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Convolution Quadrature and Discretized Operational Calculus. II *

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Summary. Operational quadrature rules are applied to problems in numerical integration and the numerical solution of integral equations: singular integrals (power and logarithmic singularities, finite part integrals), multiple time-scale convolution, Volterra integral equations, Wiener-Hopf integral equations. Frequency domain conditions, which determine the stability of such equations, can be carried over to the discretization.

Subject Classifications: AMS(MOS): 65R20, 65D30; CR: G1.9.

6. Introduction

In this paper we give applications of the operational quadrature rules of Part I to numerical integration and the numerical solution of integral equations. It is the aim to indicate problem classes where such methods can be applied successfully, and to point out features which distinguish them from other approaches.

Section 8 deals with the approximation of integrals with singular kernel. This comes as a direct application of the results of Part I.

In Sect. 9 we study the numerical approximation of convolution integrals whose kernel has components at largely differing time-scales.

In Sect. 10 we give a case study for a problem in chemical absorption kinetics. There a system of an ordinary differential equation coupled with a diffusion equation is reduced to a single Volterra integral equation. This reduction is achieved in a standard way via Laplace transform techniques and, as is typical in such situations, it is the Laplace transform of the kernel (rather than the kernel itself) which is known a priori, and which is of a simple form. Operational quadrature rules lend themselves conveniently to the discretization of such problems. The numerical solution obtained in this way can be re-interpreted as that of a semi-discretization in time of the original problem.

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In various classes of integral and integro-differential equations of convolution type the stability properties (or even existence) of the solution are determined by conditions on the range of the Laplace or Fourier transform of the kernel ("frequency domain conditions"), see e.g. Corduneanu (1973), Desoer and Vidyasagar (1975), Londen and Staffans (1979), Hannsgen et al. (1982). In a discretization by an operational quadrature rule the generating function (i.e., discrete Laplace transform) of the discrete kernel is, by construction, closely related to the Laplace transform of the continuous kernel. It is thus possible to carry over frequency domain conditions from the continuous problem to the discretization and thereby obtain stable approximations. In Sect. 11 this is illustrated at Wiener-Hopf integral equations with symmetric kernel.

Any application of the methods relies, of course, on the availability of efficient algorithms for the computation of the quadrature weights. Using Fast Fourier Transform techniques, N weights can actually be computed with $O(N)$ transfer function evaluations and $O(N \cdot \log N)$ arithmetical operations. This is discussed in Sect. 7.

Meanwhile, operational quadratures have been used in Lubich (1985, 1987), Hairer and Maass (1987), Hairer et al. (1986), Sanz-Serna (1986), Gienger (1987), Lopez-Castillo et al. (1987). The reader will find in these papers further theoretical aspects as well as extensive numerical examples. We have therefore not concentrated on numerical examples in the present paper (although one is given in Sect. 10).

On part of the reader we assume only some familiarity with the introduction to Part I. Further material from Part I will be explicitly referenced where it is required. The sections of Part II can be read independently of each other.

7. Implementation

In this section we describe briefly how to numerically evaluate the approximations (1.2), (1.3), which we recall for convenience:

$$\sum_{j=0}^{n} \omega_{n-j}(h)\, g(j\,h), \qquad n=0,1,\ldots,N \tag{7.1}$$

where

$$F(\delta(\zeta)/h)=\sum_{n=0}^{\infty} \omega_n(h)\,\zeta^n. \tag{7.2}$$

Several techniques are available, using Fast Fourier Transforms (FFT).

i) Formal power series methods to compute $\omega_n(h)$ in (7.2).

ii) Trapezoidal rule approximation of the Cauchy integral for $\omega_n(h)$ with contour $|\zeta|=\rho$.

Once the weights $\omega_n(h)$ and the required values of g are computed, the discrete convolution (7.1) can be evaluated in $O(N \log N)$ arithmetical operations, using FFT. A further approach avoids the explicit computation of the weights $\omega_n(h)$:

iii) Trapezoidal rule approximation of the Cauchy integrals on $|\zeta|=\rho$ for the coefficients of $F(\delta(\zeta)/h)\cdot g(\zeta)$ $\left(\text{where } g(\zeta)=\sum_{0}^{N} g(jh)\zeta^j\right)$.

Method i) is efficient if $F(s)$ is of a simple structure, e.g., a rational function, a fractional power, a logarithm (cf. Sect. 8) or not too complicated an expression composed of such ingredients, see Henrici (1979) for details. Methods ii) and iii) are algorithmically very similar. So we restrict our attention to ii) in the following.

