LATTICE DEFINABILITY OF CERTAIN MATRIX ALGEBRAS

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Introduction

We fix a field k and denote by $\mathfrak{A}$ the class of all associative algebras over this field. We denote by $S(A)$ the lattice of subalgebras of the algebra A. Suppose A and B are two algebras of the class $\mathfrak{A}$ and suppose the lattices $S(A)$ and $S(B)$ are isomorphic. We are interested in the relation in this case between the algebras A and B.

Lattice-isomorphic algebras (algebras with isomorphic lattices of subalgebras) were studied by D. Barnes [1] who showed that if A is a ring of matrices of order $\geqslant 3$ over a finite-dimensional (skew) field, then there exists a 1-1 mapping of A to B which is either multiplicative or antimultiplicative.

Articles [3] and [4] border on our theme in dealing with lattice-isomorphic Lie algebras, as does the work [2]. We note also a result on semilinear isomorphisms of those rings of matrices (of order $\geqslant 3$) over algebras with divisors in which the isomorphic lattices are right ideals [5, Theorem 21].

We formulate now the basic result of this work. Consider the case in which one of the lattice-isomorphic algebras is matrix over an arbitrary (skew) field. We shall assume that the basic field k contains at least three elements.

THEOREM. Suppose that D is an algebra with divisor over k, that n is a natural number, $n \geqslant 3$, that $A = D_n$ is an algebra of matrices over D, and that B is an algebra of class $\mathfrak{A}$. Then the lattices $S(A)$ and $S(B)$ are isomorphic if and only if the algebra B is isomorphic or antiisomorphic to an algebra $\tilde{D}_n$ over a (skew) field $\tilde{D}$ such that between the (skew) fields D and $\tilde{D}$ there exists a k-semilinear correspondence.

We point out that for $D = k$ this theorem is true as well for $n = 2$ [1, Theorem 2]. The situation in the general case is not known. For $n = 1$ there are trivial counterexamples:

a) $A = k$ and B a one-dimensional nilpotent algebra,

b) A and B subfields of the field C of complex numbers, $[A : Q] = 2 = [B : Q], A \neq B$. Here $k = Q$, the field of rational numbers.

Before proving our theorem we introduce the following notation: small Greek letters (with or without subscripts) indicate elements of the field k. The letter φ is reserved for isomorphisms of lattices. The algebra generated by the elements $a, b, c, \ldots$ is denoted by $\langle a, b, c \ldots \rangle$.

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§1. Algebras Lattice-Isomorphic to Algebras with Zero Products

The purpose of this section is to establish a property of algebras lattice-isomorphic to algebras with zero products.

An important observation constantly used in our considerations is given in the following obvious lemma:

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LEMMA 1. A (non-null) algebra is one-dimensional if and only if it contains no proper subalgebras.

It is clear also that a one-dimensional algebra either is an algebra with zero products or is generated by an idempotent.

We now establish the following proposition.

LEMMA 2. Let $\mathcal{B}$ and $\mathcal{L}$ be lattice-isomorphic algebras over k, and let $\mathcal{B}$ be an algebra with zero products. Then each (non-null) singly generated subalgebra of the algebra $\mathcal{L}$ is one-dimensional.

Proof. We can assume that $\dim \mathcal{B} \geqslant 2$. We note that $\mathcal{L}$ is an algebraic algebra (in each non-null subalgebra there is a non-null subalgebra without proper subalgebras). Suppose $\varphi\colon S(\mathcal{B}) \longrightarrow S(\mathcal{L})$ is an isomorphism and let $0 \neq c \in \mathcal{L}$. Then $\varphi^{-1}(\langle c\rangle)$ is a finite-dimensional algebra since it has zero products and there is in it no infinite strictly increasing chain of subalgebras (in view of finite-dimensionality of $\langle c\rangle$). Suppose $\{b_1, \dots, b_m\}$ is a basis of the algebra $\varphi^{-1}(\langle c\rangle)$. Then $\langle c\rangle = \langle c_1, \dots, c_m\rangle$, where $\langle c_j\rangle = \varphi(\langle b_j\rangle)$ is a one-dimensional algebra, $1 \leqslant j \leqslant m$. Since c is an arbitrary non-null element of the algebra $\mathcal{L}$, it follows that the algebra $\mathcal{L}$ is generated by some product of its one-dimensional subalgebras.

