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Arithmetic Distance Functions and Height Functions in Diophantine Geometry

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In virtually all areas of Diophantine Geometry, the theory of height functions plays a crucial role. This theory associates to each (Cartier) divisor D on a projective variety V a function h_D mapping the group of rational points of V to the real numbers. This function, which is defined up to a bounded function on V, has very nice functorial properties. For example, it is linear in D and depends only on the linear equivalence class of D. As a consequence, any relation between divisor classes, such as the theorem of the square for abelian varieties, will yield a corresponding relation for height functions.

Further, it is possible to write the height function h_D as a sum of local height functions $\lambda_D(\cdot\,;v)$, where v ranges over the distinct absolute values of the given field. (These local heights are also known as logarithmic distance functions or Weil functions.) Each λ_D is defined away from the support of D, and gives a measure of the v-adic distance from the given point of V to the divisor D.

In this paper we propose to define local height functions λ_X for every closed subscheme X of a projective variety V. As above, $\lambda_X(\cdot\,;v)$ will give a measure of the v-adic distance from a point of V to X. These functions will have many nice functorial properties; for example, $\lambda_{X\cap Y}$ will equal the minimum of λ_X and λ_Y. In this way, relations between closed subschemes (or equivalently, between ideal sheaves) will yield relations between local height functions. We will thus be able to convert geometric statements into arithmetic statements relatively painlessly.

More generally, we will deal with the case that V is merely quasi-projective by defining a function $\lambda_{\partial V}(\cdot\,;v)$ which will measure the v-adic distance to the "boundary" of V. We will then be able to assign to each closed subscheme X of V a local height function λ_X which will be well defined up to addition of a multiple of $\lambda_{\partial V}$. This extra generality is useful, for example, when one has a complete family of varieties and wishes to discard the "bad" fibers.

As one application of the machinery that we have developed, we prove a quantitative version of the inverse function theorem. Thus given a finite map, we use local height functions to describe how far away from the ramification locus

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(and boundary) one must stay so as to define a local inverse; and when the inverse is defined, we show that it is essentially distance preserving. In the final section, we consider the behaviour of local height functions on proper families of varieties, especially abelian varieties. Finally, in an appendix, we describe (without proof) the relation between our local height functions and sections to metrized vector bundles.

Our original motivation for studying such functions was the desire to give uniform estimates for some of the important finiteness theorems in Diophantine geometry, such as Siegel's theorem. This theorem says that there are only finitely many integral points on any curve having positive genus. In a subsequent paper we will study Siegel's theorem for an algebraic family of curves; and using local height functions as one of our tools, we will describe how the finite set of integral points may vary for different curves in the family.

More generally, our theory of local height functions should be useful anytime one is studying a Diophantine problem on an algebraic family of varieties and wishes to keep track of how the solutions vary in terms of the chosen point in the parameter space. In order to make these tools as accessible as possible, the author has tried to keep the algebro-geometric prerequisites to a minimum (almost everything used is in [3]).

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1. Local Height Functions Associated to Divisors

In this section we set notation and quickly review the facts about local height functions associated to divisors which will be used in the sequel. All of this material can be found in [4].

K will always be a field equipped with a proper set of absolute values M_K. We fix an algebraic closure K^a for K, and let $M = M(K^a)$ be the set of absolute values on K^a extending those on K. We write our absolute values additively; thus if $v \in M$, then the triangle inequality reads

$$v(x+y) \geqq \min\{v(x), v(y)\} + \log(2),$$

the $\log(2)$ being superfluous if (and only if) v is non-archimedean.

All of our varieties will be assumed to be irreducible and defined over K. If V is a variety, we use V also to denote $V(K^a)$, the set of points of V defined over K^a. A divisor on V will always mean a Cartier divisor. (I.e. A positive divisor is a locally principal codimension 1 subscheme. See [3, II.6].)

Definition. The set $\mathscr{C}$ of *(positive) M_K-constants* is defined by

$$\mathscr{C} = \{\text{functions } \gamma : M_K \to [0, \infty) | \gamma(v) = 0 \text{ for almost all } v \in M_K\}.$$

We extend each $\gamma \in \mathscr{C}$ to a map $\gamma : M \to [0, \infty)$ by setting $\gamma(v) = \gamma(v|_K)$. The set $\mathscr{F}(V)$ of *(positive) M_K-functions* on a variety V is given by

$$\mathscr{F}(V) = \{\lambda : V \times M \to [0, \infty] | \text{for all } P \in V, \text{ either } \lambda(P; \cdot) \in \mathscr{C} \text{ or } \lambda(P; \cdot) = \infty\}.$$

We define an equivalence relation $\sim$ on $\mathscr{F}(V)$ by setting $\lambda_1 \sim \lambda_2$ if there is an M_K-constant $\gamma \in \mathscr{C}$ such that

$$\lambda_1(P;v)-\gamma(v) \leqq \lambda_2(P;v) \leqq \lambda_1(P;v)+\gamma(v) \quad \text{for all} \quad (P,v) \in V \times M.$$

(If $\lambda \sim 0$, then we say that λ is *M_K-bounded.*) Then the set $\mathscr{H}(V)$ of *local height functions* on V is the quotient

$$\mathscr{H}(V)=\mathscr{F}(V)/\sim.$$

Notice we can define a partial ordering on $\mathscr{H}(V)$ by setting $\lambda_1 \leqq \lambda_2$ if there are representatives for λ_1 and λ_2 in $\mathscr{F}(V)$ (which we again denote by λ_1 and λ_2) and an M_K-constant $\gamma \in \mathscr{C}$ such that $\lambda_1(P;v) \leqq \lambda_2(P;v)+\gamma(v)$ for all $(P,v) \in V \times M$.