Here one seeks approximations to $\omega_n(h)$ of the form

$$\hat{\omega}_n(h)=\rho^{-n}\frac{1}{L}\sum_{l=0}^{L-1} F_l\cdot e^{-in\tau_l}, \qquad n=0,1,\ldots,N \tag{7.3}$$

where $F_l=F(\delta(\rho e^{i\tau_l})/h)$, $\tau_l=2\pi l/L$.

These coefficients can be computed simultaneously using FFT. This requires L evaluations of F and $O(L\log L)$ arithmetical operations. There remains the problem how to choose ρ and L. For this we note that by (1.5), (1.10) there exists $\gamma\in\mathbb{R}$ such that

$F(\delta(\zeta)/h)$ is analytic in a neighbourhood of $|\zeta|\leqq e^{-\gamma h}$ and is bounded by

$$|F(\delta(\zeta)/h)|\leqq M \qquad \text{for } |\zeta|\leqq e^{-\gamma h}, \text{ uniformly for } h\in(0,\bar{h}]. \tag{7.4}$$

We remark that γ is negative if $F(s)$ is exponentially stable, at least for sufficiently small $\bar{h}>0$. If we assume that one can actually compute only $\tilde{\omega}_n(h)$, which results from an approximate evaluation of F_l in (7.3) with a relative error of at most ε, the following error estimate is derived from the aliasing formula (Theorem 2a in Henrici (1979)):

$$|\tilde{\omega}_n(h)-\omega_n(h)|\leqq\frac{M}{1-(\rho e^{\gamma h})^L}e^{\gamma hn}\cdot(\rho e^{\gamma h})^L+\rho^{-n}M\varepsilon \qquad (n=0,1,\ldots,L-1). \tag{7.5}$$

In order to compute $\omega_n(h)$ $(n=0,1,\ldots,N;\ Nh=\bar{x}$ fixed) with an error of magnitude $O(\varepsilon)$ one has thus to choose $\log\rho=O(h)$ and $L=N|\log O(\varepsilon)|$. For accuracy requirements up to $O(\sqrt{\varepsilon})$ a choice of $L=N$ (and $\rho^N=\sqrt{\varepsilon}$) is sufficient. The situation improves for exponentially stable transfer functions $(\gamma<0)$.

It has been shown in Theorem 4.1 that $f_n(h)=\omega_n(h)/h$ is a p-th order approximation to the inverse Laplace transform $f(t)$ of $F(s)$ at $t=nh$ bounded away from 0. As a consequence, the weights $\omega_n(h)$ need in practice be computed by (7.3) only on rather short intervals $0\leqq nh\leqq\bar{x}$ and can be replaced by $hf(nh)$ else. (Values of f may be obtained by any of the common Laplace inversion algorithms, if necessary.) The quality of the approximation is improved by adding a few correction terms to (7.1), see Corollaries 3.2 and 4.2, and Sect. 10 below for an example.

8. Singular Integrals

Integrals of the form

$$\int_0^x f(t)\, g(x-t)\, dt$$

can be approximated conveniently with the help of operational quadratures whenever $f(t)$ has a known and hopefully simple Laplace transform. This includes power singularities, $f(t)=t^{\mu-1}$ $(\mu>0)$, for which $F(s)=\Gamma(\mu)\cdot s^{-\mu}$, and logarithmic singularities, $f(t)=t^{\mu-1}\cdot\log t$, for which $F(s)=\Gamma(\mu)s^{-\mu}\cdot(\log s - \Gamma'(\mu)/\Gamma(\mu))$. Such discretizations can have distinct advantages over traditional product integration rules, as is pointed out in Lubich (1985, 1987), Hairer and Maass (1987). Other classes of singular kernels (e.g. exponential integral, Hankel functions, ...) are equally amenable to approximation. Stronger types of singularities can be treated in the same way, e.g., the finite part integrals

$$\begin{aligned}\oint_0^x \frac{1}{t^k}\, g(x-t)\, dt &= \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{d}{dx}\right)^k \int_0^x \log t\cdot g(x-t)\, dt\\ &= \frac{(-1)^k}{(k-1)!}\, s^{k-1}(\log s+\gamma)\, g(x) \qquad (k=1,2,3,\ldots),\end{aligned}$$

where $\gamma=0.5772\ldots$ is Euler's constant.

9. Multiple Time-Scale Convolution

We now consider the approximation of convolution integrals $k*g$ with kernels of the form

$$k(t)=\frac{1}{\varepsilon_1}k_1\left(\frac{t}{\varepsilon_1}\right)+\ldots+\frac{1}{\varepsilon_I}k_I\left(\frac{t}{\varepsilon_I}\right)$$

where $k_i\in L^1(0,\infty)$ are "well-behaved", and ε_i are scalars with $0<\varepsilon_I<\ldots<\varepsilon_1\leqq 1$, $\varepsilon_I/\varepsilon_1 \ll 1$; cf., e.g., Hoppensteadt (1983) and the applications quoted there. In contrast to quadrature methods which are based on pointwise evaluation of the kernel k, no difficulties are encountered with operational quadratures. Here the accuracy even improves when ε_i become small, irrespective of the ratio ε_i/h.