Let $\langle c_1\rangle, \langle c_2\rangle$ be arbitrary one-dimensional subalgebras of the algebra $\mathcal{L}$. We show that $c_1c_2 = \gamma_1 c_1 + \gamma_2 c_2$. We can assume that $\langle c_1\rangle \neq \langle c_2\rangle$, since otherwise there would be nothing to prove. The algebra $\langle c_1, c_2\rangle = \langle\langle c_1\rangle, \langle c_2\rangle\rangle$ is generated by any two of its distinct non-null subalgebras since this property is possessed by the algebra $\langle\langle b_1\rangle, \langle b_2\rangle\rangle$, $\langle b_j\rangle = \varphi^{-1}(\langle c_j\rangle)$. Hence if $c_1c_2 \notin kc_1 + kc_2$, then $\langle c_1, c_2\rangle = \langle c_1, c_1c_2\rangle$ whence

$$c_2 = f(c_1, c_1c_2) = c_1 g(c_1, c_2), \tag{1}$$

where $f(X,Y)$ and $g(X,Y)$ are some polynomials over k in the noncommutative variables X and Y. Let $c_1^2 = \gamma c_1$; multiplying both sides of (1) on the left by c_1 we get $c_1c_2 = \gamma c_2$, a contradiction. The assertion is proved.

Let x_1, x_2 be non-null elements of the algebra $\mathcal{L}$ where further $x_1^2 = x_1$, $x_2^2 = x_2$, $\langle x_1\rangle \neq \langle x_2\rangle$. Then there is a non-null nilpotent element in the algebra $\langle x_1, x_2\rangle$. Suppose the contrary. Then from $x_i x_j \in kx_i + kx_j$, $i,j = 1,2$, follows that the algebra $\langle x_1, x_2\rangle$ is two-dimensional. Hence it is isomorphic to the algebra $k \oplus k$ in which there are exactly three one-dimensional k-subalgebras. But in the algebra $\varphi^{-1}(\langle x_1, x_2\rangle)$ lattice-isomorphic to the algebra $\langle x_1, x_2\rangle$, there are more than three one-dimensional subalgebras [in view of the fact that $k \neq GF(2)$]. The contradiction obtained shows that there is in the algebra $\langle x_1, x_2\rangle$ a nilpotent element $N(x_1, x_2)$ which generates a one-dimensional algebra.

It is clear that $\langle x_1, x_2\rangle = \langle x_1, N(x_1,x_2)\rangle = \langle x_2, N(x_1,x_2)\rangle$.

Let the one-dimensional algebras $\langle c_i\rangle \subseteq \mathcal{L}$ $(i \in I)$ generate the algebra $\mathcal{L}$. We can suppose that $c_{i_1} \neq c_{i_2}$ for $i_1 \neq i_2$ and that c_i^2 is equal to 0 or c_i for all $i \in I$. We can present two cases:

(1) $c_i^2 = 0$ for all $i \in I$. Then for $i_1 \neq i_2$ in the relation $c_{i_1}c_{i_2} = \gamma_1 c_{i_1} + \gamma_2 c_{i_2}$ in the light of

$$c_{i_1} \cdot c_{i_1}c_{i_2} = \gamma_1 c_{i_1}^2 + \gamma_2 c_{i_1}c_{i_2} \text{ and } c_{i_1}c_{i_2}^2 = \gamma_1 c_{i_1}c_{i_2} + \gamma_2 c_{i_2}^2,$$

there follows $c_{i_1}c_{i_2} = 0$. That is, $\mathcal{L}$ is an algebra with zero multiples, which implies $\dim\langle c\rangle = 1$ for $0 \neq c \in \mathcal{L}$.

(2) $c_{i_0}^2 = c_{i_0}$ for some $i_0 \in I$. We set $I_0 = I \setminus \{i_0\}$, $c_0' = c_{i_0}$, $c_i' = N(c_{i_0}, c_i)$ for $i \in I_0$. Then $\mathcal{L} = \langle\langle c_0'\rangle, \langle c_i'\rangle : i \in I_0\rangle$. As above, we obtain that $\mathcal{L}_0 = \langle\langle c_i'\rangle : i \in I_0\rangle$ is an algebra with a zero multiple. Clearly each element of the algebra $\mathcal{L}$ has the form $\gamma c_0' + c'$ where $c' \in \mathcal{L}_0$. We suppose that $\langle \gamma c_0' + c'\rangle$ is not a one-dimensional algebra and that $\gamma c_0' + c' \neq 0$. Then $\gamma \neq 0$ and $c' \neq 0$. In the algebra $\varphi^{-1}(\langle c_0', c'\rangle)$ there are no non-one-dimensional proper subalgebras. Hence $\langle \gamma c_0' + c'\rangle = \langle c_0', c'\rangle$, $c_0'c' = c'c_0'$. Using the equalities $c_0'^2 = c_0'$, $c'^2 = 0$, $c_0'c' \in kc_0' + kc'$, it is readily shown that $c_0'c'$ is either 0 or c'. In both cases the algebra $\langle c_0', c'\rangle$ contains only two one-dimensional subalgebras: $\langle c_0'\rangle$ and $\langle c'\rangle$. The contradiction obtained shows that each non-null singly generated subalgebra of the algebra $\mathcal{L}$ is one-dimensional. The lemma is proved.

