \in \Gamma)(\Box\alpha \vee' \Box\beta \in M_2)\}.$$

Let us write $M = [\mathcal{D}]$ and prove that $M_1 \supset M_2 = M$. We shall prove that $M_1 \wedge M \subseteq M_2$. Indeed, by virtue of Theorem 1,

$$M_1 \wedge M = [\{\Box\alpha \vee' \Box\gamma / \alpha \in \Gamma, \gamma \in \mathcal{D}\}].$$

For any $\alpha \in \Gamma$, we have $\gamma \in \mathcal{D}$ and $\Box\alpha \vee' \Box\gamma \in M_2$ by virtue of the definition of $\mathcal{D}$. We have proved that all the axioms of the logic $M_1 \wedge M$ are contained in M_2, and hence $M_1 \wedge M \subseteq M_2$.

Now let us prove that M is a maximal element of the set

$$\{M_0 / M_0 \in \mathcal{M}, \quad M_1 \wedge M_0 \subseteq M_2\}.$$

Let $M_0 \in \mathcal{M}$, $M_1 \wedge M_0 \subseteq M_2$ and $M_0 = [\Delta]$. It follows from Theorem 1 that $M_1 \wedge M_0 = [\{\Box\alpha \vee' \Box\delta / \alpha \in \Gamma, \delta \in \Delta\}]$. For any $\alpha \in \Gamma$, we have $\delta \in \Delta, \Box\alpha \vee' \Box\delta \in M_2$. It then follows that for any $\delta \in \Delta$ we have $\delta \in \mathcal{D}$ by virtue of the definition of $\mathcal{D}$. Hence $M_0 = [\Delta] \subseteq [\mathcal{D}]$. Therefore $M_1 \supset M_2 = M$. This completes the proof of the theorem.

COROLLARY. The lattice $\mathcal{M}$ is distributive.

It is well known (see, for example, [7]) that any PBA is a distributive lattice.

SECTION 2

This section is devoted to problems of embedding various lattices and partially ordered sets in $\mathcal{M}$, and to other structural problems. Hosoi [13] has introduced the mapping $T : \Phi_J \rightarrow \Phi_M$ which is defined inductively:

1) If p is a propositional variable, then $T(p) = \Box p$;

2) $T(A \& B) = T(A) \& T(B)$;

3) $T(A \vee B) = T(A) \vee T(B)$;

4) $T(A \supset B) = \Box(T(A) \rightarrow T(B))$;

5) $T(\neg A) = \Box(\sim T(A))$.

LEMMA 2.1. Let $A \in \Phi_J$; then

$$\vdash \Box T(A) \leftrightarrow T(A).$$

The proof is carried out by induction on the length of A.

For any $\Gamma \subseteq \Phi_J$, we have $T(\Gamma) \rightleftharpoons \{T(A)/A \in \Gamma\}$. Let us define the mapping $\tau: \mathcal{L} \to \mathcal{M}$: $\tau(L) \rightleftharpoons [T(L)]$.

LEMMA 2.2. 1. $\tau: \mathcal{L} \to \mathcal{M}$ is one-to-one.

2. If $A \in \Phi_J$, $\Gamma \subseteq \Phi_J$, then

$$\Gamma \models_J A \Longleftrightarrow T(\Gamma) \models T(A).$$

The proof follows directly from Theorem 1 of [10].

COROLLARY. $\mathcal{M}$ has the power of the continuum.

It is well known [8] that $\mathcal{L}$ has the power of the continuum.

THEOREM 3. The mapping $\tau: \mathcal{L} \to \mathcal{M}$ is a lattice isomorphism " f " that preserves infinite unions.

Proof. By virtue of Lemma 2.2 it is only necessary to prove that τ preserves operations. At first we shall prove that

1) $\tau(L_1 \wedge L_2) = \tau(L_1) \cap \tau(L_2)$.

$$\tau(L_1 \wedge L_2) = [T(L_1 \wedge L_2)], \quad \tau(L_1) \wedge \tau(L_2) = [T(L_1)] \wedge [T(L_2)].$$

It is evident that $[T(L_1 \wedge L_2)] \subseteq [T(L_1)] \wedge [T(L_2)]$. Let us prove the inverse inclusion. By virtue of Theorem 1,

$$[T(L_1)] \wedge [T(L_2)] = [\{\Box\alpha \vee' \Box\beta / \alpha \in T(L_1),\ \beta \in T(L_2)\}].$$

If $\alpha \in T(L_1)$, $\beta \in T(L_2)$, then $\alpha = T(A)$, $\beta = T(B)$, where $A \in L_1$, $B \in L_2$. Then $A \vee' B \in L_1$ and $A \vee' B \in L_2$, i.e., $A \vee' B \in L_1 \wedge L_2$. Hence $T(A \vee' B) = T(A) \vee' T(B) \in T(L_1 \wedge L_2)$. But it follows from Lemma 2.1 that $\vdash \Box T(A) \leftrightarrow T(A)$, $\vdash \Box T(B) \leftrightarrow T(B)$. Therefore $\Box T(A) \vee' \Box T(B) \in [T(A) \vee' T(B)]$, and hence also $\Box\alpha \vee' \Box\beta \in [T(L_1 \wedge L_2)]$.

Thus all the axioms of the logic $[T(L_1)] \wedge [T(L_2)]$ are contained in the logic $[T(L_1 \wedge L_2)]$, and hence $[T(L_1)] \wedge [T(L_2)] \subseteq [T(L_1 \wedge L_2)]$. We have proved that τ preserves intersections. Let us prove that τ preserves infinite unions.

2. $\tau(\bigvee_{\kappa \in \mathcal{K}} L_\kappa) = \bigvee_{\kappa \in \mathcal{K}} \tau(L_\kappa)$.

We have

$$\tau(\bigvee_{\kappa \in \mathcal{K}} L_\kappa) = [T(\bigvee_{\kappa \in \mathcal{K}} L_\kappa)] = [T([\bigcup_{\kappa \in \mathcal{K}} L_\kappa]_J)],$$

$$\bigvee_{\kappa \in \mathcal{K}} \tau(L_\kappa) = \Big[\bigcup_{\kappa \in \mathcal{K}} \tau(L_\kappa)\Big] = \Big[\bigcup_{\kappa \in \mathcal{K}} [T(L_\kappa)]\Big].$$

It is evident that $\Big[\bigcup_{\kappa \in \mathcal{K}} [T(L_\kappa)]\Big] \subseteq [T([\bigcup_{\kappa \in \mathcal{K}} L_\kappa]_J)]$. Let us prove the inverse inclusion. $T(\bigcup_{\kappa \in \mathcal{K}} L_\kappa)$ is a set of axioms of the logic $[T([\bigcup_{\kappa \in \mathcal{K}} L_\kappa]_J)]$ by virtue of Lemma 2.2. But $T(\bigcup_{\kappa \in \mathcal{K}} L_\kappa) \subseteq \Big[\bigcup_{\kappa \in \mathcal{K}} [T(L_\kappa)]\Big]$, and therefore

$$[T([\bigcup_{\kappa \in \mathcal{K}} L_\kappa]_J)] \subseteq \Big[\bigcup_{\kappa \in \mathcal{K}} [T(L_\kappa)]\Big].$$

This completes the proof of the lemma.

COROLLARY. a) In $\mathcal{M}$ it is possible to isomorphically embed any finite distributive lattice.

b) In $\mathcal{M}$ it is possible to isomorphically embed any countable partially ordered set.

c) In $\mathcal{M}$ it is possible to isomorphically embed a free distributive lattice of countable rank.

This follows from similar results for $\mathcal{L}$ [1, 2]. A set of formulas $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \ldots$ is said to be superindependent if for any natural numbers $m, n, i_1, \ldots, i_n, j_1, \ldots, j_m$ such that $\{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_m\} = \emptyset$, we do not have $\alpha_{i_1} \& \ldots \& \alpha_{i_n} \models \Box\alpha_{j_1} \vee' \ldots \vee' \Box\alpha_{j_m}$. Such a definition of superindependent modal formulas is similar to the definition of superindependent formulas of intuitionistic propositional calculus [1, 2]. As in [1], it is possible to show that a set $\alpha_1, \alpha_2, \ldots$ is superindependent if and only if the logics $[\alpha_1], [\alpha_2], \ldots$ generate in $\mathcal{M}$ a free distributive sublattice in which they constitute free generatrices.

<u>LEMMA 2.3.</u> There exists an infinite set of superindependent formulas $\{\alpha_n\}_{n \in N} \subseteq \mathcal{F}_M$.

<u>Proof.</u> There exists an infinite set of superindependent formulas $\{A_n\}_{n \in N} \subseteq \mathcal{F}_J$ [2, 1]. Gerchiu [1] has shown that $\{A_n\}_{n \in N} \subseteq \mathcal{F}_J$ is superindependent if and only if $[A_n]_J$ $(n \in N)$ generate in $\mathcal{L}$ a free distributive sublattice F in which they are free generatrices. It then follows from Theorem 3 that $\tau(F)$ is a free distributive sublattice of the lattice $\mathcal{M}$ with free generatrices $\tau([A_n]_J)$ $(n \in N)$. $\tau([A_n]_J) = [T([A_n]_J)]$. By virtue of Lemma 2.2 we have $[T([A_n]_J)] = [T(A_n)]$. Hence the $[T(A_n)]$ $(n \in N)$ are free generatrices in $\tau(F)$. It hence follows from the remark preceding the lemma that $\{T(A_n)\}_{n \in N}$ is a superindependent set of formulas. This completes the proof of the lemma.