We recall from [4] that a subset $B \subset V \times M$ is called *affine M_K-bounded* if there is an affine open subset $U \subset V$ with affine coordinates $x_1, \ldots, x_n$ and an M_K-constant $\gamma \in \mathscr{C}$ such that $B \subset U \times M$ and

$$\min\{v(x_i(P))\} \geqq -\gamma(v) \quad \text{for all} \quad (P,v) \in B.$$

Clearly the choice of affine coordinates for U will not affect whether or not a given subset $B \subset U \times M$ is bounded. However, if U' is an affine subset of U such that $B \subset U' \times M$, it need not be true that B will be bounded with respect to affine coordinates for U'. (For example, take $U=\mathbb{A}^1$ and $B=\{(x,v): 0<|x|_v \leqq 1\}$. If $U'=\mathbb{A}^1-\{0\}$, then $\{x, 1/x\}$ are affine coordinates for U', so B will not be bounded as a subset of U'.) If $B \subset U \times M$ is bounded for affine coordinates on U, we will say that B is *M_K-bounded inside U.*

In order to construct our generalized local height functions, we will use the following result for divisors, which is essentially due to Weil.

Theorem 1.1. *Let V be a projective variety, and let* $\mathrm{Div}^+(V)$ *be the set of positive (Cartier) divisors on V. There exists a unique map*

$$\lambda: \mathrm{Div}^+(V) \longrightarrow \mathscr{H}(V)$$

with the following properties:

(a) *For all $D, E \in \mathrm{Div}^+(V)$,*

$$\lambda_{D+E}=\lambda_D+\lambda_E.$$

(b) *Let $D \in \mathrm{Div}^+(V)$, $U \subset V$ an affine open on which D is principal, and let $B \subset U \times M$ be M_K-bounded inside U. Write $D|_U=(f)$ for some function $f \in \Gamma(U, \mathcal{O}_V)$. Then*

$$\lambda_D \sim v \circ f \quad \textit{on } B.$$

[*I.e. There is an M_K-constant $\gamma \in \mathscr{C}$ such that for all $(P,v) \in B$,*

$$\lambda_D(P,v)-\gamma(v) \leqq v \circ f(P) \leqq \lambda_D(P,v)+\gamma(v).]$$

Proof. This (and more) is contained in [4] Chapter 10.

2. Local Height Functions Associated to Closed Subschemes

Let V be a variety, and let $Z(V)$ denote the set of closed subschemes of V. Note that the subschemes $X \in Z(V)$ are in one-to-one correspondence with ideal sheaves $\mathscr{I}_X$

$\subset \mathcal{O}_V$ (cf. [3, II.5.9]). We will often implicitly identify a subscheme X with its ideal sheaf $\mathcal{I}_X$. Our goal is to assign to each $X \in Z(V)$ a local height function $\lambda_X \in \mathcal{H}(V)$ (at least when V is projective). Intuitively, for each $(P, v) \in V \times M$, we want

$$\lambda_X(P; v) = -\log(v\text{-adic distance from } P \text{ to } X).$$

In order to describe the desired functorial properties of λ_X, we start by defining a number of natural operations on $Z(V)$.

Definition. Let $X, Y \in Z(V)$.

(i) The *sum of X and Y*, denoted $X + Y$, is the subscheme of V with ideal sheaf

$$\mathcal{I}_{X+Y} = \mathcal{I}_X \mathcal{I}_Y .$$

(ii) The *intersection of X and Y*, denoted $X \cap Y$, is the subscheme of V with ideal sheaf

$$\mathcal{I}_{X \cap Y} = \mathcal{I}_X + \mathcal{I}_Y .$$

(iii) The *union of X and Y*, denoted $X \cup Y$, is the subscheme of V with ideal sheaf

$$\mathcal{I}_{X \cap Y} = \mathcal{I}_X \cap \mathcal{I}_Y .$$

(iv) *Y is contained in X*, denoted $Y \subset X$, if

$$\mathcal{I}_X \subset \mathcal{I}_Y .$$

(v) Let $\varphi : W \to V$ be a morphism of varieties. The *inverse image of X*, denoted $\varphi^* X$, is the subscheme of W with ideal sheaf

$$\mathcal{I}_{\varphi^* X} = \varphi^* \mathcal{I}_X .$$

(We are using the notation $\varphi * \mathcal{I}_X$ to denote the *inverse image ideal sheaf of $\mathcal{I}_X$*. More standard notation for this sheaf would be $\varphi^{-1} \mathcal{I}_X \cdot \mathcal{O}_W$, which is the image in $\mathcal{O}_W$ of the usual sheaf $\varphi^* \mathcal{I}_X$. See [3, II.7.12.2] for a fuller discussion.)

We are now ready to prove the existence of local height functions corresponding to each closed subscheme of V, and to describe their functorial properties.

Theorem 2.1. *Let V be a projective variety. There exists a unique map*

$$\lambda : Z(V) \longrightarrow \mathcal{H}(V)$$

satisfying the following conditions:

(a) *If $D \in Z(V)$ is a positive divisor (i.e. $D \in \operatorname{Div}^+(V)$), then λ_D is the usual (Weil) function associated to D given by Theorem 1.1.*

(b) *For all $X, Y \in Z(V)$,*

$$\lambda_{X \cap Y} = \min\{\lambda_X, \lambda_Y\} .$$

The map λ also has the following properties:

(c) *For all $X, Y \in Z(V)$,*

$$\lambda_{X+Y} = \lambda_X + \lambda_Y .$$

(d) *If $X, Y \in Z(V)$ satisfy $X \subset Y$, then*

$$\lambda_X \leqq \lambda_Y .$$

(e) *For all* $X, Y \in Z(V)$,

$$\max\{\lambda_X, \lambda_Y\} \leqq \lambda_{X \cap Y} \leqq \lambda_X + \lambda_Y.$$

Further, either of the inequalities may be an equality.