We now give a definition necessary for formulating Lemma 3. Suppose that $\mathcal{E}=\{0, e_{ij} : i,j=1,\dots,n\}$ is a semigroup consisting of zero and matrix units and that $\mathcal{F}=\{0, f_{ij} : i,j=1,\dots,n\}$ is a subsemigroup of multiplicative semigroups of some (associative) algebra. We shall say that the system $\{f_{ij}\}_{i,j=1,\dots,n}$ is an isomorphism (antiisomorphism) of the semigroups if the mapping

$$\begin{cases} 0 \longrightarrow 0 \\ e_{ij} \longrightarrow f_{ij}, \quad i,j=1,\dots,n, \end{cases}$$

is an isomorphism (antiisomorphism) of the semigroups $\mathcal{E}$ and $\mathcal{F}$.

Suppose A and B are the algebras referred to in the theorem. Denote by φ the isomorphism

$$S(A) \longrightarrow S(B).$$

Suppose $\langle e_{ij}\rangle$ is the subalgebra in A generated by e_{ij}. Then by Lemma 1

$$\varphi : \langle e_{ij}\rangle \longrightarrow \langle f_{ij}\rangle$$

for some element $f_{ij} \in B$.

Corresponding to each system of n^2 nonzero elements ν_{ij} of the field k we put

$$\bar{f}_{ij}(\nu_{ij}) = \bar{f}_{ij} = \begin{cases} \nu_{ij} f_{ij} & \text{for } i \neq j, \\ f_{ij} & \text{for } i = j. \end{cases}$$

<u>LEMMA 3.</u> For generators $\nu_{ij} \in k$ the system $\{\bar{f}_{ij}\}_{i,j=1,\dots,n}$ is isomorphic or antiisomorphic to the system of matrix units.

The proof of this assertion is contained in the article of Barnes [1] in which it appears as Lemma 12.

In the light of Lemma 3 we can consider that the system $\{f_{ij}\}_{i,j=1,\dots,n}$ is isomorphic or antiisomorphic to the system of matrix units. Let B be the algebra referred to in the Theorem. In the linear space of B we prescribe a new multiplication operation $b_1 \circ b_2 = b_2 b_1$, $b_1, b_2 \in B$. The algebra obtained we denote by B'. Now we define an algebra B''. Set

$$B'' = \begin{cases} B, & \text{if } f_{11} f_{12} = f_{12}, \\ B', & \text{if } f_{11} f_{12} = 0. \end{cases}$$

Thus we have selected in the capacity of the algebra B'' either the algebra B or B' depending on whether there is isomorphism or antiisomorphism of $\{f_{ij}\}_{i,j=1,\dots,n}$ and the system $\{e_{ij}\}_{i,j=1,\dots,n}$ of matrix units.

§2. Matricity of Algebras Lattice-Isomorphic to the Algebra D_n

In this section we demonstrate that an algebra lattice-isomorphic to the algebra D_n is isomorphic or antiisomorphic to an algebra R_n, where R is an algebra over k. We shall suppose that the system $\{f_{ij}\}_{i,j=1,\dots,n}$ is isomorphic to the system of matrix units. Consideration of the algebra B'' introduced in §1 shows that this assumption does not imply loss of generality.

Put $D_{ij} = D e_{ij}$, $S_{ij} = \varphi(D_{ij})$. There holds the following lemma.

<u>LEMMA 4.</u> For each i the element f_{ii} is a unit of the algebra S_{ii}.

Proof. Suppose $\mathcal{S}=\mathcal{S}_{ii}$, $f=f_{ii}$. $e=e_{ii}$ Note that for each $s\in\mathcal{S}$, $fs=sf$. In fact

$$(fs-fsf)^2=fs\cdot fs-fs\cdot fsf-fsf\cdot fs+fsf\cdot fsf=0,$$
$$(sf-fsf)^2=sf\cdot sf-sf\cdot fsf-fsf\cdot sf+fsf\cdot fsf=0,$$

whence $fs=fsf=sf$ (there is in $\mathcal{S}$ a unique one-dimensional subalgebra $<f>$ and hence nilpotent elements are absent).