<u>COROLLARY.</u> There exists a continuum of modal logics with axiomatization from an infinite set of superindependent formulas that are not finitely axiomatizable.

It suffices to take logics generated by infinite subsets $\{\alpha_n\}_{n \in N}$. It is evident that they will be distinct and not finitely axiomatizable.

<u>LEMMA 2.4.</u> Let $M_1, M_2 \in \mathcal{M}$ and $M_1 \neq S4$, $M_2 \neq S4$; then $M_1 \wedge M_2 \neq S4$.

The proof follows directly from Theorem 1 and the assertion XI. 9.5 of [7].

<u>THEOREM 4.</u> The lattice $\mathcal{M}$ does not have atoms.

<u>Proof.</u> Suppose there exists an atom M, $M \neq S4$.

We shall consider the class

$$K \rightleftharpoons \{ \mathcal{B}_i \,/\, \mathcal{B}_i \text{ is a finite TBA} \}.$$

It is well known [11] that $M(\mathcal{B}_i) \neq S4$, $M(\mathcal{B}_i) \supseteq S4$.

Let us prove that $M(\mathcal{B}_i) \supseteq M$. We have $M \wedge M(\mathcal{B}_i) \subseteq M$ and by virtue of Lemma 2.4 we have $M \wedge M(\mathcal{B}_i) \neq S4$; hence $M(\mathcal{B}_i) \wedge M = M$, i.e., $M(\mathcal{B}_i) \supseteq M$. We have proved that for any $\mathcal{B}_i \in K$ we have $M(\mathcal{B}_i) \supseteq M$; hence $\bigwedge_{\mathcal{B}_i \in K} M(\mathcal{B}_i) \supseteq M$. On the other hand, $\bigwedge_{\mathcal{B}_i \in K} M(\mathcal{B}_i) = S4$ by virtue of XI. 9.1 of [7]. Hence $M = S4$; we have arrived at a contradiction. This completes the proof of the theorem.

SECTION 3

In this section we shall study in detail the connection between superintuitionistic and modal logics expressed by the relation

$$(\forall A \in \mathcal{F}_J)(L \models_J A \Longleftrightarrow M \models T(A)). \tag{T}$$

For this purpose we shall examine certain relationships between PBA characterizing the superintuitionistic logic L and TBA characterizing M if L and M are connected by the relation (T).

Let us recall ([7], IV, 1.4) that to any TBA $\mathcal{B} = \langle \mathcal{B}; \&, \vee, \rightarrow, \sim, \Box; 1 \rangle$ there corresponds a PBA

$$G(\mathcal{B}) = \langle G(\mathcal{B}); \&, \vee, \supset, \neg; 1 \rangle, \tag{1}$$

defined as follows:

$G(\mathscr{B})$ is the set of all open elements of the algebra $\mathscr{L}$, i.e., $G(\mathscr{B}) = \{x / x \in \mathscr{B}, \Box x = x\}$; $\langle G(\mathscr{B}); \&, \vee; 1 \rangle$ is a subalgebra of $\langle \mathscr{B}; \&, \vee; 1 \rangle$;

$$x \supset y = \Box(\sim x \vee y);$$
$$\neg x = \Box(\sim x).$$

The relationship between truth in $\mathscr{L}$ and in $G(\mathscr{L})$ can be expressed by the following:

LEMMA 3.1. For any TBA $\mathscr{L}$ and any formula $A \in \Phi_J$ we have the equivalence

$$\mathscr{L} \models T(A) \Longleftrightarrow G(\mathscr{L}) \models A.$$

LEMMA 3.2. Let $\mathscr{L} = \langle \mathscr{B}; \&, \vee, \rightarrow, \sim, \Box; 1 \rangle$ be a TBA, let $\mathfrak{A}_1 = \langle \mathscr{A}_1; \&, \vee, \supset, \neg; 1 \rangle$ be a subalgebra of the PBA $G(\mathscr{L})$, and let $\mathscr{L}_1 = \langle \mathscr{B}_1; \&, \vee, \rightarrow, \sim, \Box; 1 \rangle$ be a subalgebra of the TBA $\mathscr{L}$ generated by the set $\mathscr{A}_1$. Then:

a) $\mathscr{B}_1$ is the set of all elements of the form

$$x = (\sim a_1 \vee b_1) \& \ldots \& (\sim a_n \vee b_n), \qquad (*)$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathscr{A}_1$;

b) $G(\mathscr{L}_1) = \mathfrak{A}_1$.

Proof. It is evident that $\langle \mathscr{A}_1; \&, \vee; 1 \rangle$ is a sublattice of the lattice $\langle \mathscr{B}; \&, \vee; 1 \rangle$. Therefore the set $\mathscr{B}_2$ of all elements of the form (*) constitutes by virtue of ([7], P. 2.2) a subalgebra of the algebra $\langle \mathscr{B}; \&, \vee, \rightarrow, \sim; 1 \rangle$ generated by the set $\mathscr{A}_1$ with the aid of $\&, \vee, \rightarrow, \sim$. Now let us note that for x of the form (*) we have

$$\Box x = \Box(\sim a_1 \vee b_1) \& \ldots \& \Box(\sim a_n \vee b_n) = (a_1 \underset{G(\mathscr{L})}{\supset} b_1) \& \ldots \& (a_n \underset{G(\mathscr{L})}{\supset} b_n),$$

and hence $\Box x \in \mathscr{A}_1$. Therefore $\mathscr{B}_2$ will be closed under the operation $\Box$ and hence it coincides with $\mathscr{B}_1$, whereas $G(\mathscr{B}_1) = \mathscr{A}_1$.

Now we shall assign to each PBA $\mathfrak{A}$ a topological Boolean algebra $\mathfrak{s}(\mathfrak{A})$ constructed as follows. Let $S_{\mathfrak{A}}$ be the Stone space of all simple filters in $\mathfrak{A}$ and let the sets

$$\varphi(a) = \{\Phi / a \in \Phi \in S_{\mathfrak{A}}\} \qquad (2)$$

for $a \in \mathfrak{A}$, form a basis of this space. The mapping φ is an isomorphism into the algebra $G(\mathscr{P}(S_{\mathfrak{A}}))$, where

$$\mathscr{P}(S_{\mathfrak{A}}) = \langle P(S_{\mathfrak{A}}), \&, \vee, \rightarrow, \sim, \Box; 1 \rangle \qquad (3)$$

is a topological Boolean algebra of all the subsets of the space $S_{\mathfrak{A}}$, the operations $\&, \vee, \sim$ are defined as intersection, union, and completion respectively, $X \rightarrow Y = \sim X \vee Y, 1 = S_{\mathfrak{A}}$, and $\Box$ is the operation of taking the interior.

Let us denote by

$$\mathfrak{s}(\mathfrak{A}) = \langle \mathfrak{s}(\mathscr{A}); \&, \vee, \rightarrow, \sim, \Box; 1 \rangle \qquad (4)$$

a subalgebra of the algebra $\mathscr{P}(S_{\mathfrak{A}})$ generated by the set $\varphi(\mathscr{A})$. By virtue of Lemma 3.2 we have

$$G(\mathfrak{s}(\mathfrak{A})) \cong \mathfrak{A}. \qquad (5)$$

It is evident that if $\mathfrak{A} \cong \mathfrak{A}_1$, then $\mathfrak{s}(\mathfrak{A}) \cong \mathfrak{s}(\mathfrak{A}_1)$.

LEMMA 3.3. Let $\mathcal{L}$ be a TBA, let $\mathcal{L}_1$ be a subalgebra of $\mathcal{L}$ generated by the set $G(\mathcal{L})$, and let h be a homomorphism from $G(\mathcal{L})$ into $\mathfrak{A}$. Then there exists a homomorphism h_1 from $\mathcal{L}_1$ into $\mathfrak{s}(\mathfrak{A})$.

Proof. By virtue of Lemma 3.2, all the elements of $\mathcal{L}_1$ can be represented in the form

$$x = (\sim a_1 \vee b_1) \,\&\ldots\&\, (\sim a_n \vee b_n), \tag{*}$$

where $a_1,\ldots,a_n, b_1,\ldots,b_n \in G(\mathcal{L})$. Let us write

$$h_1(x) = (\sim\varphi h(a_1) \vee \varphi h(b_1)) \,\&\ldots\&\, (\sim\varphi h(a_n) \vee \varphi h(b_n)), \tag{**}$$

where φ is a Stone isomorphism (2) from $\mathfrak{A}$ into $G(\mathcal{P}(S_{\mathfrak{A}}))$, and $\&$, $\vee$, and $\sim$ are operations in the algebra $\mathcal{P}(S_{\mathfrak{A}})$. Then $h_1(x) \in \mathfrak{s}(\mathfrak{A})$ for $x \in \mathcal{L}_1$.

Let us show that for $x, y \in \mathcal{L}_1$:

1. $x \le y \Rightarrow h_1(x) \le h_1(y)$.

Let $x \le y$ and

2. $y = (\sim c_1 \vee d_1) \,\&\ldots\&\, (\sim c_m \vee d_m)$,

where $c_1,\ldots,c_m, d_1,\ldots,d_m \in G(\mathcal{L})$. By virtue of distributivity, $h_1(x)$ can be expressed as a disjunction of all possible conjunctions of the form

3. $\underset{i\in I}{\&} \sim\varphi h(a_i) \,\&\, \underset{j\in J}{\&} \varphi h(b_j)$, where $I \cup J = \{1,\ldots,n\}$, $I \cap J = \emptyset$.