To see this, assume (1.5) with $c<0$ (so that $F(s)$ is exponentially stable) and (1.10). Consider

$$F(\varepsilon s)\, g(x)=\frac{1}{\varepsilon}\int_0^x f\left(\frac{t}{\varepsilon}\right) g(x-t)\, dt \tag{9.1}$$

and its discretization

$$F(\varepsilon s_h)\, g(x)=\sum_{0\leqq jh\leqq x} \omega_j(h/\varepsilon)\, g(x-jh), \tag{9.2}$$

where, in accordance with the notation (1.12), $\omega_j(h/\varepsilon)$ are the coefficients of

$$F(\varepsilon\,\delta(\zeta)/h)=\sum_{j=0}^{\infty}\omega_j(h/\varepsilon)\,\zeta^j.$$

It is well-known that for $g\in C^1$

$$F(\varepsilon\, s)\, g(x)\to F(0)\cdot g(x)=\int_0^{\infty} f(t)\,dt\cdot g(x) \qquad \text{as } \varepsilon\to 0+.$$

For fixed $h>0$ and $x>0$ we have also

$$F(\varepsilon\, s_h)\, g(x)\to F(0)\cdot g(x) \qquad \text{as } \varepsilon\to 0+,$$

which follows immediately upon taking the limit in the finite sum (9.2).

This simple result can be considerably sharpened. We assume

$$g\in C^{p+1}[0,\infty), \qquad g^{(p+1)} \text{ bounded on } [0,\infty). \tag{9.3}$$

Theorem 9.1. *Assume* (1.5) *with* $c<0$, (1.10) *and* (9.3). *Then*

$$|F(\varepsilon\, s_h)\, g(x)-F(\varepsilon\, s)\, g(x)|\leqq C\cdot\varepsilon\cdot h^p$$

where the constant C *is independent of* $h\in(0,1]$, ε *with* $0<\varepsilon^m\leqq h$ *for some fixed, arbitrary* $m>0$, *and* $x\in[x_0,\infty)$ *with fixed* $x_0>0$.

Remark. The behaviour near 0 is easily seen from the estimate (9.4) below and formula (3.1).

Proof. By (1.13),

$$F(\varepsilon\, s_h)\, g(x)=\frac{1}{2\pi i}\int_{\Gamma} F(\lambda)\cdot(\varepsilon\, s_h-\lambda)^{-1}\, g(x)\cdot d\lambda,$$

and the same formula also holds with s formally in place of s_h.

By Lemma 2.1 we have for $q=0,1,\ldots,p$

$$\left|\left(s_h-\frac{\lambda}{\varepsilon}\right)^{-1} t^q(x)-\left(s-\frac{\lambda}{\varepsilon}\right)^{-1} t^q(x)\right|\leqq C\cdot\frac{\varepsilon}{|\lambda|}\left(h\,|e^{\kappa x\lambda/\varepsilon}|+\frac{\varepsilon}{|\lambda|}\,\rho^{x/h}\right)\cdot h^{q-1}$$

for some $0<\kappa<1$, $0<\rho<1$. Hence

$$|F(\varepsilon\, s_h)\, t^q(x)-F(\varepsilon\, s)\, t^q(x)|\leqq C\cdot(h\cdot\tilde\rho^{x/\varepsilon}+\varepsilon\cdot\tilde\rho^{x/h})\cdot h^{q-1} \qquad (q=0,1,\ldots,p) \tag{9.4}$$

with $\tilde\rho=\max\{e^{\kappa c},\rho\}<1$ and C independent of $\varepsilon\in(0,1]$, $h\in(0,1]$ and $x\geqq 0$. The result now follows from (2.1), (2.2), and (2.3) with $p+1$ instead of p. □

10. A Boundary Integral Equation for an Absorption-Diffusion Problem

The following example arises in chemical absorption kinetics, see Gelbin (1985). Find $y(t)$ given by the coupled system of ordinary differential equation and diffusion equation

$$\frac{dy}{dt}(t) = -\alpha \frac{\partial u}{\partial r}(1, t), \qquad y(0) = y_0 \tag{10.1}$$

where $u(r, t)$ satisfies

$$\frac{\partial u}{\partial t} = \beta\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) \quad \text{in } 0<r<1, \ t>0 \tag{10.2}$$

with nonlinear boundary conditions (b smooth)

$$\begin{aligned} u(1,t) &= b(y(t)), \qquad t>0 \\ \frac{\partial u}{\partial r}(0,t) &= 0, \qquad\qquad t>0 \end{aligned} \tag{10.3}$$

and initial condition

$$u(r,0) = 0, \qquad 0<r<1. \tag{10.4}$$

Boundary and initial conditions are incompatible, $b(y_0) > 0$.