We denote $\mathcal{A}=\{s\in\mathcal{S}: fs=0\}$. It suffices to prove that $\mathcal{A}=0$ since $fs-s\in\mathcal{A}$ for all $s\in\mathcal{S}$. Suppose the contrary, i.e., let $\mathcal{A}\neq 0$, $\mathcal{A}$ a subalgebra of $\mathcal{S}$, where moreover $\mathcal{A}\not\supseteq<f>$. Let $t\in\mathcal{D}$ be such that $0\neq t\in\varphi^{-1}(\mathcal{A})$.

Consider $\mathcal{B}=\varphi(<t^{-1}>)$.

We assume that $\mathcal{B}\supseteq<f>$. Then

$$e=\sum_{m<m_0}\lambda_m t^{-m}+\lambda_{m_0}t^{-m_0}\quad,\quad\lambda_{m_0}\neq 0\;;$$
$$t^{m_0}=\sum\lambda_m t^{m_0-m}+\lambda_{m_0}e.$$

The latter equation demonstrates that $e\in\varphi^{-1}(\mathcal{A})$, i.e., $<f>\subseteq\mathcal{A}$, a contradiction. This shows that $\mathcal{B}\not\supseteq<f>$.

Assume $\mathcal{B}\cap\mathcal{A}\neq 0$. Then we have

$$\sum_{m\leq m_0}\nu_m(t^{-1})^m\in\varphi^{-1}(\mathcal{A})\cap\varphi^{-1}(\mathcal{B}),\qquad\nu_{m_0}\neq 0.$$

Hence

$$t^{m_0}\sum_{m\leq m_0}\nu_m t^{-m}\in\varphi^{-1}(\mathcal{A}),$$
$$\nu_{m_0}e+\sum_{m<m_0}\nu_m t^{m_0-m}\in\varphi^{-1}(\mathcal{A}).$$

This implies $e\in\varphi^{-1}(\mathcal{A})$, i.e., $<f>\subseteq\mathcal{A}$, a contradiction.

Thus $\mathcal{B}\not\supseteq<f>$, $\mathcal{B}\cap\mathcal{A}=0$, $<\mathcal{A},\mathcal{B}>\supseteq<f>$. That is to say $f=b+a+\sum\lambda_j w_j(\mathcal{A},\mathcal{B})$ where in each word w_j there enters some $a_k\in\mathcal{A}$. In view of centrality of f and $\mathcal{A}f=0$ we have

$$f^2=bf+af+\sum\lambda_j w_j(\mathcal{A},\mathcal{B})f=bf.$$

Thus $bf=f$ which is to say $b^2f=f$, $(b^2-b)f=0$, i.e.,

$$b^2-b\in\mathcal{A}.$$

But $\mathcal{A}\cap\mathcal{B}=0$ whence $b^2-b=0$. From the equality $f=bf$ there follows $b\neq 0$. It is clear that $<b>\neq<f>$. Thus in $\mathcal{S}$ there are two one-dimensional subalgebras, a contradiction.

LEMMA 5. Suppose $i\neq j$. Then $f_{ii}s_{ij}=s_{ij}$, $s_{ij}f_{ii}=0$, $s_{ij}f_{jj}=s_{ij}$, $f_{jj}s_{ij}=0$ for each element $s_{ij}\in\mathcal{S}_{ij}$.

Proof. Suppose $s_{12}\in\mathcal{S}_{12}$. If $s_{12}=0$, then the assertion of the lemma is trivial. Hence suppose $s_{12}\neq 0$. In view of Lemmas 1 and 2 there exists $r\in\mathcal{D}$, $r\neq 0$ such that $<re_{12}>=\varphi^{-1}(<s_{12}>)$. Suppose $<s_{31}>=\varphi(<r^{-1}e_{31}>)$, $<s_{21}>=\varphi(<r^{-1}e_{21}>)$, $<s_{13}>=\varphi(<re_{13}>)$. Then for

$$C = \langle e_{11}, re_{12}, r^{-1}e_{31}, r^{-1}e_{21}, re_{13}, e_{32}, e_{23}, e_{22}, e_{33} \rangle$$

we have

$$\varphi : C \longrightarrow \langle f_{11}, s_{12}, s_{31}, s_{21}, s_{13}, f_{32}, f_{23}, f_{22}, f_{33} \rangle.$$

But $C \cong k_3$. Lemma 5 now follows from Lemma 3 in view of the correspondences $\langle e_{11} \rangle \longleftrightarrow \langle f_{11} \rangle$, $\langle re_{12} \rangle \longleftrightarrow \langle s_{12} \rangle$ and the fact that $f_{22}f_{23} = f_{23}$ [since the corresponding system of elements of the algebra $\varphi(C)$ is isomorphic to the system of matrix units].