(f) *If* $X, Y \in Z(V)$ *satisfy* Support$(X) \subset$ Support(Y), *then there exists a constant c such that*

$$\lambda_X \leqq c\lambda_Y.$$

(g) *Let* $X, Y \in Z(V)$, *let* $U \subset V$ *be an open set, and let* $Z \subset V$ *be the complement of* U. *Suppose that*

$$X|_U \subset Y|_U.$$

Then there exists a constant $c \geqq 0$ *such that*

$$\lambda_X \leqq \lambda_Y + c\lambda_Z.$$

(h) *Let* $\varphi: W \to V$ *be a morphism of varieties, and let* $X \in Z(V)$. *Then*

$$\lambda_{W, \varphi^* X} = \lambda_{V, X} \circ \varphi.$$

(Here we write $\lambda_{V,X}$ *in place of* λ_X *so as to better indicate the underlying variety.)*

Before giving the proof of Theorem 1.1, we prove two lemmas dealing with the intersection of divisors.

Lemma 2.2. *Let* $X \in Z(V)$. *There exist positive divisors* $D_1, \ldots, D_r$ *such that*

$$X = \cap D_i.$$

Proof. Fix an ample invertible sheaf $\mathcal{O}_V(1)$ on V, and choose n sufficiently large so that $\mathcal{I}_X(n)$ is generated by global sections, say by $s_1, \ldots, s_r \in \Gamma(V, \mathcal{I}_X(n))$. Since $\mathcal{I}_X(n) \subset \mathcal{O}_V(n)$, each s_i is a global section of the invertible sheaf $\mathcal{O}_V(n)$, so it has a divisor $(s_i) = D_i \geqq 0$. We claim that

$$\mathcal{I}_X = \mathcal{I}_{D_1} + \ldots + \mathcal{I}_{D_r},$$

which will complete the proof of the lemma.

To prove this equality of sheaves, it suffices to check it on stalks, so let $P \in V$. Pick any $s_0 \in \mathcal{O}_{V,P}$ with $s_0(P) \neq 0$; then the map $s \to s/s_0$ gives an isomorphism $\mathcal{O}_V(n)_P \to \mathcal{O}_{V,P}$. Now $\mathcal{I}_X(n)_P$ maps isomorphically onto $\Sigma(s_i/s_0)\mathcal{O}_{V,P}$, while each $\mathcal{I}_{D_i}(n)$ maps to $(s_i/s_0)\mathcal{O}_{V,P}$. Therefore $\mathcal{I}_{X,P} = \Sigma \mathcal{I}_{D_i,P}$ as desired.

Lemma 2.3. *Let U be an affine open subset of V, and let* $D, D_1, \ldots, D_r$ *be positive divisors on V which are principal on U and satisfy*

$$D|_U \supset \cap D_i|_U.$$

Let $B \subset U \times M$ *be* M_K*-bounded inside U. Then*

$$\lambda_D \geqq \min\{\lambda_{D_i}\} \quad \text{on } B.$$

Proof. Choose functions $f, f_1, f_2, \ldots \in \Gamma(U, \mathcal{O}_V)$ so that

$$D|_U = (f) \quad \text{and} \quad D_i|_U = (f_i).$$

Then from Theorem 1.1,

$$\lambda_D \sim v \circ f \quad \text{and} \quad \lambda_{D_i} \sim v \circ f_i \quad \text{on } B.$$

We now use the fact that $D|_U \supset \cap D_i|_U$. Since D and the D_i's are principal on U, this means that there exist $g_1, g_2, \ldots \in \Gamma(U, \mathcal{O}_V)$ such that

$$f = \Sigma g_i f_i .$$

Hence on B we have the estimate

$$\begin{aligned} v \circ f &= v \circ (\Sigma g_i f_i) \\ &\geqq \min v \circ (g_i f_i) - \gamma \\ &\geqq \min v \circ f_i + \min v \circ g_i - \gamma \\ &\geqq \min v \circ f_i - \gamma' \end{aligned}$$

for some M_K-constants γ, $\gamma' \in \mathscr{C}$. Here the last line follows from [4, Chap. 10, Proposition 1.3], which says that since each g_i is regular on U, each $v \circ g_i$ (restricted to B) is bounded below by an M_K-constant. This implies that

$$\lambda_D \geqq \min\{\lambda_{D_i}\} \quad \text{on } B \text{, as desired.}$$

Proof of Theorem 2.1. The uniqueness of λ follows immediately from Lemma 2.2. For if we write X as an intersection of positive divisors $\cap D_i$, then (a) and (b) imply that $\lambda_X = \min\{\lambda_{D_i}\}$ for functions λ_{D_i} which have already been defined in Theorem 1.1. We are thus forced to define

$$\lambda_X = \min\{\lambda_{D_i}\} .$$

The next step is to check that λ_X is well-defined (in $\mathscr{H}(V)$), independent of the choice of the D_i's. After that, we will verify properties (a)–(g).

Thus suppose that $\{D_i\}$ and $\{E_j\}$ are two sets of effective divisors satisfying $X = \cap D_i = \cap E_j$. We must show that

$$\min\{\lambda_{D_i}\} = \min\{\lambda_{E_j}\} .$$

Since X also equals $(\cap D_i) \cap (\cap E_j)$, it suffices to assume that one set of divisors is a subset of the other; and then by adding on one divisor at a time, we may assume that the larger set has only one more divisor than the smaller. Now, since the minimum over a larger set of numbers is necessarily smaller, this reduces us to the following problem.