Let us show that for any such I, J, and any k $(1 \le k \le m)$ we have in $\mathfrak{s}(\mathfrak{A})$ the inequality

4. $\underset{i\in I}{\&} \sim\varphi h(a_i) \,\underset{j\in J}{\&}\, \varphi h(b_j) \le \sim\varphi h(c_k) \vee \varphi h(d_k)$.

From (*) and 2 it follows that in the algebra $\mathcal{L}$,

$$\underset{i\in I}{\&} \sim a_i \,\&\, \underset{j\in J}{\&} b_j \le \sim c_k \vee d_k,$$

or, in equivalent form,

$$\underset{j\in J}{\&} b_j \,\&\, c_k \le d_k \vee \bigvee_{i\in I} a_i.$$

Since φh is a homomorphism from $G(\mathcal{L})$ into $G(\mathcal{P}(S_{\mathfrak{A}}))$, we have in the algebra $G(\mathcal{P}(S_{\mathfrak{A}}))$, and hence in $\mathcal{P}(S_{\mathfrak{A}})$:

$$\underset{j\in J}{\&} \varphi h(b_j) \,\&\, \varphi h(c_k) \le \varphi h(d_k) \vee \bigvee_{i\in I} \varphi h(a_i).$$

Therefore we have in $\mathcal{P}(S_{\mathfrak{A}})$:

$$\underset{i\in I}{\&} \sim \varphi h(a_i) \,\&\, \underset{j\in J}{\&} \varphi h(b_j) \le \sim\varphi h(c_k) \vee \varphi h(d_k).$$

Since all $\varphi h(a_i)$, $\varphi h(b_j)$, $\varphi h(c_k)$, $\varphi h(d_k)$ belong to $\mathfrak{s}(\mathfrak{A})$, we have proved the inequality 4. Hence follows directly that $h_1(x) \le h_1(y)$. In particular, we find that $h_1(x)$ does not depend on the adopted representation of x in the form (*).

Since by virtue of Lemma 3.2 any element $X \in \mathfrak{s}(\mathfrak{A})$ can be represented in the form

$$X = (\sim\varphi(u_1) \vee \varphi(v_1)) \,\&\ldots\&\, (\sim\varphi(u_n) \vee \varphi(v_n)),$$

where $u_1,\dots,u_n$, $v_1,\dots,v_n \in \mathfrak{A}$, there exist elements $a_1,\dots,a_n, b_1,\dots,b_n \in G(\mathfrak{L})$ such that $u_1 = h(a_1),\dots, u_n = h(a_n)$, $v_1 = h(b_1),\dots, v_n = h(b_n)$, and hence $X = h_1(x)$, where $x = (\sim a_1 \vee b_1) \& \dots \& (\sim a_n \vee b_n) \in \mathfrak{L}_1$. Thus h_1 maps $\mathfrak{L}_1$ into $\Delta(\mathfrak{A})$. It is evident that

5. $h_1(x \& y) = h_1(x) \& h_1(y)$.

Next,

$$h_1(\sim(\mathop{\&}_{i=1}^{n}(\sim a_i \vee b_i))) = h_1(\bigvee_{i=1}^{n}(a_i \& \sim b_i)) =$$

$$= h_1(\mathop{\&}_{I}((\bigvee_{i\in I} a_i) \vee \sim (\mathop{\&}_{j\in J} b_j))) = \mathop{\&}_{I}((\bigvee_{i\in I} \varphi h(a_i)) \vee \sim (\mathop{\&}_{j\in J} \varphi h(b_j))),$$

where

$$I \cap J = \varnothing, \quad I \cup J = \{1,\dots,n\}.$$

$$\sim h_1(\mathop{\&}_{i=1}^{n}(\sim a_i \vee b_i)) = \sim \mathop{\&}_{i=1}^{n}(\sim \varphi h(a_i) \vee \varphi h(b_i)) =$$

$$= \bigvee_{i=1}^{n}(\varphi h(a_i) \& \sim \varphi h(b_i)) = \mathop{\&}_{I}((\bigvee_{i\in I} \varphi h(a_i)) \vee \sim (\mathop{\&}_{i\notin I} \varphi h(b_i))).$$

Hence

6. $h_1(\sim x) = \sim h_1(x)$.

From 5 and 6 we obtain

7. $h_1(x \vee y) = h_1(\sim(\sim x \& \sim y)) = h_1(x) \vee h_1(y)$,

8. $h_1(x \rightarrow y) = h_1(\sim x \vee y) = h_1(x) \rightarrow h_1(y)$.

Let us show that

9. $h_1(\Box x) = \Box h_1(x)$.

In fact, for an x of the form (*) we have

$$\Box h_1(x) = \Box((\sim \varphi h(a_1) \vee \varphi h(b_1)) \& \dots \& (\sim \varphi h(a_n) \vee \varphi h(b_n))) =$$

$$= (\varphi h(a_1) \mathbin{\underset{\varphi(\mathfrak{A})}{\supset}} \varphi h(b_1)) \& \dots \& (\varphi h(a_n) \mathbin{\underset{\varphi(\mathfrak{A})}{\supset}} \varphi h(b_n)) =$$

$$= \varphi h((a_1 \mathbin{\underset{G(\mathfrak{L})}{\supset}} b_1) \& \dots \& (a_n \mathbin{\underset{G(\mathfrak{L})}{\supset}} b_n)) = \varphi h(\Box x) = h_1(\Box x).$$

LEMMA 3.4. Let $\mathfrak{A} = G(\mathfrak{L})$. Then the subalgebra $\mathfrak{L}_1$ of the algebra $\mathfrak{L}$ generated by the set $\mathfrak{A}$ will be isomorphic to the algebra $\Delta(\mathfrak{A})$.

Proof. By taking h in Lemma 3.3 in the form of an identity isomorphism of $G(\mathfrak{L})$ into $\mathfrak{A}$, we find that the mapping (**) is a homomorphism of $\mathfrak{L}_1$ into $\Delta(\mathfrak{A})$. Let us show that h_1 is an isomorphism.

Suppose that x has the form (*), y has the form 2, and $x \nleq y$. Then there exist I, J, and κ such that $I \cup J = \{1,\dots,n\}$, $I \cap J = \varnothing$, $1 \le \kappa \le m$ and $\mathop{\&}_{i\in I} \sim a_i \& \mathop{\&}_{j\in J} b_j \nleq \sim c_\kappa \vee d_\kappa$. It hence follows that in the algebra $\mathfrak{A}$,

$$c_\kappa \& \mathop{\&}_{j\in J} b_j \nleq \bigvee_{i\in I} a_i \vee d_\kappa.$$

Then there exists ([7], I. 9.2) a simple filter $\Phi \in S_{\mathfrak{A}}$ such that $c_\kappa \& \mathop{\&}_{j\in J} b_j \in \Phi$, $\bigvee_{i\in I} a_i \vee d_\kappa \notin \Phi$. Hence

$$\Phi \in \varphi(c_k),\ \Phi \in \mathop{\&}_{j\in J} \varphi(b_j),\ \Phi \notin \varphi(d_k),\ \Phi \notin \bigvee_{i\in I} \varphi(a_i).$$

In $\varphi(\mathfrak{A})$ we therefore have

$$\varphi(c_k) \mathbin{\&} \mathop{\&}_{j\in J} \varphi(b_j) \nleq \bigvee_{i\in I} \varphi(a_i) \vee \varphi(d_k),$$

and hence in $\Delta(\mathfrak{A})$: $\mathop{\&}_{i\in I} \sim \varphi(a_i) \mathbin{\&} \mathop{\&}_{j\in J} \varphi(b_j) \nleq \sim \varphi(c_k) \vee \varphi(d_k)$, and consequently, $h_1(x) \nleq h_1(y)$. This completes the proof of the lemma.

Now let us construct a mapping of the lattice $\mathcal{M}$ into $\mathcal{L}$. To any modal logic M we shall assign a set

$$\rho(M) = \{A / A \in \Phi_J \text{ and } M \models T(A)\}. \tag{6}$$

From Lemma 3.1 we directly obtain

LEMMA 3.5. $M = M(\mathfrak{B}) \implies \rho(M) = LG(\mathfrak{B})$.

In fact, if $M = M(\mathfrak{B})$, then

$$A \in \rho(M) \iff M \models T(A) \iff \mathfrak{B} \models T(A) \iff G(\mathfrak{B}) \models A.$$

Since for any logic M the Lindenbaum algebra $\mathfrak{B}_M$ is a characteristic TBA, i.e., $M = M(\mathfrak{B}_M)$, it follows that $\rho(M)$ is a superintuitionistic logic. Thus ρ will be a mapping from $\mathcal{M}$ into $\mathcal{L}$.

LEMMA 3.6. For any logic $M \in \mathcal{M}$ and any PBA $\mathfrak{A}$ all the formulas belonging to $\rho(M)$ will be true in $\mathfrak{A}$ if and only if there exists a TBA $\mathfrak{B}$ such that $M(\mathfrak{B}) \supseteq M$ and $\mathfrak{A} \cong G(\mathfrak{B})$.

Proof. Let us recall [4] that any superintuitionistic logic L is complete with respect to the variety $\mathfrak{M}_L$ of all PBA in which all the formulas belonging to L are true. Now let us consider the class

$$K = \{\mathfrak{A} / \exists \mathfrak{B}\ (M(\mathfrak{B}) \supseteq M \text{ and } \mathfrak{A} \cong G(\mathfrak{B}))\}.$$

At first we shall show that $\rho(M)$ is complete with respect to the class K, i.e., $\rho(M) = \bigwedge_{\mathfrak{A}\in K} L\mathfrak{A}$.