Assertion 6. $s_{ii} f_{ij} \in S_{ij}$ and $f_{ji} s_{ii} \in S_{ji}$ for each $s_{ii} \in S_{ii}$.

Proof. For $i = j$ there is nothing to prove. Suppose $i = 1, j = 2$. If $s_{11} \in k$ then the assertion is obvious. Suppose $s_{11} \notin k$. Consider the algebra $\mathcal{A} = \langle s_{11}, f_{12} \rangle$. We assert that each one-dimensional subalgebra $\mathcal{B} \subseteq \mathcal{A}$ lies in $\langle f_{11}, S_{12} \rangle$. Indeed, suppose $C = \varphi^{-1}(\langle s_{11} \rangle)$. Then $\mathcal{A} = \langle \varphi(C), f_{12} \rangle$. Suppose $xe_{11} \in C$, $x \in D$ and suppose for some $\zeta \in k$

$$(xe_{11} + ye_{12})^2 = \zeta(xe_{11} + ye_{12}).$$

Then $x(x - \zeta) = 0$ and if $x \neq 0$ (i.e., $\langle xe_{11} + ye_{12} \rangle \nsubseteq D_{12}$) then $x = \zeta \in k$. That is to say $\langle xe_{11} + ye_{12} \rangle \subseteq \langle e_{11}, D_{12} \rangle$, i.e., $\varphi(\langle xe_{11} + ye_{12} \rangle) \subseteq \langle f_{11}, S_{12} \rangle$.

In view of the relation $(s_{11}f_{12})^2 = s_{11}f_{12} \cdot f_{11} s_{11} f_{12} = 0$ we have

$$s_{11} f_{12} \in \langle f_{11}, S_{12} \rangle.$$

On the strength of Lemma 5, $s_{11}f_{12} = \lambda f_{11} + s_{12}$. Multiplying both sides of this equality on the right by f_{22} we get $s_{11}f_{12} = s_{12} \in S_{12}$. In exactly the same way we show that $f_{ji}s_{ii} \in S_{ji}$.

LEMMA 6. For arbitrary $s_{ij} \in S_{ij}$ and $s'_{ji} \in S_{ji}$ there holds the inclusion $s_{ij}s'_{ji} \in S_{ii}$.

Proof. We can of course consider that $i \neq j$. Suppose $\mathcal{A} = \langle s_{12}, s_{21} \rangle$. We show that for each algebra $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \nsubseteq \langle S_{11}, S_{12}, S_{22} \rangle$ there holds the relation $\langle S_{22}, \mathcal{B} \rangle \supseteq \langle s_{21} \rangle \neq 0$ where $s_{21} \in S_{21}$. Indeed, suppose

$$xe_{11} + ye_{12} + ze_{22} + ve_{21} \in \varphi^{-1}(\mathcal{B}), \quad v \neq 0, \quad x, y, z, v \in D.$$

Then $\langle D_{22}, \varphi^{-1}(\mathcal{B}) \rangle \supseteq \langle ve_{12} \rangle \neq 0$, whence the assertion follows.

We put $\mathcal{B} = \langle s_{12} s_{21} \rangle$. If there held the relation $\mathcal{B} \nsubseteq \langle S_{11}, S_{12}, S_{22} \rangle$, then for some $0 \neq \bar{s}_{21} \in S_{21}$ there would be fulfilled the equality

$$\bar{s}_{21} = s_{22} + \sum \lambda_j (s_{12}s_{21})^j.$$

Multiplying both sides of this equality on the left by f_{22} and on the right by f_{11} we are led to the relation $\bar{s}_{21} = 0$, a contradiction. This means that $\mathcal{B} \subseteq \langle S_{11}, S_{12}, S_{22} \rangle$. Suppose

$$s_{12}s_{21} = \tilde{s}_{11} + \tilde{s}_{12} + \tilde{s}_{22} + \sum \lambda_q v_q s_{12} + \sum \mu_q w_q s_{22}$$

(in the light of Lemmas 4 and 5, $s_{12}s_{11} = 0 = s_{22}s_{11}$ for $s_{ij} \in S_{ij}$). Multiplying both sides of the equality on the right by f_{11} we get $s_{12}s_{21} = \tilde{s}_{11} \in S_{11}$.

Assertion 7. $S_{ik}S_{kj} \subseteq S_{ij}$ and $S_{ik}S_{mj} = 0$ for $k \neq m$.