If $D, D_1, D_2, \ldots \in \mathrm{Div}^+(V)$ and $D \supset \cap D_i$, then

$$\lambda_D \geqq \min\{\lambda_{D_i}\} .$$

From Lemma 2.3, we know this is true on any set $B \subset V \times M$ which is M_K-bounded inside an affine open subset U of V on which $D, D_1, D_2, \ldots$ are all principal. But using [4, Chap. 10, Proposition 1.2], we see that it is possible to cover $V \times M$ by finitely many sets B of this sort. This completes the proof that λ_X is well-defined.

We now verify that the function $\lambda_X \in \mathscr{H}(V)$, defined as above by $\lambda_X = \min\{\lambda_{D_i}\}$ for any set of effective divisors such that $\cap D_i = X$, satisfies properties (a)–(g).

(a) Clear.

(b) If $X = \cap D_i$ and $Y = \cap E_j$, then $X \cap Y = (\cap D_i) \cap (\cap E_j)$, so

$$\lambda_{X \cap Y} = \min\{\lambda_{D_i}, \lambda_{E_j}\} = \min\{\lambda_X, \lambda_Y\} .$$

(c) Again writing $X = \cap D_i$ and $Y = \cap E_j$, we see that $X + Y = \cap(D_i + E_j)$. Hence

$$\lambda_{X+Y} = \min\{\lambda_{D_i+E_j}\} = \min\{\lambda_{D_i}\} + \min\{\lambda_{E_j}\} = \lambda_X + \lambda_Y.$$

Here we have used [4, Chap. 10, Proposition 2.1], which says that $\lambda_{D+E} = \lambda_D + \lambda_E$ for divisors D and E.

(d) If $X \subset Y$, then $X = X \cap Y$, so from (b)

$$\lambda_X = \lambda_{X \cup Y} = \min\{\lambda_X, \lambda_Y\}.$$

(e) Since $X, Y \subset X \cup Y \subset X + Y$, applying (c) and (d) yields

$$\max\{\lambda_X, \lambda_Y\} \leqq \lambda_{X \cap Y} \leqq \lambda_X + \lambda_Y.$$

Further, if $X = Y$, so $X = X \cup Y$, then the lower inequality will be an equality; while if for example X and Y are distinct irreducible subvarieties of V which intersect properly, then $X \cup Y = X + Y$, so the upper inequality will be an equality.

(f) If Support$(X) \subset$ Support(Y), then the Nullstellensatz implies that there is an integer $m > 0$ such that $\mathscr{I}_X \supset \mathscr{I}_Y^m$. Thus $X \subset mY$, so using (c) and (d) yields

$$\lambda_X \leqq m\lambda_Y.$$

(g) Using (c) and (d), it suffices to find an integer m such that $X \subset Y + mZ$; or equivalently, such that $\mathscr{I}_X \supset \mathscr{I}_Y \mathscr{I}_Z^m$. Since this may be checked locally, we may assume that $V = \mathrm{Spec}(A)$ is affine, and write $\mathscr{I}_X$, $\mathscr{I}_Y$, $\mathscr{I}_Z$ as ideals of A. Writing $\mathscr{I}_Z = (f_1, \ldots, f_r)$, we see that U is the union of the open sets $U_i = \mathrm{Spec}(A_{f_i})$. Now the given inclusion $X|_U \supset Y|_U$ says precisely that

$$\mathscr{I}_X A_{f_i} \supset \mathscr{I}_Y A_{f_i} \quad \text{for each } 1 \leqq i \leqq r.$$

Hence we can find an integer m such that $\mathscr{I}_X \supset \mathscr{I}_Y f_i^m$ for all i; and so

$$\mathscr{I}_X \supset \mathscr{I}_Y(f_1^m, \ldots, f_r^m) \supset \mathscr{I}_Y(f_1, \ldots, f_r)^{m'} = \mathscr{I}_Y \mathscr{I}_Z^{m'}.$$

(Here we may take $m' = rm - r + 1$.)

(h) Write $X = \cap D_i$. Then $\varphi^* X = \cap \varphi^* D_i$, so (f) follows from the corresponding property for divisors [4, Chap. 10, Proposition 2.5].

The next result shows how one can actually compute the local height function corresponding to a closed subscheme.

Proposition 2.4. *Let $X \in Z(V)$, and let $U \subset V$ be an affine open subset such that $\mathscr{I}_X|_U$ is generated by global sections, say by $f_1, \ldots, f_r \in \Gamma(U, \mathscr{I}_X)$. Let $B \subset U \times M$ be M_K-bounded inside U. Then*

$$\lambda_X \sim \min v \circ f_i \quad \textit{on } B.$$

Proof. For each i, let $D_i \geqq 0$ be the divisor of zeros of f_i. By assumption, $X|_U = \cap D_i|_U$. Further, let $Y \in Z(V)$ be the (reduced) subscheme of V whose support is the complement of U. Then by Theorem 2.1 (b, g), there is a constant $c \geqq 0$ such that

$$\lambda_X - c\lambda_Y \leqq \min \lambda_{D_i} \leqq \lambda_X + c\lambda_Y.$$

Further, since $(f_i)|_U = D_i|_U$, Theorem 1.1(b) implies that

$$\lambda_{D_i} \sim v \circ f_i \quad \text{on } B.$$

Thus to complete the proof of the proposition, it suffices to prove that λ_Y restricted to B is bounded by an M_K-constant.

Write $Y = \cap E_j$ as an intersection of positive divisors (Lemma 2.2). Since $Y|_U = 0$, we can apply Lemma 2.3 (with $D = 0$) to conclude that

$$\lambda_0 \geqq \min \lambda_{E_j} = \lambda_Y .$$

But λ_0 is M_K-bounded [4, Chap. 10, Proposition 2.2], which gives the desired result.