Here $u(r, t)$ represents a concentration profile in a spherical absorbing particle, and $y(t)$ is the concentration in the surrounding aqueous solution. (Actually one is interested in the inverse problem of determining the positive constants α, β from measurements of $y(t)$.)

The above system can be reduced to a single Volterra integral equation, roughly by the following (standard) arguments: Taking Laplace transforms in (10.2)–(10.4) leads to the boundary value problem

$$sU = \beta\left(U'' + \frac{2}{r}U'\right), \qquad U'(0,s) = 0, \qquad U(1,s) = B(s) \qquad \left(' = \frac{\partial}{\partial r}\right)$$

which can be solved analytically for U. In particular, one obtains

$$U'(1,s) = \frac{s}{\beta} K(s) \cdot B(s) \tag{10.5}$$

with

$$K(s) = \frac{1}{\sqrt{s/\beta}\tanh\sqrt{s/\beta}} - \frac{1}{s/\beta}. \tag{10.6}$$

By (10.1) the Laplace transform of $y(t)$ satisfies

$$s \cdot Y(s) - y_0 = -\alpha U'(1,s).$$

Inserting (10.5) and applying the inverse transform gives the (weakly singular) Volterra integral equation

$$y(t) = y_0 - \frac{\alpha}{\beta}\int_0^t k(t-\tau)\, b(y(\tau))\, d\tau, \qquad t>0, \tag{10.7}$$

where the Laplace transform $K(s)$ of the kernel $k(t)$ is known from (10.6), rather than the kernel itself.

This equation has been discretized by

$$y_n = y_0 - \frac{\alpha}{\beta}\left(\sum_{j=0}^{n} \omega_{n-j}(h)\, b(y_j) + \sum_{j=0}^{4} w_{nj}(h)\, b(y_j)\right), \qquad n \geqq 1 \tag{10.8}$$

where

$$\sum_0^\infty \omega_n(h)\, \zeta^n = K(\delta(\zeta)/h)$$

with

$$\delta(\zeta) = (1-\zeta) + \tfrac{1}{2}(1-\zeta)^2 + \tfrac{1}{3}(1-\zeta)^3,$$

the third order backward differentiation formula. The correction quadrature weights $w_{nj}(h)$ have been chosen such that for $\gamma = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$

$$\sum_{j=0}^{n} \omega_{n-j}(h) \cdot (j\,h)^\gamma + \sum_{j=0}^{4} w_{nj}(h) \cdot (j\,h)^\gamma = \int_0^t k(t-\tau)\, \tau^\gamma\, d\tau \qquad (t = n\,h). \tag{10.9}$$

Method (10.8) yields a 4-dimensional nonlinear system of equations for the starting values $y_1, \ldots, y_4$. The approximations y_n for $n \geqq 5$ can then be computed one after another by solving a 1-dimensional nonlinear equation at each step.

The order of convergence is that of the underlying multistep method:

Theorem 10.1. *The error of the above method satisfies*

$$y_n - y(n\,h) = O(h^3) \qquad \textit{uniformly for } 0 \leqq n\,h \leqq \text{const}.$$

Proof. From the asymptotic behaviour of $K(s)$ for $|s| \to \infty$, $\operatorname{Re} s \geqq 0$

$$K(s) = (s/\beta)^{-1/2} - (s/\beta)^{-1} + O(|s|^{-M}) \qquad (M > 0 \text{ arbitrarily large}) \tag{10.10}$$

one derives the behaviour of $k(t)$ for $t \to 0+$ (see Doetsch (1955), p. 174):

$$k(t) = \sqrt{\frac{\beta}{\pi\,t}} - \beta + O(t^{M-1}).$$

Picard iteration in (10.7) then shows that the solution $y(t)$ has at $t=0$ an asymptotic expansion in powers of $\sqrt{t}$. The correction weights have been constructed so that for such a solution the consistency error of method (10.8) is $O(h^3)$. This follows from Theorem 5.2 with the proof of Corollary 3.2. The result is then obtained using, e.g., the proof of Theorem 1 in Lubich (1985). □

The convolution quadrature weights have been computed with the FFT techniques described in Sect. 7. The values of the right-hand side of (10.9) were obtained by using for $\beta t \geqq 0.1$ Talbot's (1979) method for the numerical inversion of the Laplace transform $K(s)\, s^{-1-\gamma} \cdot \Gamma(1+\gamma)$. For $\beta t < 0.1$ we have instead analytically inverted the simple transform $((s/\beta)^{-\frac{1}{2}} - (s/\beta)^{-1}) \cdot s^{-1-\gamma} \cdot \Gamma(1+\gamma)$

(obtained from the approximation (10.10) of $K(s)$) which gives more accurate approximations in this range of βt.