If $\mathfrak{A} \in K$, then $L\mathfrak{A} \supseteq \rho(M)$ by virtue of Lemma 3.1 and the definition of $\rho(M)$. Next, if $\rho(M) \not\models A$, then $M \not\models T(A)$ and hence $\mathfrak{B} \not\models T(A)$ for a TBA $\mathfrak{B}$ such that $M(\mathfrak{B}) \supseteq M$. Then $G(\mathfrak{B}) \in K$ and $G(\mathfrak{B}) \not\models A$ by virtue of Lemma 3.1. Hence $\rho(M) = \bigwedge_{\mathfrak{A}\in K} L\mathfrak{A}$, and therefore the class K generates the variety $\mathfrak{M}_{\rho(M)}$.

Now let us note that

$$\prod_{i\in I} G(\mathfrak{B}_i) = G(\prod_{i\in I} \mathfrak{B}_i),$$

and therefore the Cartesian product of systems belonging to K will likewise belong to K. It follows from Lemma 3.2 that any subalgebra of the algebra $G(\mathfrak{B})$ can be represented in the form $G(\mathfrak{B}_1)$ for a suitable subalgebra $\mathfrak{B}_1$ of the algebra $\mathfrak{B}$. If $M(\mathfrak{B}) \supseteq M$, then $M(\mathfrak{B}_1) \supseteq M$ and therefore K is closed under the operation of taking subalgebras.

It follows from Lemma 3.3 and (5) that if $M(\mathfrak{B}) \supseteq M$ and $\mathfrak{A}$ is a homomorphism image of $G(\mathfrak{B})$, then $M(\Delta(\mathfrak{A})) \supseteq M$ and $\mathfrak{A} \cong G(\Delta(\mathfrak{A}))$, i.e., K is closed with respect to homomorphic images.

It hence follows from Birkhoff's theorem [3] that the family K forms a variety, and hence $K = \mathfrak{M}_{\rho(M)}$. This completes the proof of the lemma.

Let us construct yet another mapping $\sigma: \mathcal{L} \to \mathcal{M}$. Let $L \in \mathcal{L}$. By $\sigma(L)$ we shall denote a subset of Φ_M defined as follows:

$$\sigma(L) = \{\alpha \,/\, (\forall \mathfrak{A})(L\mathfrak{A} \supseteq L \Rightarrow \mathfrak{s}(\mathfrak{A}) \models \alpha)\}. \tag{7}$$

It is evident that $\sigma(L) \in \mathcal{M}$.

THEOREM 5. For any superintuitionistic logic L, $\sigma(L)$ will be a maximal modal logic M possessing the property

$$(\forall A \in \Phi_J)(L \models_J A \Leftrightarrow M \models T(A)). \tag{T}$$

Proof. First of all let us note that from Lemma 3.1 and formula (5) it follows that

$$\mathfrak{A} \models_J A \Leftrightarrow \mathfrak{s}(\mathfrak{A}) \models T(A) \tag{8}$$

for any $A \in \Phi_J$. Therefore

$$L \models_J A \Leftrightarrow (\forall \mathfrak{A})(L\mathfrak{A} \supseteq L \Rightarrow \mathfrak{A} \models_J A) \Leftrightarrow$$

$$\Leftrightarrow (\forall \mathfrak{A})(L\mathfrak{A} \supseteq L \Rightarrow \mathfrak{s}(\mathfrak{A}) \models T(A)) \Leftrightarrow \sigma(L) \models T(A),$$

i.e., (T) holds for $M = \sigma(L)$.

Now suppose that the logic M has the property (T). Then $L = \rho(M)$. Let us assume that $L\mathfrak{A} \supseteq L$. By virtue of Lemma 3.6 there exists a TBA $\mathfrak{B}$ such that $M(\mathfrak{B}) \supseteq M$ and $\mathfrak{A} \cong G(\mathfrak{B})$. From Lemma 3.4 it follows directly that $M(\mathfrak{s}(\mathfrak{A})) \supseteq M(\mathfrak{B}) \supseteq M$. By virtue of the definition of $\sigma(L)$ this signifies that $M \subseteq \sigma(L)$, which completes the proof.

COROLLARY. If $L = L\mathfrak{A}$, then $\sigma(L) = M(\mathfrak{s}(\mathfrak{A}))$.

Proof. Let $L = L\mathfrak{A}$. From the definition of $\sigma(L)$ it follows directly that $\sigma(L) \subseteq M(\mathfrak{s}(\mathfrak{A}))$. On the other hand it follows from (8) that

$$L \models_J A \Leftrightarrow \mathfrak{A} \models_J A \Leftrightarrow \mathfrak{s}(\mathfrak{A}) \models T(A) \Leftrightarrow M(\mathfrak{s}(\mathfrak{A})) \models T(A)$$

and by virtue of the theorem we have $M(\mathfrak{s}(\mathfrak{A})) \subseteq \sigma(L)$.

Thus we have a mapping $\rho : \mathcal{M} \to \mathcal{L}$ and two mappings τ and σ from $\mathcal{L}$ into $\mathcal{M}$, with

$$\text{(P1)} \quad M_1 \subseteq M_2 \Rightarrow \rho(M_1) \subseteq \rho(M_2),$$

$$\text{(P2)} \quad \tau(L_1 \wedge L_2) = \tau(L_1) \wedge \tau(L_2),$$
$$\tau(\bigvee_{i \in I} L_i) = \bigvee_{i \in I} \tau(L_i),$$

$$\text{(P3)} \quad L_1 \subseteq L_2 \Rightarrow \sigma(L_1) \subseteq \sigma(L_2).$$

From the definition of ρ and τ it follows directly that

$$\text{(P4)} \quad \tau\rho(M) \subseteq M,$$

since all the axioms of $\tau\rho(M)$ belong to M. It follows from Lemma 2.2 that for any $L \in \mathcal{L}$ the logic $M = \tau(L)$ has the property (T); hence

$$\text{(P5)} \quad \rho\tau(L) = L.$$

It follows from Theorem 5 that

$$\text{(P6)} \quad \rho\sigma(L) = L,$$

$$\text{(P7)} \quad M \subseteq \sigma\rho(M).$$

From (P6) it is evident that σ is one-to-one and that $\rho(\mathcal{M}) = \mathcal{L}$.

Now let us find some more relations characterizing ρ and σ.

THEOREM 6. a) ρ is a homomorphism of $\mathcal{M}$ into $\mathcal{L}$ that preserves infinite unions and intersections,

$$\text{b)}\ \sigma(\bigwedge_{i\in I} L_i) = \bigwedge_{i\in I} \sigma(L_i).$$

Proof. a) It follows directly from (P1) that $\rho(\bigwedge_{i\in I} M_i) \subseteq \bigwedge_{i\in I} \rho(M_i)$. On the other hand, by successively using (P5), (P2), and (P1), P(4), and (P1), we obtain

$$\bigwedge_{i\in I} \rho(M_i) = \rho\tau(\bigwedge_{i\in I} \rho(M_i)) \subseteq \rho(\bigwedge_{i\in I} \tau\rho(M_i)) \subseteq \rho(\bigwedge_{i\in I} M_i).$$

Hence $\rho(\bigwedge_{i\in I} M_i) = \bigwedge_{i\in I} \rho(M_i)$.

Next, by successively using (P7) and (P1), (P3), (P6), and (P1), we obtain

$$\rho(\bigvee_{i\in I} M_i) \subseteq \rho(\bigvee_{i\in I} \sigma\rho(M_i)) \subseteq \rho\sigma(\bigvee_{i\in I} \rho(M_i)) = \bigvee_{i\in I} \rho(M_i) \subseteq \rho(\bigvee_{i\in I} M_i),$$

$$\rho(\bigvee_{i\in I} M_i) = \bigvee_{i\in I} \rho(M_i);$$

b) by using (P3), (P7), (P1), and (P6), we obtain

$$\sigma(\bigwedge_{i\in I} L_i) \subseteq \bigwedge_{i\in I} \sigma(L_i) \subseteq \sigma\rho(\bigwedge_{i\in I} \sigma(L_i)) \subseteq \sigma(\bigwedge_{i\in I} \rho\sigma(L_i)) = \sigma(\bigwedge_{i\in I} L_i).$$

Hence

$$\sigma(\bigwedge_{i\in I} L_i) = \bigwedge_{i\in I} \sigma(L_i).$$

COROLLARY. a) If M is tabular, then $\rho(M)$ is also tabular;

b) if M is finitely approximable, then $\rho(M)$ is also finitely approximable;

c) if $\rho(M)$ is tabular, then M is finitely approximable;

d) if L is tabular, then $\sigma(L)$ is tabular and $\tau(L)$ is finitely approximable;

e) if L is finitely approximable, then $\sigma(L)$ is also finitely approximable.

Proof. a) follows from Lemma 3.5; b) follows from a) and Theorem 6a; c) is the assertion of Lemma 4.9; d) follows from the corollary of Theorem 5 and from c), since $\rho\tau(L) = L$; e) follows from d) and Theorem 6b.