Proof. We show that $S_{12}S_{23} \subseteq S_{13}$. Suppose $s_{12} \in S_{12}$, $s_{23} \in S_{23}$, $s_{12}s_{23} \neq 0$. It is clear that $\langle s_{12}, s_{23}\rangle$ has but one one-dimensional subalgebra lying neither in S_{12} nor in S_{23} and that it lies in S_{13}. The algebra $\langle s_{12}s_{23}\rangle$ is one-dimensional. Clearly $S_{12} \not\ni \langle s_{12}s_{23}\rangle \notin S_{23}$. This means that $s_{12}s_{23} \in S_{13}$.

We show that $S_{11}S_{12} \subseteq S_{12}$:

$$s_{11}s_{12} = s_{11}f_{13} \cdot f_{31}s_{12} = s'_{13}s'_{32} \in S_{12}.$$

Analogously, $S_{12}S_{22} \subseteq S_{12}$. The remaining inclusions are obvious.

Suppose m and κ are natural numbers, $m \neq \kappa$. Then

$$s_{i\kappa}s_{mj} = s_{i\kappa}f_{\kappa\kappa} \cdot f_{mm}s_{mj} = 0.$$

LEMMA 7. The algebra $B = \varphi(A)$ is isomorphic to R_n where R is an algebra over k.

Proof. In the light of Assertion 7, for each element $b \in B$

$$b = \sum s_{ij}, \quad s_{ij} \in S_{ij}.$$

If $\sum s_{ij} = \sum s'_{ij}$ then $f_{\kappa\kappa}\sum s_{ij}f_{mm} = f_{\kappa\kappa}\sum s'_{ij}f_{mm}$, whence $s_{\kappa m} = s'_{\kappa m}$ for all κ and m.

Fix a pair (κ, m). In the light of the equality

$$s_{pq} = f_{p\kappa}(f_{\kappa p}s_{pq}f_{qm})f_{mq}$$

we have the relation

$$S_{pq} = \{f_{p\kappa}s_{\kappa m}f_{mq} : s_{\kappa m} \in S_{\kappa m}\},$$

where the presentation of each element of S_{pq} in the form $f_{p\kappa}s_{\kappa m}f_{mq}$ is unique.

Suppose $R = S_{11}$. If $b \in B$ then $b = \sum_{i,j=1}^{n} f_{i1}r_{ij}f_{1j}$, where $r_{ij} \in R$. It is apparent that the mapping

$$\sum f_{i1}r_{ij}f_{1j} \longrightarrow \| r_{ij} \|_{i,j=1,\dots,n}$$

is an isomorphism of the algebra B to R_n.

§3. Algebras Lattice-Isomorphic to Matrix Algebras over (Skew) Fields

In this section we conclude the proof of our Theorem, formulated in the Introduction. We shall assume that the system $\{f_{ij}\}_{i,j=1,\dots,n}$ is isomorphic to the system of matrix units.

Assertion 8. Let $a \in D$ and $\varphi : \langle ae_{12}\rangle \to \langle \bar{a}f_{12}\rangle$. Then $\varphi : \langle ae_{13}\rangle \to \langle \bar{a}f_{13}\rangle$ and $\varphi : \langle ae_{23}\rangle \to \langle \bar{a}f_{23}\rangle$.

Proof. We can obviously assume that $a \neq 0$. For some b, $c \in R$ we have

$$\varphi : \langle ae_{13}\rangle \to \langle bf_{13}\rangle \text{ and } \varphi : \langle ae_{23}\rangle \to \langle cf_{23}\rangle.$$

Put $A_1 = \langle ae_{12}, ae_{13}, e_{23}\rangle$. It is clear that $A_1 \cap D_{13} = \langle ae_{13}\rangle$. Suppose $\varphi : A_1 \to B_1 = \langle \bar{a}f_{12}, bf_{13}, f_{23}\rangle$. Then $(R_{ij} = Rf_{ij})$

$$\mathcal{B}_1 \cap R_{13} \supseteq \langle \bar{a} f_{13} \rangle \text{ and } \mathcal{B}_1 \cap R_{13} \supseteq \langle b f_{13} \rangle,$$

whence $\langle b f_{13} \rangle = \langle \bar{a} f_{13} \rangle$.

Now consider $\mathcal{A}_2 = \langle e_{12},\ a e_{23},\ a e_{13} \rangle$. Suppose $\varphi : \mathcal{A}_2 \to \mathcal{B}_2 = \langle f_{12}, c f_{23}, \bar{a} f_{13} \rangle$. As above we obtain that $\langle c f_{23} \rangle = \langle a f_{23} \rangle$.

<u>Assertion 9.</u> Suppose a, $b \in \mathcal{D}$. We assume that $\varphi : \langle a e_{12} \rangle \to \langle \bar{a} f_{12} \rangle$ and $\varphi : \langle b e_{12} \rangle \to \langle \bar{b} f_{12} \rangle$. Then $\varphi : \langle a b\, e_{12} \rangle \to \langle \bar{a} \bar{b} f_{12} \rangle$.