We give numerical results for

$$y_0=10, \quad \alpha=1, \quad \beta=10^{-2}, \quad b(y)=y/(1+y^{0.75}).$$

In the interval $[0, 10]$ the solution decreases monotonically from $y(0)=10$ to $y(10)=0.50522$. The results at $t=2$ are given below.

Table 1

h	Numerical solution
0.4	1.042462948
0.1	1.043427639
0.025	1.043427277

Finally we remark that the numerical solution y_n is still closely related to the original problem. Apart from the correction quadrature (which can be viewed as a perturbation), it can be interpreted as the solution of the semi-discretization in time which is obtained by applying a linear multistep method directly to (10.1)–(10.4). This can be seen by repeating the arguments which led to (10.7) with s replaced by s_h.

11. Wiener-Hopf Integral Equations

We consider the Wiener-Hopf integral equation with symmetric kernel,

$$y(x)=f(x)+\int_0^\infty k(|x-t|)\, y(t)\, dt, \qquad x\geqq 0 \tag{11.1}$$

with real-valued $k\in L^1(0, \infty)$. (k may be unbounded at 0 throughout this section.) We assume that the Laplace transform $K(s)$ of $k(t)$ satisfies the following:

$$\text{There exists } \alpha>0 \text{ such that } 1-2\operatorname{Re} K(s)\geqq\alpha \text{ for } \operatorname{Re} s\geqq 0. \tag{11.2}$$

The following facts are known from (or can easily be derived from) the classical work of Krein (1958), or Gochberg and Feldman (1974): The condition (11.2) is necessary and sufficient that for every $f\in L^2(0, \infty)$ the Eq. (11.1) has a unique solution $y\in L^2(0, \infty)$. (The same holds for the spaces $L^p(0, \infty)$ with $1\leqq p\leqq\infty$, and $C_0[0, \infty)$.) Moreover, one has the estimate

$$\|y\|_{L^2(0,\infty)}\leqq\frac{1}{\alpha}\|f\|_{L^2(0,\infty)}. \tag{11.3}$$

If $k(t)=O(e^{-\beta t})$ as $t\to\infty$ for some $\beta>0$, one also has for sufficiently small $\gamma>0$ that

$$y(t)=O(e^{-\gamma t}) \quad \text{whenever} \quad f(t)=O(e^{-\gamma t}) \quad \text{as} \quad t\to\infty. \tag{11.4}$$

We now consider the truncated equation

$$y^T(x)=f(x)+\int_0^T k(|x-t|)\,y^T(t)\,dt, \qquad 0\leqq x\leqq T. \tag{11.5}$$

It can be shown (cf. the discrete analogue Theorem 11.2 below) that for every $T>0$ and $f\in L^2(0,T)$ the Eq. (11.5) has a unique solution $y^T\in L^2(0,T)$, which satisfies

$$\|y^T\|_{L^2(0,T)}\leqq\frac{1}{\alpha}\|f\|_{L^2(0,T)}. \tag{11.6}$$

Subtracting Eq. (11.1) and (11.5) we have

$$y(x)-y^T(x)=\int_T^\infty k(|x-t|)\,y(t)\,dt+\int_0^T k(|x-t|)\,(y(t)-y^T(t))\,dt, \qquad 0\leqq x\leqq T. \tag{11.7}$$

Let M denote the bound

$$|2\,\mathrm{Re}\,K(s)|\leqq M \qquad \text{for } \mathrm{Re}\,s\geqq 0. \tag{11.8}$$

Applying the estimate (11.6) to the truncated Wiener-Hopf equation (11.7) for $y-y^T$ and using Parseval's formula we obtain

$$\|y-y^T\|_{L^2(0,T)}\leqq\frac{M}{\alpha}\|y\|_{L^2(T,\infty)}. \tag{11.9}$$

For sufficiently large T the solution y^T of (11.5) is therefore a good approximation to the solution y of (11.1). In principle, (11.5) could be discretized by any of the existing numerical methods for Fredholm integral equations of the second kind. It may, however, be desirable to choose a method whose error does not blow up as T becomes large. As we will show in the following, this is easily achieved with operational quadratures. Other approaches to the approximation of Wiener-Hopf integral equations have been given by Gochberg and Feldman (1974), Stenger (1972), Sloan and Spence (1986).