Example. It is quite possible for distinct closed subschemes to have the same local height function. For example, let D and E be distinct lines in $\mathbb{P}^2$, and let

$$X = 2D \cap 2E \quad \text{and} \quad Y = 2D \cap 2E \cap (D+E).$$

[If D and E intersect at $x = y = 0$, then locally around $(0, 0)$ we have $\mathcal{I}_X = (x^2, y^2)$ and $\mathcal{I}_Y = (x^2, y^2, xy)$, so X and Y are distinct subschemes.] Using Theorem 2.1 (b, c) we compute

$$\lambda_Y = \min\{2\lambda_D, 2\lambda_E, \lambda_D + \lambda_E\} = \min\{2\lambda_D, 2\lambda_E\} = \lambda_X .$$

(*Exercise.* If $\lambda_X = \lambda_Y$, then $X^{\mathrm{red}} = Y^{\mathrm{red}}$.)

3. Arithmetic Distance Functions

For this section, we let V be a projective variety. Let $P, Q \in V$ and $v \in M$. We wish to define a measure of the v-adic distance from P to Q. More precisely, we want a local height function $\delta \in \mathcal{H}(V \times V)$ which is intuitively defined by

$$\delta(P, Q; v) = -\log(v\text{-adic distance from } P \text{ to } Q).$$

Since any point of V can be considered as a closed subscheme, one possibility is to use $\lambda_P(Q; v)$. However, this choice does not have good functorial properties, since the M_K-constant inherent in the various transformation formulas would depend on P. We obviate this problem by the usual trick of considering the distance to the diagonal of $V \times V$.

Definition. The *diagonal map* of V is denoted $\Delta : V \to V \times V$. We let $\Delta(V)$ denote the image of Δ considered as a closed subscheme of $V \times V$ (with the usual induced-reduced subscheme structure.) The *(arithmetic) distance function* on V is the local height function $\delta_V \in \mathcal{H}(V \times V)$ given by

$$\delta_V = \lambda_{\Delta(V)} .$$

[Any function in $\mathcal{F}(V \times V)$ in the equivalence class of δ_V will also be called a distance function.]

We start by proving some of the elementary properties of arithmetic distance functions.

Proposition 3.1. (a) *(Symmetry)*

$$\delta_V(P, Q; v) = \delta_V(Q, P; v) \quad \textit{for all} \quad (P, Q; v) \in V^2 \times M .$$

(b) *(Triangle Inequality 1)*

$$\delta_V(P,R;v) \geqq \min\{\delta_V(P,Q;v), \delta_V(Q,R;v)\} \quad \textit{for all} \quad (P,Q,R;v) \in V^3 \times M.$$

(c) *(Triangle inequality 2) Let* $X \in Z(V)$. *Then*

$$\lambda_X(Q;v) \geqq \min\{\lambda_X(P;v), \delta_V(P,Q;v)\} \quad \textit{for all} \quad (P,Q;v) \in V^2 \times M.$$

(d) *Fix a point* $P \in V$. *Then*

$$\delta_V(P,Q;v) = \lambda_P(Q;v) \quad \textit{for all} \quad (Q,v) \in V \times M.$$

Proof. (a) Let $i: V^2 \to V^2$ be the map $i(P, Q) = (Q, P)$. Then $i^*\Delta(V) = \Delta(V)$, so using Theorem 2.1(h) gives

$$\delta_V = \lambda_{\Delta(V)} = \lambda_{i^*\Delta(V)} = \lambda_{\Delta(V)} \circ i = \delta_V \circ i.$$

(b) For each $1 \leqq i < j \leqq 3$, let $\pi_{ij}: V^3 \to V^2$ be projection on the $(i,j)^{\text{th}}$ components. Then we must show that

$$\delta_V \circ \pi_{13} \geqq \min\{\delta_V \circ \pi_{12}, \delta_V \circ \pi_{23}\}.$$

Now using Theorem 2.1(h),

$$\delta_V \circ \pi_{ij} = \lambda_{\Delta(V)} \circ \pi_{ij} = \lambda_{\pi_{ij}^*\Delta(V)};$$

and so from Theorem 2.1(b),

$$\min\{\delta_V \circ \pi_{12}, \delta_V \circ \pi_{23}\} = \min\{\lambda_{\pi_{12}^*\Delta(V)}, \lambda_{\pi_{23}^*\Delta(V)}\}$$
$$= \lambda_{\pi_{12}^*\Delta(V) \cap \pi_{23}^*\Delta(V)}.$$

We thus must prove that

$$\lambda_{\pi_{13}^*\Delta(V)} \geqq \lambda_{\pi_{12}^*\Delta(V) \cap \pi_{23}^*\Delta(V)};$$

so from Theorem 2.1(d) it suffices to show that

$$\pi_{13}^*\Delta(V) \supset \pi_{12}^*\Delta(V) \cap \pi_{23}^*\Delta(V).$$

But $\pi_{12}^*\Delta(V) \cap \pi_{23}^*\Delta(V)$ equals the diagonal of V^3, which is clearly contained in $\pi_{13}^*\Delta(V)$.

(c) Let $\pi_1, \pi_2: V^2 \to V$ be the two projections. We must show that

$$\lambda_X \circ \pi_2 \geqq \min\{\lambda_X \circ \pi_1, \delta_V\}.$$

Now $\lambda_X \circ \pi_2 = \lambda_{\pi_2^*X}$ and

$$\min\{\lambda_X \circ \pi_1, \delta_V\} = \min\{\lambda_{\pi_1^*X}, \lambda_{\Delta(V)}\} = \lambda_{\pi_1^*X \cap \Delta(V)},$$

so it suffices to check that

$$\pi_2^*X \supset \pi_1^*X \cap \Delta(V).$$

Since $\pi_2^*X = V \times X$ and $\pi_1^*X \cap \Delta(V) = X \times X$, this inclusion is clear.