In concluding this section, let us recall that Grzegorczyk [17] has constructed a logic $G \in \mathcal{M}$ that differs from S4 and is such that $\rho(G) = \rho(S4) = \mathcal{J}$. This logic can be obtained by adjoining to S4 the axiom

$$(\gamma) \quad (\Box(\Box(\alpha \to \Box\beta) \to \Box\beta) \,\&\, \Box(\Box(\sim\alpha \to \Box\beta) \to \Box\beta)) \to \Box\beta.$$

It follows from (P7) that $G \subseteq \sigma(\mathcal{J})$. In [17] it is shown that $S5 \not\supseteq G$. For any consistent $L_1, L_2 \in \mathcal{L}$ we therefore obtain $\sigma(L_1) \nsubseteq \tau(L_2)$, since $S5 = \tau(K)$, where K is a classical logic. It hence follows that $\sigma(L)$ differs from $\tau(L)$ for any consistent L. Moreover, we have the following

ASSERTION. For any consistent logic L the set $\rho^{-1}(L)$ contains an infinitely descending chain.

For the proof let us recall [16] that consistent logics that contain S5 form a strictly descending chain $M_1 \supset M_2 \supset M_3 \supset \ldots$; each of these logics can be axiomatized by one formula δ_n, i.e., $M_n = [\delta_n]$. Now let us consider for any $L \in \mathcal{L}$ the chain $M_1' \supseteq M_2' \supseteq M_3' \supseteq \ldots$, where $M_n' = \sigma(L) \wedge M_n$. We have $\gamma \in G \subseteq \sigma(\mathcal{J}) \subseteq$

$\sigma(L)$; therefore $\Box\delta_n \vee' \Box\gamma \in M_n'$ (see Theorem 1). By using characteristic TBA for M_{n+1} [16], it is easy to show that $\gamma \notin M_{n+1}$ [17] and $\Box\delta_n \vee' \Box\gamma \notin M_{n+1}$; hence $M_n' \not\subseteq M_{n+1}'$. Next, $K = \rho(S5) \subseteq \rho(Mn)$; therefore $\rho(M_n) = K$ and $\rho(M_n') = \rho\sigma(L) \wedge \rho(Mn) = L \wedge K = L$. This completes the proof of the assertion.

SECTION 4

LEMMA 4.1. a) If $\{\mathfrak{L}_i\}_{i\in J}$ is a family of TBA, then

$$M(\prod_{i\in J}\mathfrak{L}_i) = \bigwedge_{i\in J} M(\mathfrak{L}_i);$$

b) if $\mathfrak{L}_1$ is a subalgebra of the algebra $\mathfrak{L}_2$, then

$$M(\mathfrak{L}_2) \subseteq M(\mathfrak{L}_1);$$

c) if $\mathfrak{L}_1$ is a homomorphic image of $\mathfrak{L}_2$, then

$$M(\mathfrak{L}_2) \subseteq M(\mathfrak{L}_1).$$

All this follows directly from the definitions.

LEMMA 4.2. a) Let $M = M(\mathfrak{L})$; then there exists for any $\alpha \notin M$ a subdirectly indecomposable TBA $\mathfrak{L}_1$ such that $\alpha \notin M(\mathfrak{L}_1)$, $M(\mathfrak{L}_1) \supseteq M$, and $\bar{\bar{\mathfrak{L}}}_1 \leqslant \bar{\bar{\mathfrak{L}}}$;

b) if $\Gamma \subseteq \mathfrak{F}_M$ and $\alpha \notin [\Gamma]$, then there exists a subdirectly indecomposable TBA $\mathfrak{L}$ such that $M(\mathfrak{L}) \supseteq [\Gamma]$ and $\mathfrak{L} \not\models \alpha$.

Proof. a) From Birkhoff's representation theorem [3, 9] it follows that $\mathfrak{L} \cong \tilde{\prod}_{i\in J}\mathfrak{L}_i$, where the $\mathfrak{L}_i$ are subdirectly indecomposable [3], and $\tilde{\prod}_{i\in J}\mathfrak{L}_i$ is a subdirect product. $M(\tilde{\prod}_{i\in J}\mathfrak{L}_i) \supseteq M(\prod_{i\in J}\mathfrak{L}_i)$, since $\tilde{\prod}_{i\in J}\mathfrak{L}_i$ is a subalgebra of the algebra $\prod_{i\in J}\mathfrak{L}_i$. $\alpha \notin M(\mathfrak{L})$, and hence $\alpha \notin M(\prod_{i\in J}\mathfrak{L}_i)$. It then follows from Lemma 4.1 that there exists a $\mathfrak{L}_i$ such that $\alpha \notin M(\mathfrak{L}_i)$; $M(\mathfrak{L}_i) \supseteq M$, since $\mathfrak{L}_i$ is a homomorphic image of $\mathfrak{L}$ (by the definition of a subdirect product). It is evident that $\bar{\bar{\mathfrak{L}}}_i \leqslant \bar{\bar{\mathfrak{L}}}$. As $\mathfrak{L}_1$ we then take $\mathfrak{L}_i$. This completes the proof of a).

b) follows from a) if we take $\mathfrak{L}$ in the form of Lindenbaum's algebra for $[\Gamma]$. This completes the proof of the lemma.

LEMMA 4.3. Let $M = M(\mathfrak{L})$; then $M = M(\prod_{i\in J}\mathfrak{L}_i) = \bigwedge_{i\in J} M(\mathfrak{L}_i)$, where $\{\mathfrak{L}_i\}_{i\in J}$ is the family of all subdirectly indecomposable pairwise-nonisomorphic TBA such that $M(\mathfrak{L}_i) \supseteq M$ and $\bar{\bar{\mathfrak{L}}}_i \leqslant \bar{\bar{\mathfrak{L}}}$.

Proof. It follows from Lemma 4.1 that $M(\prod_{i\in J}\mathfrak{L}_i) \supseteq M$. Let us prove the inverse inclusion. Let $\alpha \notin M$. By virtue of Lemma 4.2 there exists a $\mathfrak{L}_1$ that is subdirectly indecomposable and such that $\alpha \notin M(\mathfrak{L}_1)$, $M(\mathfrak{L}_1) \supseteq M(\mathfrak{L})$, and $\bar{\bar{\mathfrak{L}}}_1 \leqslant \bar{\bar{\mathfrak{L}}}$. Hence $\mathfrak{L}_1$ is isomorphic to an algebra belonging to $\{\mathfrak{L}_i\}_{i\in J}$. It then follows from Lemma 4.1 that $\alpha \notin M(\prod_{i\in J}\mathfrak{L}_i)$. This completes the proof of the lemma.

For any $n \geqslant 1$ let us define the following formula:

$$\alpha(n) \rightleftharpoons \bigvee_{1\leqslant i<j\leqslant n+1} \Box(p_i \leftrightarrow p_j).$$

LEMMA 4.4. Let $\mathfrak{L}$ be a completely connected (CC) TBA; then

$$\alpha(n) \in M(\mathfrak{L}) \iff \bar{\bar{\mathfrak{L}}} \leqslant n.$$

Proof. If $\bar{\bar{\mathfrak{L}}} \leqslant n$, then evidently $\alpha(n) \in M(\mathfrak{L})$. If $\bar{\bar{\mathfrak{L}}} > n$, it is possible to interpret all the variables $\alpha(n)$ as distinct elements of $\mathfrak{L}$. Then every disjunctive term of the formula $\alpha(n)$ will differ from 1, and by virtue of the completely connected $\mathfrak{L}$ the $\alpha(n)$ will not be equal to 1 in this interpretation, i.e., $\alpha(n) \notin M(\mathfrak{L})$. This completes the proof of the lemma.

Remark. Any subdirectly indecomposable TBA is completely connected [9].

LEMMA 4.5. A logic M will be tabular if and only if there exists an n such that $\alpha(n) \in M$.

Proof. Let M be tabular; then there evidently exists an n such that $\alpha(n) \in M$. Now let $\alpha(n) \in M$. It follows from Lemma 4.3 that $M = M(\prod_{i \in J} \mathcal{B}_i)$, where the $\mathcal{B}_i$ are subdirectly indecomposable and pairwise nonisomorphic. Since $\alpha(n) \in M$, it follows (by virtue of Lemma 4.1) that $\alpha(n) \in M(\mathcal{B}_i)$ for any $i \in J$. From Lemma 4.4 it follows that the $\mathcal{B}_i$ are finite and $\overline{\overline{\mathcal{B}}}_i \leq n$. Since there are finitely many such pairwise-non-isomorphic $\mathcal{B}_i$, it follows that $M = M(\mathcal{B}_1 \times \ldots \times \mathcal{B}_K)$, where $\overline{\overline{\mathcal{B}}}_i \leq n$ $(i \in \{1, \ldots, K\})$. This completes the proof of the lemma.

THEOREM 7. Let $\mathcal{B}$ be a finite TBA. Then:

1) if $M \supseteq M(\mathcal{B})$, then M will be a tabular logic;

2) there exist only finitely many extensions of $M(\mathcal{B})$.

Proof. 1) follows directly from Lemma 4.5 and 2) follows directly from Lemmas 4.3 and 4.4.

THEOREM 8.* Let $\mathcal{B}$ be a finite TBA; then $M(\mathcal{B})$ will be finitely axiomatizable.

Proof. By virtue of Lemma 4.5 there exists an $n: \alpha(n) \in M(\mathcal{B})$. Let $M(\mathcal{B}) = \{\alpha_n\}_{n \in N}$. Let us define a set of formulas $\{\beta_n\}_{n \in N}: \beta_n \leftrightharpoons \alpha(n) \,\&\, \alpha_1 \,\&\, \alpha_2 \ldots \&\, \alpha_n$. It is evident that $[\{\beta_n\}_{n \in N}] = M(\mathcal{B})$. If $M(\mathcal{B})$ is not finitely axiomatizable, then there exists for any β_n a β_{n+K} such that $\beta_n \not\vdash \beta_{n+K}$. Let us inductively construct $\{\gamma_n\}_{n \in N}: \gamma_1 = \beta_1$; if γ_{n-1} has been constructed, then $\gamma_{n-1} = \beta_K$ and there exists a β_{K+m} such that $\beta_K \not\vdash \beta_{K+m}$; then we take $\gamma_n = \beta_{K+m}$.