We consider a multistep method (1.10) which is A-stable, i.e. (cf. Dahlquist (1963)),

$$\mathrm{Re}\,\delta(\zeta)\geqq 0 \qquad \text{for } |\zeta|\leqq 1. \tag{11.10}$$

We expand

$$K(\delta(\zeta)/h)=\sum_{n=0}^\infty \omega_n(h)\,\zeta^n \tag{11.11}$$

and put

$$\tilde\omega_0(h)=2\omega_0(h), \qquad \tilde\omega_n(h)=\omega_{|n|}(h) \qquad \text{for } n\in\mathbb{Z},\ n\neq 0. \tag{11.12}$$

We then discretize (11.1) by the discrete Wiener-Hopf equation

$$y_n=f(nh)+\sum_{j=0}^\infty \tilde\omega_{n-j}(h)\,y_j, \qquad n\geqq 0, \tag{11.13}$$

where y_n is to approximate $y(nh)$. All the following results rely on the crucial fact that

$$1-\sum_{n=-\infty}^{\infty}\tilde{\omega}_n(h)\,\zeta^n=1-2\,\mathrm{Re}\,K(\delta(\zeta)/h)\geqq\alpha>0 \qquad \text{for } |\zeta|=1, \tag{11.14}$$

with α of (11.2). The properties of the discretization (11.13) and its truncated version are determined by (11.14) in much the same way as those of the continuous problem by (11.2).

For ease of notation we identify the sequences $y_h=\{y_n\}_0^\infty$ and $f_h=\{f(nh)\}_0^\infty$ with step functons on $[0,\infty)$ and thus write

$$\|y_h\|_{L^2(0,\infty)}=\left(h\sum_0^\infty y_n^2\right)^{1/2}$$

and correspondingly for f_h. We have the following discrete analogue of (11.3).

Theorem 11.1. *Consider the discretization* (11.10)–(11.13) *of the Wiener-Hopf integral equation* (11.1), (11.2). *Then the Eq.* (11.13) *has for every* $f_h\in l^2\,(\subset L^2(0,\infty))$ *a unique solution* $y_h\in l^2$, *which satisfies*

$$\|y_h\|_{L^2(0,\infty)}\leqq\frac{1}{\alpha}\,\|f_h\|_{L^2(0,\infty)}.$$

Proof. Let A_h denote the discrete Wiener-Hopf operator in (11.13),

$$A_h v=\left\{v_n-\sum_{j=0}^{\infty}\tilde{\omega}_{n-j}(h)\,v_j\right\}_{n=0}^{\infty} \qquad \text{for } v=\{v_n\}_0^\infty\in l^2,$$

and let $(\cdot,\cdot)$ denote the l^2 inner product. Via Parseval's formula

$$(A_h v, w)=\frac{1}{2\pi}\int_0^{2\pi}\left(1-\sum_{-\infty}^{\infty}\tilde{\omega}_n(h)\,e^{-in\tau}\right)\hat{v}(\tau)\,\overline{\hat{w}(\tau)}\,d\tau$$

(with $\hat{v},\hat{w}\in L^2(0,2\pi)$ the Fourier transforms of $v, w\in l^2$) the relation (11.14) implies (with M of (11.8))

$$\begin{aligned} |(A_h v, w)|&\leqq(1+M)\,\|v\|\cdot\|w\| \qquad &&\text{for } v, w\in l^2\\ \alpha\,\|v\|^2&\leqq(A_h v, v) &&\text{for } \quad v\in l^2\end{aligned} \tag{11.15}$$

so that A_h is l^2-elliptic. By the Lax-Milgram lemma, see e.g. Ciarlet (1978), A_h is invertible on l^2, and $\|A_h^{-1}\|\leqq 1/\alpha$. □

Remark. The above derivation of (11.15) is immediate only if $\{\omega_n(h)\}_0^\infty\in l^1$ (e.g., if $k(t)$, $t\,k(t)\in L^1(0,\infty)$). Otherwise the infinite sum in the discrete Wiener-Hopf operator has to be interpreted as

$$A_h v=\lim_{r\to 1-}\left\{v_n-\sum_{j=0}^{\infty}r^{|n-j|}\,\tilde{\omega}_{n-j}(h)\,v_j\right\}_{n=0}^{\infty}.$$

Since $\sum_{-\infty}^{\infty} r^{|n|}\,\tilde{\omega}_n(h)\,\zeta^n \to 2\,\mathrm{Re}\,K(\delta(\zeta)/h)$ uniformly on $|\zeta|=1$ as $r\to 1-$, the limit exists in the l^2 sense, and one obtains again (11.15).