(d) Let $\varphi: V \to V \times V$ be the map $\varphi(Q) = (P, Q)$. Then as a function on $V \times M$,

$$\delta_V(P, \cdot\,; \cdot) = \delta_V \circ \varphi = \lambda_{\Delta(V)} \circ \varphi = \lambda_{\varphi^*\Delta(V)} = \lambda_P.$$

Example. Let us show that on an abelian variety, distance is translation invariant. In other words, we wish to prove that if A/K is an abelian variety, then

$$\delta_A(P,Q;v)=\delta_A(P+R,Q+R;v) \quad \text{for all} \quad (P,Q,R;v)\in A^3\times M.$$

The obvious approach is to use the "translation-by-R map" $\tau_R: A\to A$. Since τ_R is an isomorphism, we have

$$\begin{aligned}\delta_A(P,Q;v)&=\lambda_{A\times A,\Delta}(P,Q;v)=\lambda_{A\times A,(\tau_R\times\tau_R)*\Delta}(P,Q;v)\\&=\lambda_{A\times A,\Delta}(P+R,Q+R;v)=\delta_A(P+R,Q+R;v).\end{aligned}$$

Unfortunately, this derivation only shows that the difference $\delta_A(P+R,\ Q+R;v) - \delta_A(P,Q;v)$ is bounded as P and Q vary; the bound will depend on R. To obtain the stated result, we look instead at the subtraction map,

$$\sigma: A\times A\to A \qquad \sigma(P,Q)=P-Q.$$

Then

$$\lambda_{A,O}(P-Q;v)=\lambda_{A\times A,\sigma^*O}(P,Q;v)=\lambda_{A\times A,\Delta}(P,Q;v)=\delta_A(P,Q;v).$$

Now this equality holds (up to a bounded amount) for all P and Q. Since the left-hand side does not change if we replace P and Q by $P+R$ and $Q+R$, it follows that the right-hand side has the same property.

4. Global Height Functions

In this section we add up the local height functions defined in Sect. 2 so as to obtain a global height function corresponding to each closed subscheme of a projective variety V. We will assume, for this section only, that K is a finite extension of a fixed base field F whose proper set of absolute values have multiplicities assigned to them; and hence the absolute values on every finite extension of F also have attached multiplicities. (Generally, one also assumes that the product formula holds, but this will not be necessary for our purposes. For all of this formalism, see [4].)

Definition. Denote the set of functions $h: V\to[0,\infty]$ by $\mathscr{F}^*(V)$. We define an equivalence relation on $\mathscr{F}^*(V)$ by setting $h_1\sim h_2$ if the function $|h_1-h_2|$ is bounded on V. (By convention, $\infty-\infty=0$.) We will also sometimes denote this relation by $h_1=h_2+O(1)$. The set $\mathscr{H}^*(V)$ of *global height functions* on V is defined to be the quotient

$$\mathscr{H}^*(V)=\mathscr{F}^*(V)/\sim.$$

As usual, we put a partial order on $\mathscr{F}^*(V)$ by saying $h_1\leqq h_2$ if h_2-h_1 is bounded below on V; this induces a partial order on $\mathscr{H}^*(V)$, also.

Definition. Let $X\in Z(V)$. The *(global) height function associated to* X is the function $h_X\in\mathscr{F}^*(V)$ defined as follows:

For $P\in V$, choose a finite extension L/K so that $P\in V(L)$. Then

$$h_X(P)=[L:F]^{-1}\sum_{v\in M_L} N_v\lambda_X(P;v).$$

(Here N_v is the multiplicity attached to v.) As usual, h_X is independent of the choice of the field L. If the dependence on V is important, we use the notation $h_{V,X}$.

We now give the basic functorial properties of global height functions, all of which follow by addition from the corresponding properties for local height functions proven in Theorem 2.1.

Theorem 4.1. *The map*

$$h: Z(V) \longrightarrow \mathscr{H}(V)$$

has the following properties:

(a) *If $D \in Z(V)$ is a positive divisor (i.e. $D \in \mathrm{Div}^+(V)$), then h_D is the usual Weil height associated to D. (Cf. [4, Chap. 4].)*

(b) *For all $X, Y \in Z(V)$,*

$$h_{X \cap Y} \leqq \min\{h_X, h_Y\}.$$

(Note the inequality, which occurs because $\Sigma \min\{a_i, b_i\}$ need not equal $\min\{\Sigma a_i, \Sigma b_i\}$.)

(c) *For all $X, Y \in Z(V)$,*

$$h_{X+Y} = h_X + h_Y.$$

(d) *If $X, Y \in Z(V)$ satisfy $X \subset Y$, then*

$$h_X \leqq h_Y.$$

(e) *For all $X, Y \in Z(V)$,*

$$\max\{h_X, h_Y\} \leqq h_{X \cap Y} \leqq h_X + h_Y.$$

Further, either of the inequalities may be an equality.

(f) *If $X, Y \in Z(V)$ satisfy* $\mathrm{Support}(X) \subset \mathrm{Support}(Y)$, *then there exists a constant c such that*

$$h_X \leqq c h_Y.$$

(g) *Let $X, Y \in Z(V)$, let $U \subset V$ be an open set, and let $Z \subset V$ be the complement of U. Suppose that*

$$X|_U \subset Y|_U.$$

Then there exists a constant $c \geqq 0$ such that

$$h_X \leqq h_Y + c h_Z.$$

(h) *Let $\varphi: W \to V$ be a morphism of varieties, and let $X \in Z(V)$. Then*

$$h_{\varphi^* X} = h_X \circ \varphi.$$

Remark. Global height functions associated to divisors have two further important properties. First, at least for number fields, there are only finitely many points of bounded height (and degree) in V. Second, linearly equivalent divisors give equivalent height functions. As the following example shows, height functions associated to subschemes of codimension greater than 1 do not have the analogous properties. For this reason, such global height functions are principally useful as a bookkeeping tool; in order to obtain finiteness theorems, one must ultimately relate everything back to height functions corresponding to divisors.