<u>Proof.</u> We can obviously assume that $a \neq 0, b \neq 0$. Then $ab \neq 0$. Put $\mathcal{A} = \langle a e_{12},\ b e_{23},\ (ab) e_{13} \rangle$. In view of Assertion 8

$$\varphi : \mathcal{A} \to \mathcal{B} = \langle \bar{a} f_{12}, \bar{b} f_{23}, \overline{ab} f_{13} \rangle,$$

which implies $\mathcal{B} \cap R_{13} \supseteq \langle \bar{a} \bar{b} f_{13} \rangle$ and $\mathcal{B} \cap R_{13} \supseteq \langle \overline{ab} f_{13} \rangle$, so that it suffices to prove that $\bar{a}\bar{b} \neq 0$. But if equality $\bar{a}\bar{b} = 0$ held, then in view of the relation $\mathcal{B} = \langle \bar{a} f_{12}, \bar{b} f_{23} \rangle$ we would have $\mathcal{B} \cap R_{13} = 0$, a contradiction. The assertion is proved.

Now assume an element $t \in \mathcal{D}$. Consider the algebra

$$\mathcal{O}_t = \langle e_{11} + t e_{12} + e_{33} \rangle .$$

It is one-dimensional. If $\mathcal{O}$ is a one-dimensional algebra, $\mathcal{O} \subseteq \langle\langle e_{11} \rangle, \mathcal{D}_{12}, \langle e_{33} \rangle\rangle$, then $\mathcal{O} = \mathcal{O}_t$ for some $t \in D$ (provided of course that $\mathcal{O} \not\subseteq \langle e_{11}, \mathcal{D}_{12} \rangle$, $\mathcal{O} \not\subseteq \langle e_{33}, \mathcal{D}_{12} \rangle$). Suppose $\varphi : \mathcal{O}_t \to \tilde{\mathcal{O}}_r$, $r \in R$ (the existence and uniqueness of r are obvious). Then define $f : t \to r$. The ensuing situation corresponds to the diagram

$$\begin{tikzcd}
\mathcal{O}_t \arrow[r, "\varphi"] & \tilde{\mathcal{O}}_r \arrow[d] \\
t \arrow[u] \arrow[r, dashed, "f"] & r
\end{tikzcd}$$

<u>LEMMA 8.</u> Suppose s and t are elements of the (skew) field $\mathcal{D}$, linearly independent over k. Then $(s+t)f = sf + tf$.

<u>Proof.</u> Suppose $t_1, t_2 \in \mathcal{D}$. Put $a = e_{11} + t_1 e_{12} + e_{33}$, $b = e_{11} + t_2 e_{12} + e_{33}$. Then $a^2 = a$, $b^2 = b$, $ab = b$, $ba = a$, so that $\langle a, b \rangle = \{\lambda a + \mu b : \lambda, \mu \in k\}$. Therefore

$$\langle \mathcal{O}_{t_1}, \mathcal{O}_{t_2} \rangle \cap \mathcal{D}_{12} = \{\lambda (t_1 - t_2) e_{12} : \lambda \in k\}.$$

For $x \in \mathcal{D}$ we have

$$(\langle \mathcal{O}_x, \mathcal{O}_t \rangle \cap \mathcal{D}_{12} \supseteq \langle \mathcal{O}_s, e_{11}, e_{33} \rangle \cap \mathcal{D}_{12}) \,\&$$
$$\& \,(\langle \mathcal{O}_x, \mathcal{O}_s \rangle \cap \mathcal{D}_{12} \supseteq \langle \mathcal{O}_t, e_{11}, e_{33} \rangle \cap \mathcal{D}_{12}) \Longleftrightarrow x = s + t.$$

Indeed, the condition on the left side of $\Longleftrightarrow$ can be expressed so: for some $\lambda, \mu \in k$

$$\begin{cases} x - s = \lambda t \\ x - t = \mu s. \end{cases}$$

For $x = s + t$ we put $\lambda = \mu = 1$. Conversely, if we have (7) then $t - s = \lambda t - \mu s$ and hence by linear independence of s and t we obtain $\lambda = 1$, i.e., $x = s + t$.

Now Lemma 8 easily follows from the definitions.