By virtue of Lemma 4.2 there exists for any γ_n a subdirectly indecomposable $\mathcal{B}_n$ such that $\gamma_n \in M(\mathcal{B}_n)$, $\gamma_{n+1} \notin M(\mathcal{B}_n)$. If $n > m$, then $\mathcal{B}_n$ is not isomorphic to $\mathcal{B}_m$, since $\gamma_m \in M(\mathcal{B}_m)$ and $\gamma_n \notin M(\mathcal{B}_m)$. Hence there exists an infinite set of nonisomorphic subdirectly indecomposable TBA such that $\alpha(n)$ is true on them. But according to Lemma 4.4 there exist only finitely many such TBA. This is a contradiction, and it completes the proof of the theorem.

Now let us strengthen this theorem by explicitly writing out the axiomatization of the logic of finite TBA by using Hosoi's method [12].

THEOREM 9. Let $\mathcal{B}$ be a finite TBA; then it is possible to effectively construct a formula $\delta(\mathcal{B})$ such that

$$[\delta(\mathcal{B})] = M(\mathcal{B}).$$

Proof. Let $|\mathcal{B}| = \{\sigma_1, \ldots, \sigma_n\}$. By $\Phi_0 \subseteq \Phi_M$ we shall denote a set of formulas all whose variables are contained in the set $P_0 = \{p_1, \ldots, p_n\}$. Let us write $F = |\mathcal{B}|^{P_0}$; it is evident that F is finite. Let us fix an $f \in F$ and denote by $\bar{f}$ an interpretation of formulas from Φ_0 into $\mathcal{B}$ that continues f. $\bar{f}: \Phi_0 \to |\mathcal{B}|$. It is evident that $\bar{f}(\Phi_0)$ is a subalgebra of the algebra $\mathcal{B}$ that coincides with the subalgebra of the algebra $\mathcal{B}$ generated by the set $f(P_0)$.

Let us introduce a function that maps $\bar{f}(\Phi_0)$ into Φ_0 as follows. For any $x \in \bar{f}(\Phi_0)$, $x \neq 1$ let us specify at first a formula $\mathcal{B}_{x,f}(p_1 \ldots p_n) \in \Phi_0$ such that $\bar{f}\mathcal{B}_{x,f}(p_1, \ldots p_n) = x$. For any $x \in \bar{f}(\Phi_0)$ let us now define a $g_f: \bar{f}(\Phi_0) \to \Phi_0$:

$$g_f(x) = \begin{cases} \mathcal{B}_{x,f}(p_1, \ldots p_n), & \text{if } x \neq 1, \\ (p_1 \to p_1), & \text{if } x = 1. \end{cases}$$

*It was shown by D. M. Smirnov that this theorem follows also from the theorem on finite bases of a finite algebra that lies in a variety of algebras with distributive congruence lattices [18].

We have

$$\bar{f}(g_f(x)) = x\,. \tag{*}$$

Let us define the following formulas:

$$\alpha_{f(=)} \rightleftharpoons \mathop{\&}_{p_j \in P_0} (g_f(f(p_j)) \longleftrightarrow p_j)$$

$$\alpha_{f(\sim)} \rightleftharpoons \mathop{\&}_{x \in \bar{f}(\Phi_0)} (\sim g_f(x) \longleftrightarrow g_f(\sim x)),$$

$$\alpha_{f(\Box)} \rightleftharpoons \mathop{\&}_{x \in \bar{f}(\Phi_0)} (g_f(\Box x) \longleftrightarrow \Box g_f(x)),$$

$$\alpha_{f(\&)} \rightleftharpoons \mathop{\&}_{x,y \in \bar{f}(\Phi_0)} (g_f(x \& y) \longleftrightarrow g_f(x) \& g_f(y)),$$

$$\gamma_f \rightleftharpoons \Box(\alpha_{f(=)} \& \alpha_{f(\sim)} \& \alpha_{f(\Box)} \& \alpha_{f(\&)}),$$

$$\delta(\mathcal{B}) \rightleftharpoons (\bigvee_{f \in F} \gamma_f) \& \alpha(n).$$

Let us prove that $[\delta(\mathcal{B})] \subseteq M(\mathcal{B})$, i.e., $\delta(\mathcal{B}) \in M(\mathcal{B})$. Let φ be an interpretation on $\mathcal{B}$; then $\varphi/\Phi_0 = \bar{f}$, where $f \in F$. In this case $\bar{f}(\gamma_f) = 1$ by virtue of (*), and hence $\varphi(\gamma_f) = 1$. Thus, $\varphi(\bigvee_{f \in F} \gamma_f) = 1$ for any interpretation φ on $\mathcal{B}$, i.e., $\bigvee_{f \in F} \gamma_f \in M(\mathcal{B})$. It is evident that $\alpha(n) \in M(\mathcal{B})$; therefore $\delta(\mathcal{B}) \in M(\mathcal{B})$. Let us prove that $M(\mathcal{B}) \subseteq [\delta(\mathcal{B})]$, i.e., for any formula $\alpha \in M(\mathcal{B})$ we have $\delta(\mathcal{B}) \models \alpha$. At first let us consider the case (a). All the variables belonging to α occur in P_0.

<u>LEMMA.</u> For any $f \in F$ and any formula $\beta(p_1 \ldots p_n)$ belonging to Φ_0 we have

$$\gamma_f \vdash (\beta(p_1 \ldots p_n) \longleftrightarrow g_f(\beta(f(p_1) \ldots f(p_n)))).$$

<u>Proof.</u> At first we shall assume that $\beta(p_1 \ldots p_n)$ does not contain logical connectives other than $\sim, \&, \Box$. For such $\beta(p_1 \ldots p_n)$ it is easy to prove the lemma by induction on the length of β by using the substitution theorem for S4, the relation (*), and the form of the formula γ_f. It remains to note that any formula is equivalent in S4 to a formula that contains only the connectives $\&, \sim, \Box$. This completes the proof of the lemma.

In particular, we obtain $\gamma_f \vdash \alpha(p_1 \ldots p_n) \longleftrightarrow g_f(\alpha(f(p_1) \ldots f(p_n)))$. But $\alpha(p_1 \ldots p_n) \in M(\mathcal{B})$, and hence $\alpha(f(p_1) \ldots f(p_n)) = 1$; but in this case, $g_f(\alpha(f(p_1) \ldots f(p_n))) = p_1 \to p_1$. We obtain: $\gamma_f \vdash \alpha(p_1 \ldots p_n) \longleftrightarrow (p_1 \to p_1)$, i.e., $\gamma_f \vdash \alpha(p_1 \ldots p_n)$. By virtue of Lemma 1.1 we obtain $\vdash \Box\gamma_f \to \alpha(p_1 \ldots p_n)$, i.e., $\vdash \gamma_f \to \alpha(p_1 \ldots p_n)$ (since $\vdash \Box\gamma_f \longleftrightarrow \gamma_f$). Hence we have: $\vdash \bigvee_{f \in F} \gamma_f \to \alpha(p_1 \ldots p_n)$, and hence $\alpha(p_1 \ldots p_n) \in [\delta(\mathcal{B})]$. Thus we have proved the assertion for the case (a).

Now suppose that α depends on the variables $q_1 \ldots q_k$: $\alpha = \alpha(q_1 \ldots q_k)$. We shall define the following set of mappings: $G = \{g / g : \{q_1 \ldots q_k\} \to \{p_1 \ldots p_n\}\}$, it is finite. For any $g \in G$ let us define $\mathcal{B}_g \rightleftharpoons \alpha(g(q_1) \ldots g(q_k))$. $\alpha(q_1 \ldots q_k) \models \mathcal{B}_g$, and hence $\alpha \models \mathop{\&}_{g \in G} \mathcal{B}_g$. Let us write $\mathcal{B}(p_1 \ldots p_n) = \mathop{\&}_{g \in G} \mathcal{B}_g$. $\mathcal{B}(p_1 \ldots p_n) \in M(\mathcal{B})$; hence by using the proof for the case (a), we obtain $\delta(\mathcal{B}) \models \mathcal{B}(p_1, \ldots, p_n)$. Let us prove that $\alpha(n), \mathcal{B}(p_1 \ldots p_n) \models \alpha(q_1 \ldots q_k)$. Suppose that this is not the case; then there exists by virtue of Lemma 4.2b a subdirectly indecomposable TBA $\mathcal{B}_0$ such that $M(\mathcal{B}_0) \supseteq [\alpha(n), \mathcal{B}(p_1, \ldots, p_n)], \alpha(q_1 \ldots q_k) \notin M(\mathcal{B}_0)$. Hence there exists an interpretation φ on $\mathcal{B}_0$ such that $\alpha(\varphi(q_1) \ldots \varphi(q_k)) \neq 1$. By virtue of Lemma 4.4 we have $\bar{\bar{\mathcal{B}}}_0 \leq n$, i.e., $|\mathcal{B}_0| = \{v_1, \ldots, v_\ell\}$, $\ell \leq n$. Let us define $g_1 \in G$ as follows:

$$g_1(q_j) = p_i \quad , \quad \text{if} \quad \varphi(q_j) = v_i\,.$$

Let us take an interpretation φ_1 of $\mathcal{B}_{g_1}$ into $\mathcal{L}_0$ defined as follows:

$$\varphi_1(p_i) = \begin{cases} v_i, & \text{if } p_i \in g_1\{g_1,\dots,g_K\}, \\ v_1 & \text{otherwise.} \end{cases}$$

Then $\varphi_1\mathcal{B}_{g_1} = \varphi(\alpha(q_1,\dots q_K)) \neq 1$, i.e., $\mathcal{B}_{g_1} \notin M(\mathcal{L}_0)$ is a contradiction to $\mathcal{B}(p_1,\dots p_n) \in M(\mathcal{L}_0)$. Hence $\alpha(n)$, $\mathcal{B}(p_1,\dots p_n) \vDash \alpha(q_1,\dots q_K)$, but in this case $\delta(\mathcal{L}) \vDash \alpha(q_1,\dots q_K)$. This completes the proof of the theorem.