Example. Let $X=[0,0,1]\in Z(\mathbb{P}^2)$. For any $P\in\mathbb{P}^2(\mathbb{Q})$, write $P=[x,y,z]$ with $x,y,z\in\mathbb{Z}$ and $gcd(x,y,z)=1$. Then one easily checks that if $\max\{|x|,|y|\}\geqq|z|$, then $h_X(P)=\log[gcd(x,y)]$. (A similar formula holds for $\max\{|x|,|y|\}\leqq|z|$.) In particular, if x and y are relatively prime integers, then $h_X([x,y,1])=0$, so $\mathbb{P}^2(\mathbb{Q})$ clearly contains infinitely many points of bounded height.

Next let $Y=[0,1,1]\in Z(\mathbb{P}^2)$. Then X and Y are rationally equivalent (as are any two points in $\mathbb{P}^2$). But using notation as above, we have (assuming say $|x|\geqq|z|$)

$$h_X([x,y,z])=\log[gcd(x,y)] \quad\text{and}\quad h_Y([x,y,z])=\log[gcd(x,y-z)].$$

Thus h_X and h_Y are clearly not equivalent functions.

We now show that the height function corresponding to an ample divisor will dominate any other height function. We recall the fact that it is possible to assign a (height) function $h_\xi: V\to\mathbb{R}$ to every Cartier divisor class $\xi\in CaCl(V)$ in such a way that if ξ is effective and $D\in\mathrm{Div}^+(V)$ is in the class of ξ, then

$$h_\xi=h_D+O(1) \quad\text{on } V\text{-Support}(D).$$

(See [4, Chap. 5 and/or Chap. 10, Sect. 4].)

Proposition 4.2. *Let $\xi\in CaCl(V)$ be an ample divisor class, and let $X\in Z(V)$. Then there exist constants $c_1, c_2>0$ such that*

$$h_X\leqq c_1h_\xi+c_2 \quad\textit{on } V\text{-Support}(X).$$

[The intuition for this result is that if P is v-adically close to X, but not on X, then its coordinates (relative to a projective embedding induced by a multiple of ξ) must be v-adically complicated, thereby making $h_\xi(P)$ large.]

Proof. Write $X=\cap E_i$ as an intersection of divisors (Lemma 2.2). From [4, Chap. 4, Proposition 5.4] (which gives our result when X is a divisor), there are constants $c_1, c_2>0$ such that for each i,

$$h_{E_i}\leqq c_1h_\xi+c_2 \quad\text{on } V\text{-Support}(E_i).$$

Since $X\subset E_i$, (4.1d) gives

$$h_X\leqq c_1h_\xi+c_2 \quad\text{on } V\text{-Support}(E_i);$$

and since this inequality holds for each i (and $\mathrm{Support}(X)=\cap\mathrm{Support}(E_i)$), we see that it holds on all of V-Support(X).

Proposition 4.2 is the main estimate needed to prove Faltings' finiteness lemma [2, Lemma 3] concerning metrized line bundles with logarithmic singularities (although this fact is somewhat concealed in the midst of his proof.) In terms of divisor classes, Faltings' result may be formulated as follows. Let $\xi\in CaCl(V)$, $U\subset V$ an open set, and $X\in Z(V)$ with $\mathrm{Support}(X)=V-U$. A (height) function $h: U\to\mathbb{R}$ is said to have *logarithmic singularities along X (relative to ξ)* if there is a constant $c>0$ such that

$$|h-h_\xi|\leqq c\log(2+h_X) \quad\text{on } U.$$

Corollary 4.3 (Faltings). *Let U and X be as above. Assume that K is a number field. Let $h: U \to \mathbb{R}$ have logarithmic singularities along X relative to some ample divisor class $\xi \in CaCl(V)$. Then for any constant C, the set*

$$\{P \in U(K) : h(P) \leqq C\}$$

is finite.

Proof. Let $P \in U(K)$ with $h(P) \leqq C$. Then from (4.2) and the definition of logarithmic singularities, we see that

$$\begin{aligned} h_\xi(P) &\leqq h(P) + c\log(2 + h_X(P)) \\ &\leqq C + c\log(2 + c_1 h_\xi(P) + c_2). \end{aligned}$$

From this one easily deduces that there is a constant $\kappa = \kappa(C, c, c_1, c_2)$ such that $h_\xi(P) \leqq \kappa$. Hence

$$\{P \in U(K) : h(P) \leqq C\} \subset \{P \in V(K) : h_\xi(P) \leqq \kappa\},$$

and the latter set is well-known to be finite [4, Chap. 3, Theorem 2.6].

5. Height Functions on Quasi-Projective Varieties

Let V be a quasi-projective variety; that is, a Zariski open subset of a projective variety. As above, we would like to associate to each closed subscheme $X \in Z(V)$ a local height function $\lambda_X \in \mathscr{H}(V)$, this association to have nice functorial properties. Unfortunately, because V is not complete, a problem arises due to the fact that points can "move out to ∞." The solution will be to define a function $\lambda_{\partial V} \in \mathscr{H}(V)$ which will measure the distance to the "boundary" of V. It will be defined up to multiplication by a non-zero constant, and then the functions λ_X will be defined up to addition of a multiple of $\lambda_{\partial V}$. This prompts us to set the following notation.