LEMMA 9. Let $\gamma = (1f)^{-1}$. We suppose that $[\mathcal{D}:k] \geq 2$. Then for all x, $y \in \mathcal{D}$ there hold the equalities

$$(xy)f = \gamma \cdot xf \cdot yf \quad \text{and} \quad (x+y)f = xf + yf.$$

Proof. For $x \in \mathcal{D}$ we put $\bar{x} = xf$. We prove the equality $\overline{xy} = \gamma \bar{x}\bar{y}$. We can assume that $x \neq 0$, $y \neq 0$. Consider the cases:

1) $x \notin k$; then $x = 1 + a$ where 1, a are linearly independent over k. We have

$$\overline{xy} = \overline{(1+a)y} = \lambda\ \overline{1+a}\,\bar{y} = \lambda(\gamma^{-1} + \bar{a})\bar{y} = \lambda\gamma^{-1}\bar{y} + \lambda\bar{a}\bar{y}.$$

On the other hand, since y and ay are linearly independent, it follows that

$$\overline{xy} = \overline{y + ay} = \bar{y} + \mu\bar{a}\bar{y}.$$

If the relation $\lambda\gamma^{-1} \neq 1$ held, then $\bar{y} = \nu\bar{a}\bar{y}$ would be fulfilled, whence $0 = (1 - \nu\bar{a})\bar{y}$. This would mean that $1 - \nu\bar{a} = 0$, $\bar{a} = \nu^{-1}$, whence $a \in k$, a contradiction. Thus $\overline{xy} = \gamma\bar{x}\bar{y}$.

2) $y \notin k$. The proof in this case proceeds analogously to that in case 1).

3) Suppose α, $\beta \in k$, $x \in \mathcal{D} \setminus k$, $x + y = \beta$. Then $y \notin k$. We have

$$\overline{\alpha\beta} = \overline{\alpha(x+y)} = \overline{\alpha x} + \overline{\alpha y} = \gamma\bar{\alpha}\bar{x} + \gamma\bar{\alpha}\bar{y} = \gamma\bar{\alpha}(\bar{x} + \bar{y}) = \gamma\bar{\alpha}\overline{x+y} = \gamma\bar{\alpha}\bar{\beta}.$$

Thus, in all cases $\overline{xy} = \gamma\bar{x}\bar{y}$.

The equality $\overline{x+y} = \bar{x} + \bar{y}$ requires proof only in case of linearly dependent x and y.

We note that $-\bar{1} = -\bar{1} = -\gamma^{-1}$: $\bar{1} = \overline{(-1)(-1)} = \gamma\,\overline{-1}\,\overline{-1}$, i.e., $\bar{1}^2 = \bar{1}\gamma^{-1} = \overline{-1}^2$. But $\bar{1} \neq \overline{-1}$ in case $\chi(k) \neq 2$, whence $\overline{-1} = -\bar{1}$. In the case where the characteristic $\chi(k)$ of the field k is equal to 2, the equation $\overline{-1} = -\bar{1}$ is trivial. Hence

$$\overline{-x} = \overline{(-1)x} = \gamma\,\overline{-1}\,\bar{x} = \gamma(-\gamma^{-1})\bar{x} = -\bar{x}.$$

Suppose now $x = \alpha z$, $y = \beta z$, $x + y \neq 0$. In view of the relation $[\mathcal{D}:k] \geq 2$ there exists $t \in \mathcal{D}$ such that t and z are linearly independent. It is clear that $\alpha z + t$ and $\beta z - t$ are also linearly independent. Hence $(x \neq 0,\ y \neq 0)$

$$\overline{(x+y)} = \overline{(\alpha z + t) + (\beta z - t)} = \overline{\alpha z + t} + \overline{\beta z - t} = \overline{\alpha z} + \bar{t} + \overline{\beta z} - \bar{t} = \bar{x} + \bar{y}.$$

Finally, if $x = 0$ or $y = 0$, then $\overline{x+y} = \bar{x} + \bar{y}$ in the light of the equality $\bar{0} = 0$. The lemma is proved.

We return to the proof of the theorem. If $\mathcal{D} = k$, then in view of Lemma 3 there is nothing to prove. If $\mathcal{D} \neq k$, then $[\mathcal{D}:k] \geq 2$. From Lemma 9 follows that the mapping $\bar{f}: \mathcal{D} \to R$ provided by the equality $x\bar{f} = \gamma^{-1} \cdot xf$ is a semilinear isomorphism of $\mathcal{D}$ to R. From Lemma 7 now follows that $B = \tilde{\mathcal{D}}_n$, where the (skew) field $\tilde{\mathcal{D}}$ is semilinearly isomorphic to the (skew) field $\mathcal{D}$. Conversely, if the (skew) field $\tilde{\mathcal{D}}$ is semilinearly isomorphic to the (skew) field $\mathcal{D}$, then obviously the lattice $S(\tilde{\mathcal{D}}_n)$ is isomorphic to the lattice $S(\mathcal{D}_n)$. The theorem is completely proved.

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