LEMMA 4.6. For any nontabular logic M there exists a pretabular logic M_0 such that $M \subseteq M_0$.

Proof. Let us use Zorn's lemma. Let $\{M_i\}_{i\in J}$ be a chain in which all M_i are nontabular logics, $M_i \supseteq M$. Then $\bigvee_{i\in J} M_i = \bigcup_{i\in J} M_i$ (since $\{M_i\}_{i\in J}$ is a chain). $\bigvee_{i\in J} M_i$ is not tabular, since otherwise there exists by virtue of Lemma 4.5 an n such that $\alpha(n) \in \bigvee_{i\in J} M_i$; but in this case there exists an i such that $\alpha(n) \in M_i$, which contradicts Lemma 4.5.

LEMMA 4.7. Let M be a pretabular logic, $M = M(\mathcal{L})$, where $\mathcal{L}$ is a subdirectly indecomposable TBA that has an infinite set $G(\mathcal{L})$. Then $\rho(M)$ will be pretabular.

Proof. $\rho(M) = LG(\mathcal{L})$ (Lemma 3.5). Let us prove that $LG(\mathcal{L})$ is pretabular.

1. $LG(\mathcal{L})$ is not tabular. Indeed, $\mathcal{L}$ is subdirectly indecomposable; hence [9] it is completely connected; but in this case $G(\mathcal{L})$ will be completely connected and have an infinite number of elements. If $LG(\mathcal{L})$ would be tabular, there would exist [12] an m such that

$$A(m) \rightleftharpoons \bigvee_{1\le i<j\le m+1} (p_i \equiv p_j) \in LG(\mathcal{L}).$$

Hence $A(m) = 1$ on $G(\mathcal{L})$ for any interpretation. But $G(\mathcal{L})$ is infinite, and hence all the variables $A(m)$ can be interpreted as distinct elements of $G(\mathcal{L})$; but since $G(\mathcal{L})$ is completely connected, it follows that $A(m) \neq 1$, which is a contradiction. Hence $LG(\mathcal{L})$ is not tabular.

2. $LG(\mathcal{L})$ is pretabular. Kuznetsov has proved that any nontabular superintuitionistic logic is contained in a pretabular superintuitionistic logic; therefore $LG(\mathcal{L}) \subseteq L$, where L is pretabular. It follows from Theorem 6 that ρ is a homomorphism "onto," and hence there exists an $M_0 \in \mathcal{M}$: $\rho(M_0) = L$ and $\rho(M \vee M_0) = \rho(M) \vee \rho(M_0) = LG(\mathcal{L}) \vee L = L$. Hence $M \vee M_0$ is not tabular [if M is tabular, then $\rho(M)$ will be likewise]. We have: $M \vee M_0$ is not tabular and $M \vee M_0 \supseteq M$; hence $M \vee M_0 = M$; but in this case $\rho(M) = \rho(M \vee M_0) = L$, where L is pretabular. This completes the proof of the lemma.

LEMMA 4.8. If M is a pretabular logic, $M = M(\mathcal{L})$, where $\mathcal{L}$ is subdirectly indecomposable and $G(\mathcal{L})$ is infinite, then there exists an infinite set of tabular logics containing M.

Proof. It follows from Lemma 4.8 that $\rho(M)$ is a pretabular logic. Hence according to [6] there exists a countable set $\{L_n\}_{n\in N}$ of tabular logics containing $\rho(M)$. Then the logics $\sigma(L_n)$ will be distinct by virtue of Theorem 5. $\sigma(L_n) \supseteq \sigma(\rho(M))$ by virtue of (P3), §3, and $\sigma(\rho(M)) \supseteq M$ by virtue of (P7), §3; hence $\sigma(L_n) \supseteq \sigma(\rho(M)) \supseteq M$, and by virtue of Theorem 5 we have $\sigma(L_n) \neq \sigma(\rho(M))$; hence all $\sigma(L_n)$ are tabular and distinct, and they contain M. This completes the proof of the lemma.

LEMMA 4.9. If $M \in \mathcal{M}$ and $\rho(M)$ is tabular, then M will be finitely approximable.

Proof. Let $\alpha \notin M$, $\alpha = \alpha(p_1 \dots p_n)$; it then follows from Lemma 4.2 that there exists a subdirectly indecomposable TBA $\mathcal{L}$ such that $M(\mathcal{L}) \supseteq M$ and $\alpha \notin M(\mathcal{L})$. By virtue of Theorem 6, ρ is a homorphism, and hence $\rho(M) \subseteq \rho(M(\mathcal{L}))$. $\rho(M)$ is tabular; hence [12] there exists an $A(n) \rightleftharpoons \bigvee_{1\le i<j\le n+1} (p_i \equiv p_j)$ such that $A(n) \in \rho(M)$. In this case $A(n) \in \rho(M(\mathcal{L}))$ and from Lemma 3.5 it follows that $\rho(M(\mathcal{L})) = LG(\mathcal{L})$. $\mathcal{L}$ is completely connected, and hence $G(\mathcal{L})$ is also completely connected. Since $A(n) \in LG(\mathcal{L})$ and $G(\mathcal{L})$ is completely connected, it follows (as in Lemma 4.7) that $\overline{\overline{G(\mathcal{L})}} \le n$. $\alpha \notin M(\mathcal{L})$, and hence there exists a $\varphi : \{p_1,\dots,p_n\} \to \mathcal{L}$ such that $\alpha(\varphi(p_1)\dots\varphi(p_n)) \neq 1$. $\{\varphi(p_1),\dots,\varphi(p_n)\} \cup G(\mathcal{L}) \subseteq \mathcal{L}$, and by taking the closure of this subset under Boolean operations, we obtain a finite Boolean algebra $\mathcal{L}_0$ ([7], P. 2.3). Let us note

that $\mathfrak{B}_0$ is closed also under $\Box$, and hence it will be a subalgebra of the TBA $\mathfrak{B}$ generated by the set $\{\varphi(p_1),\dots,\varphi(p_n)\} \cup G(\mathfrak{B})$. Hence $M(\mathfrak{B}_0) \supseteq M(\mathfrak{B}) \supseteq M$, $\alpha(p_1 \dots p_n) \notin M(\mathfrak{B}_0)$, since $\alpha(\varphi(p_1)\dots\varphi(p_n)) \neq 1$. This completes the proof of the lemma.

COROLLARY. If $M = M(\mathfrak{B})$ and $G(\mathfrak{B})$ is finite, then M will be finitely approximable.

By virtue of Lemma 3.5 we have $\rho(M) = LG(\mathfrak{B})$, and hence $\rho(M)$ is tabular; in this case it follows from Lemma 4.9 that M is finitely approximable.

THEOREM 10. Pretabular modal logics are finitely approximable.

Proof. Let M be pretabular. It follows from Lemma 4.3 that $M = \bigwedge_{i \in J} M(\mathfrak{B}_i)$, where the $\mathfrak{B}_i$ are subdirectly indecomposable and $M(\mathfrak{B}_i) \supseteq M$ for any $i \in J$. If there are no such $\mathfrak{B}_i$ so that $G(\mathfrak{B}_i)$ is infinite, it follows from the corollary of Lemma 4.9 that all $M(\mathfrak{B}_i)$ are finitely approximable. In this case M will also be finitely approximable. Let us assume that there exists a $\mathfrak{B}_i$ such that $G(\mathfrak{B}_i)$ is infinite. Then $M(\mathfrak{B}_i) \supseteq M$ and from Lemmas 4.4 and 4.5 it follows that $M(\mathfrak{B}_i)$ is not tabular, and hence $M(\mathfrak{B}_i) = M$, i.e., $M(\mathfrak{B}_i)$ is pretabular. It then follows from Lemma 4.8 that there exist infinitely many tabular logics containing it, i.e., $\{Mn\}_{n \in N}$. $\bigwedge_{n \in N} Mn \supseteq M(\mathfrak{B}_i)$ and $\bigwedge_{n \in N} Mn$ is not a tabular logic according to Lemma 4.5 and Lemma 4.3. Hence $M(\mathfrak{B}_i) = \bigwedge_{n \in N} Mn$, i.e., $M(\mathfrak{B}_i)$ is finitely approximable. But $M(\mathfrak{B}_i) = M$. This completes the proof of the theorem.

We shall say that a logic is an immediate predecessor of a logic M if it is maximal in the set $\{M_0 / M_0 \in \mathcal{M}, M_0 \subseteq M, M_0 \neq M\}$.

LEMMA 4.10. If M is a finitely axiomatizable logic, $M_1 \subseteq M$, $M_1 \neq M$, then M_1 will be contained in a logic which is an immediate predecessor of M.