Next we study the truncated discrete equation

$$y_n^T = f(nh) + \sum_{j=0}^{N} \tilde{\omega}_{n-j}(h)\, y_j^T, \qquad 0\leqq n\leqq N,\ Nh=T. \tag{11.16}$$

For its solution $y_h^T=\{y_n^T\}_0^N$ we have the following analogue of (11.6).

Theorem 11.2. *Consider the discretization* (11.10)–(11.12), (11.16) *of the truncated Wiener-Hopf equation* (11.2), (11.5). *Then the Eq.* (11.16) *has for every* $f_h=\{f_n\}_0^N$ *a unique solution* y_h^T, *which satisfies*

$$\|y_h^T\|_{L^2(0,T)} \leqq \frac{1}{\alpha}\,\|f_h\|_{L^2(0,T)}.$$

Proof. y_h^T is the solution of the Toeplitz system with matrix

$$A=\begin{bmatrix} 1-2\omega_0(h) & -\omega_1(h) & \dots & -\omega_N(h) \\ -\omega_1(h) & 1-2\omega_0(h) & & \vdots \\ \vdots & & & -\omega_1(h) \\ -\omega_N(h) & \dots & -\omega_1(h) & 1-2\omega_0(h) \end{bmatrix}.$$

Taking $v=(v_0,\dots,v_N,0,0,\dots)$ in (11.15) shows that $A-\alpha I$ is nonnegative definite. Hence A is invertible, with $\|A^{-1}\|\leqq 1/\alpha$ (and condition number $\kappa(A)=\|A\|\cdot\|A^{-1}\|\leqq(1+M)/\alpha$, independent of h and T). □

Theorems 11.1 and 11.2 show the *stability* of the discrete approximation. From the error equation one derives with their help the error estimate

$$\|y_h^T - y\|_{L^2(0,T)} \leqq \frac{1}{\alpha}\,\|d_h\|_{L^2(0,\infty)} + \frac{M}{\alpha}\,\|y_h\|_{L^2(T,\infty)} \tag{11.17}$$

where $y=\{y(nh)\}_0^\infty$ is the sampled solution of (11.1), y_h^T is the solution of (11.16), y_h that of (11.13), and $d_h=\{d_n\}_0^\infty$ is the consistency error defined by

$$d_n=\sum_{j=0}^{\infty}\tilde{\omega}_{n-j}(h)\,y(jh) - \int_0^\infty k(|x-t|)\,y(t)\,dt, \qquad x=nh.$$

The last term in (11.17) can be further estimated by

$$\|y_h\|_{L^2(T,\infty)} \leqq \|y\|_{L^2(T,\infty)} + \frac{1}{\alpha}\,\|d_h\|_{L^2(0,\infty)}$$

$\left(\text{since } \|y_h - y\|_{L^2(0,\infty)} \leqq \frac{1}{\alpha}\,\|d_h\|_{L^2(0,\infty)} \text{ by Theorem 11.1}\right).$

From the estimate (11.17) one can derive *convergence* of the approximation (11.16) under suitable assumptions in (11.1), e.g.: $K(s)$ satisfies (1.4) with $c<0$, f is continuous and exponentially decaying on $[0, \infty)$.

Remark. a) L^2-convergence as $h \to 0$, $T \to \infty$ for

$$y_h^T(x) = f(x) + \sum_{x-T \leqq jh \leqq x} \tilde{\omega}_j(h)\, y_h^T(x-jh), \qquad 0 \leqq x \leqq T,$$

can be shown under the mere assumptions $k(t)$, $t\,k(t) \in L^1(0, \infty)$, $f \in L^2(0, \infty)$.

b) Eggermont (1987) has recently shown how L^∞ error estimates can be derived on the basis of the present L^2 estimates.

As the proof of Theorem 11.2 shows, (11.16) is a symmetric, positive definite Toeplitz system for y_h^T. Such systems of linear equations can be solved efficiently, by the Levinson algorithm (see e.g. Golub and Van Loan (1983), Sect. 5.7) or using fast iterative techniques (e.g. conjugate gradient method with FFT).

In order to improve the accuracy, it may be useful to add a correction quadrature to (11.16). With suitable weights $w_{nj}(h)$ (cf. Corollaries 3.2, 4.2, and Sect. 10), an improved approximation takes the form

$$y_n^T = f(nh) + \sum_{j=0}^{N} \tilde{\omega}_{n-j}(h)\, y_j^T + \sum_{j=0}^{m} w_{nj}(h)\, y_j^T,$$
$$0 \leqq n \leqq N, \quad Nh = T, \quad m \text{ fixed.} \tag{11.18}$$

For small h the condition number of the system (11.18) is close to that of (11.16), whereas the consistency error d_h is reduced. The computational complexity for the solution of (11.18) is still of the same magnitude as that of the pure Toeplitz system (11.16).

Remark. Under additional conditions on the Laplace transform $K(s)$ the A-stability assumption (11.10) (which by Dahlquist's (1963) order barrier implies that the order, p, of the discretization cannot exceed 2) can be dispensed with. If $K(s)$ satisfies (1.4) and in addition to (11.2) also the condition

$$1 - 2\,\mathrm{Re}\, K(s) \geqq \alpha > 0 \quad \text{for } |\arg(s-\sigma)| \leqq \pi - \theta \quad \left(\text{some } \sigma \in \mathbb{R},\ \theta < \frac{\pi}{2}\right),$$

then similar results as above can be obtained (for sufficiently small h only, if $\sigma > 0$) with $A(\theta)$-stable multistep methods.

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References

Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. Amsterdam: North Holland 1978

Corduneanu, C.: Integral Equations and Stability of Feedback Systems. New York: Academic Press 1973

Dahlquist, G.: A special stability problem for linear multistep methods. BIT **3**, 27–43 (1963)

Desoer, C.A., Vidyasagar, M.: Feedback Systems: Input-Output Properties. New York: Academic Press 1975
Doetsch, G.: Handbuch der Laplace-Transformation, Vol. II. Basel: Birkhäuser 1955
Eggermont, P.P.B.: Uniform error estimates of Galerkin methods for monotone Abel-Nolterra integral equations on the half-line. Report, Univ. of Delaware, Newark (1987)
Gelbin, D.: Measuring absorption rates from an aqueous solution. AICHE J. (to appear)
Gienger, G.: On convolution quadratures and their application to Fredholm integral equations. Dissertation, Univ. Heidelberg (1987)
Gochberg, I.Z., Feldman, I.A.: Faltungsgleichungen und Projektionsverfahren zu ihrer Lösung. Stuttgart: Birkhäuser 1974
Golub, G.H., Van Loan, C.F.: Matrix Computations. Oxford: North Oxford Academic 1983
Hairer, E., Lubich, Ch., Schlichte, M.: Fast numerical solution of weakly singular Volterra integral equations. Report, Université de Genève (1986) J. Comput. Appl. Math. (to appear)
Hairer, E., Maass, P.: Numerical methods for singular nonlinear integro-differential equations. Appl. Numer. Math. **3**, 243–256 (1987)
Hannsgen, K.B., Herdman, T.L., Stech, H.W., Wheeler, R.L. (eds.): Volterra and Functional Differential Equations. New York: Marcel Dekker 1982
Henrici, P.: Fast Fourier methods in computational complex analysis. SIAM Rev. **21**, 487–527 (1979)
Hoppensteadt, F.C.: An algorithm for approximate solutions to weakly filtered synchronous control systems and nonlinear renewal processes. SIAM J. Appl. Math. **43**, 834–843 (1983)
Krein, M.G.: Integral equations on a half-line with kernel depending upon the difference of the arguments. Uspekhi Mat. Nauk **13**, 3–120 (1958) (engl.: AMS Translations **22**, 163–288 (1962))
Londen, S.O., Staffans, O.J. (eds.): Volterra Equations. Springer LNM, Vol. 737 (1979)
Lopez-Castillo, J.M., Jay-Gerin, J.P., Tannous, C.: Dynamics of electron delocalization: an exact treatment. Europhys. Lett. (to appear)
Lubich, C.: Fractional linear multistep methods for Abel-Volterra integral equations of the second kind. Math. Comput. **45**, 463–469 (1985)
Lubich, C.: Fractional linear multistep methods for Abel-Volterra integral equations of the first kind. IMA J. Numer. Anal. **7**, 97–106 (1987)
Lubich, C.: Convolution Quadrature and Discretized Operational Calculus. I. Numer. Math. **52**, 129–145 (1988)
Sanz-Serna, J.M.: A numerical method for a partial integro-differential equation. SIAM J. Numer. Anal. (to appear)
Sloan, I.H., Spence, A.: Projection methods for integral equations on the half-line. IMA J. Numer. Anal. **6**, 153–172 (1986)
Stenger, F.: The approximate solution of Wiener-Hopf integral equations. J. Math. Anal. Appl. **37**, 687–724 (1972)
Talbot, A.: The accurate numerical inversion of Laplace transforms. J. Inst. Math. Appl. **23**, 97–120 (1979)

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