The proof is based on the use of Zorn's lemma.

COROLLARY. If M is a tabular logic, $M_1 \neq M$, $M_1 \subseteq M$, then M_1 will be contained in a logic which is an immediate predecessor of M.

It follows from Theorem 8 that M is finitely axiomatizable.

Now let us prove the following theorem:

THEOREM 11. Modal logics that are immediate predecessors of a tabular modal logic are tabular logics.

Proof. Let M be a tabular logic other than $\mathfrak{F}_M$, and let M_0 be an immediate predecessor of this logic. We shall assume that M_0 is not tabular; it then follows from Lemma 4.7 that there exists a pretabular logic M_1 such that $M_0 \subseteq M_1$. Hence $M_0 \subseteq M \wedge M_1 \subseteq M$. It follows from Theorem 10 that M_1 is finitely approximable; hence $M_1 = \bigwedge_{i \in J} M(\mathfrak{B}_i)$, where the $\mathfrak{B}_i$ are finite TBA. By virtue of Lemma 4.5 there exists an n such that $\alpha(n) \in M$; since M_1 is not tabular, it follows from Lemma 4.5 that $\alpha(n) \notin M_1$. Hence there exists a $\mathfrak{B}_i$ such that $\alpha(n) \notin M(\mathfrak{B}_i)$. In this case $M_0 \subseteq M(\mathfrak{B}_i) \wedge M \subseteq M$, but $M(\mathfrak{B}_i) \wedge M \neq M$, since $\alpha(n) \in M$ and $\alpha(n) \notin M(\mathfrak{B}_i) \wedge M$. M_0 is an immediate predecessor of the logic M; hence $M(\mathfrak{B}_i) \wedge M = M_0$, i.e., M_0 is tabular. This completes the proof of the theorem.

COROLLARY. There exists an algorithm which recognizes according to any finite TBA $\mathfrak{B}$ and a finite system of formulas $\alpha_1,\dots,\alpha_n$ whether or not the equation $[\alpha_1,\dots,\alpha_n] = M(\mathfrak{B})$ holds.

Let $\mathfrak{B}$ be a finite TBA and let $\alpha_1,\dots,\alpha_n$ be a system of formulas, $\alpha \rightleftharpoons \alpha_1 \& \dots \& \alpha_n$. Let us check whether the formula $\mathfrak{B}$ is true on α. If $\mathfrak{B} \not\models \alpha$, then $[\alpha] \neq M(\mathfrak{B})$. Otherwise we shall proceed as follows. Let $\mathfrak{B}_1, \mathfrak{B}_2, \dots$ be finite TBA. By virtue of Theorem 9 we can effectively write down their axiomatizations $\delta(\mathfrak{B}_1), \delta(\mathfrak{B}_2), \dots$ and the axiomatization of $\mathfrak{B}$, namely $\delta(\mathfrak{B})$. Let us apply an algorithm which derives from α all possible corollaries and an algorithm which checks: a) $\mathfrak{B} \models \delta(\mathfrak{B}_i)$, b) $\mathfrak{B}_i \models \delta(\mathfrak{B})$, and c) $\mathfrak{B}_i \models \alpha$. It follows from Theorem 11 that either we find at some step of the first algorithm that $\delta(\mathfrak{B})$ is deducible from α, or we find at some step of the second algorithm that $\mathfrak{B} \models \delta(\mathfrak{B}_i)$, $\mathfrak{B}_i \not\models \delta(\mathfrak{B})$, $\mathfrak{B}_i \models \alpha$. In the first case we have $[\alpha] = M(\mathfrak{B})$, and in the second case $[\alpha] \neq M(\mathfrak{B})$. This completes the proof of the corollary.

The maximal logic in the lattice $\mathcal{M}$ is $\mathfrak{F}_M$, which is an absolutely inconsistent logic whose only immediate predecessor is the logic $K_\Box \rightleftharpoons [p \supset \Box p]$. Let us examine how many immediate predecessors the logic $K_\Box$ has, and which logics they are. We shall consider two TBA:

$\mathfrak{L}_1 = \langle B; \&, \vee, \rightarrow, \sim, \Box_1; 1 \rangle$ and $\mathfrak{L}_2 = \langle B; \&, \vee, \rightarrow, \sim, \Box_2; 1 \rangle$, where $B = \{0, a, b, 1\}$; $\langle B; \&, \vee, \rightarrow, \sim; 1 \rangle$ is a Boolean algebra;

$$\Box_1 x = \begin{cases} 1, x = 1 \\ 0, x \neq 1 \end{cases} \qquad \Box_2 x = \begin{cases} 1, x = 1 \\ a, x = a \\ 0, x = 0 \text{ or } x = b \end{cases}$$

It is evident that $M(\mathfrak{L}_1) \neq K_\Box$, $M(\mathfrak{L}_2) \neq K_\Box$.

Now we shall prove that $M(\mathfrak{L}_1)$ and $M(\mathfrak{L}_2)$ are precisely all the immediate predecessors of the logic $K_\Box$; moreover:

THEOREM 12. Any element of the lattice $\mathcal{M}$ other than $K_\Box$ and $\mathfrak{F}_M$ is contained in at least one of the logics $M(\mathfrak{L}_1)$ and $M(\mathfrak{L}_2)$, and $M(\mathfrak{L}_1) \neq M(\mathfrak{L}_2)$.

Proof. Let $M \in \mathcal{M}$, $M \neq K_\Box$, $M \neq \mathfrak{F}_M$. We shall consider $\rho(M)$. If $\rho(M)$ is a classical logic K, then $M \supseteq \tau\rho(M) = \tau(K) = S5$. Since $M \subset K_\Box$, $M \neq K_\Box$, it follows [16] that $M \subseteq M(\mathfrak{L}_1)$. Let us assume that $\rho(M)$ does not coincide with K; it then follows from the relation $\rho(M) \neq \mathfrak{F}_J$ that $\rho(M) \subseteq LS_2$, where the PBA $S_2 = \langle S_2, \&, \vee, \supset, \neg; 1 \rangle$ is a three-element linearly ordered PBA. Hence $M \subseteq \sigma\rho(M) \subseteq \sigma(LS_2) = M(\mathfrak{s}(S_2)) = M(\mathfrak{L}_2)$. The inclusions follow from (P7) and (P3) of §3 and from the corollary of Theorem 5. Now we note that $M(\mathfrak{L}_1) \neq M(\mathfrak{L}_2)$. Indeed, $\Diamond p \rightarrow \Box\Diamond p$ is true in $\mathfrak{L}_1$, but not in $\mathfrak{L}_2$, whereas $\Diamond\Box(p \rightarrow \Box p)$ is true in $\mathfrak{L}_2$, but not in $\mathfrak{L}_1$. This completes the proof of the theorem.

LITERATURE CITED

1. V. Ya. Gerchiu, "Superindependent superintuitionistic logics," Matem. Issled., 5, No. 3, 24-37 (1970).
2. V. Ya. Gerchiu and A. V. Kuznetsov, "On finitely axiomatizable superintuitionistic logics," Dokl. Akad. Nauk SSSR, 195, No. 6, 1263-1266 (1970).
3. P. M. Cohn, Universal Algebra, Harper and Row, New York (1966).
4. A. V. Kuznetsov, "Some structural properties of varieties of pseudo-Boolean algebras," Eleventh All-Union Algebraic Colloquium, Abstracts of Communications and Reports, Kishinev [in Russian] (1971), pp. 255-256.
5. A. V. Kuznetsov and V. Ya. Gerchiu, "On superintuitionistic logics and finite approximability," Dokl. Akad. Nauk SSSR, 195, No. 5, 1029-1032 (1970).
6. L. L. Maksimova, "Pretabular superintuitionistic logics," Algebra i Logika, 11, No. 5, 558-570 (1972).
7. H. Rasiowa and R. Sikorski, The Mathematics of Metamathematics, Monografie Mat., 41, PWN, Warsaw (1963).
8. V. A. Yankov, "Construction of sequence of strongly independent superintuitionistic propositional calculi," Dokl. Akad. Nauk SSSR, 181, No. 1, 33-34 (1968).
9. G. Birkhoff, "Subdirect unions in universal algebras," Bull. Amer. Math. Soc., 50, 764-768 (1944).
10. M. A. Dummett and E. J. Lemmon, "Modal logics between 34 and 35," Zeitschr. Math. Log. und Grundl. Math., 5, 250-264 (1959).
11. J. Dugundji, "Note on a property of matrices for Lewis and Langford's calculi of propositions," J. Symb. Logic, 5, 150-151 (1940).
12. T. Hosoi, "On axiomatic method and the algebraic method for dealing with propositional logics," J. Fac. Sci. Univ. Tokyo, Sec. IA, 14, 131-169 (1967).
13. T. Hosoi, "On intermediate logics. I-II," J. Fac. Sci. Univ. Tokyo, Sec. I, 14, 293-312 (1967); 16, 1-12 (1969).
14. J. C. McKinsey and A. Tarski, "Some theorems about the sentential calculi of Lewis and Heyting," J. Symb. Logic, 13, 1-15 (1948).
15. S. Miuras, "A remark on the intersection of two logics," Nagoya Math. J., 26, 167-171 (1966).
16. J. Scroggs, "Extensions of S5," J. Symb. Logic, 16, 112-120 (1951).
17. A. Grzegorczyk, "Some relational systems and the associated topological spaces," Fund. Math., 60, 223-231 (1967).
18. K. A. Baker, "Equational bases for finite algebras," Not. Am. Math. Soc., 19, 691-08-2 (1972).