Notation. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathscr{H}(V)$. We write

$$\lambda_1 = \lambda_2 + O(\lambda_3)$$

to mean that there exists a constant $c > 0$ such that

$$\lambda_1 - c\lambda_3 \leqq \lambda_2 \leqq \lambda_1 + c\lambda_3 .$$

Similarly, we use the notation

$$\lambda_1 \gg \ll \lambda_2$$

to mean that there is a constant $c > 0$ such that

$$c^{-1}\lambda_1 \leqq \lambda_2 \leqq c\lambda_1 .$$

We start with the following fundamental lemma, which explains the behavior of local height functions under rational maps.

Lemma 5.1. *Let $\varphi: W \to V$ be a dominating rational map of projective varieties. Let $U \subset W$ be an open set on which φ is a morphism (so $\varphi(U)$ is dense in V), and let $Z \in Z(W)$ be a subscheme with support equal to $W - U$. For $X \in Z(V)$, let $\varphi^* X$ be the*

Zariski closure, in W, of $\varphi|_U{}^*X$. *(Note* φ^*X *may depend on U.) Then*

$$\lambda_X \circ \varphi = \lambda_{\varphi^*X} + O(\lambda_Z) \quad on\ U \times M.$$

Proof. Blowing up W along Support(Z), we can find a variety W' and a birational morphism $\psi: W' \to W$ with the following properties:

(i) ψ maps $U' = \psi^{-1}(U)$ isomorphically to U.

(ii) The rational map $\varphi \circ \psi: W' \to V$ extends to a morphism $\Psi: W' \to V$.

(W' and Ψ are produced by applying [3, II.7.17.3]. The fact that W' is projective and ψ is a birational morphism is [3, II.7.16].)

Notice that since $\Psi|_{U'}$ equals $\varphi \circ \psi$, we have

$$(\Psi^*X)|_{U'} = (\psi^*(\varphi^*X))|_{U'}.$$

Hence letting $Z' \in Z(W')$ have support $W' - U'$, Theorem 2.1(g, h) implies that (note ψ and Ψ are morphisms)

$$\lambda_{\varphi^*X} \circ \psi = \lambda_X \circ \Psi + O(\lambda_{Z'}).$$

Next we note that

$$\text{Support}\, Z' = \text{Support}\, \psi^*Z,$$

so using Theorem 1.2(f) we find

$$\lambda_{\varphi^*X} \circ \psi = \lambda_X \circ \Psi + O(\lambda_Z \circ \psi).$$

Finally, since $\psi: U' \to U$ is an isomorphism, and since $\Psi(P) = \varphi(\psi(P))$ for $P \in U'$, this last inequality implies that if we restrict attention to $U \times M$, then

$$\lambda_{\varphi^*X} = \lambda_X \circ \varphi + O(\lambda_Z).$$

Next we describe a local height function which gives the distance to the boundary of a quasi-projective variety.

Lemma 5.2. *Let V be a quasi-projective variety. There exists a local height function* $\lambda_{\partial V} \in \mathscr{H}(V)$, *which we will call a boundary function, with the following property.*

Let V′ be any projective closure of V, and let $Z' \in Z(V')$ *be a subscheme of V′ with support equal to* $V' - V$. *Then*

$$\lambda_{\partial V} \gg \ll \lambda_{Z'}|_V.$$

$\lambda_{\partial V}$ *is determined (in* $\mathscr{H}(V)$*) up to* $\gg \ll$ *equivalence.*

Proof. We define $\lambda_{\partial V}$ to be $\lambda_{Z'}|_V$ for a fixed choice of V' and Z'. Now let V'' and Z'' be another choice. We must show that $\lambda_{Z'}|_V \gg \ll \lambda_{Z''}|_V$. Since V' and V'' both have V as a dense open subset, V' and V'' are birational, say by the map $\varphi: V'' \to V'$ taking V to V. Hence from Lemma 5.1,

$$\lambda_{Z'} \circ \varphi = \lambda_{\varphi^*Z'} + O(\lambda_{Z''}).$$

But φ^*Z' is the closure in V'' of $\varphi|_V{}^*Z'$, and $\varphi|_V{}^*Z'$ is trivial (i.e. the empty scheme). Therefore

$$\lambda_{Z'} \circ \varphi = O(\lambda_{Z''}), \quad \text{so} \quad \lambda_{Z'} \circ \varphi \leqq c\lambda_{Z''}.$$

The same argument applied to φ^{-1} gives the opposite inequality, which completes the proof of existence. The uniqueness (up to scalar multiplication) is clear.

The following proposition illustrates the behavior of the boundary function $\lambda_{\partial V}$. The second part is useful, for example, in cases where one has a complete family of varieties and wishes to discard the "bad" fibers.

Proposition 5.3. *Let V and W be quasi-projective varieties.*

(a) *Let π_1 and π_2 be the two projections on $V \times W$. Then*

$$\lambda_{\partial V \times W} \gg \ll \lambda_{\partial V} \circ \pi_1 + \lambda_{\partial W} \circ \pi_2 .$$

(b) *Let $\varphi: W \to V$ be a proper morphism. Then*

$$\lambda_{\partial W} \gg \ll \lambda_{\partial V} \circ \varphi .$$

Proof. Choose projective closures W' and V' for W and V.

(a) This follows immediately from Theorem 2.1(f) and the fact that

$$(W' \times V') - (W \times V) = (W' - W) \times V \cup W \times (V' - V) .$$

[I.e. Using a more suggestive notation, $\partial(W \times V) = (\partial W) \times V \cup W \times (\partial V)$.]

(b) φ induces a rational map $W' \to V'$; and by blowing-up W' along $W' - W$, we may assume that φ extends to a morphism $W' \to V'$, which we denote by ψ. (Again we are using [3, II.7.17.3]). We thus have the following diagram:

$$\begin{array}{ccc} W' & \xrightarrow{\psi} & V' \\ \cup & & \cup \\ W & \xrightarrow{\varphi} & V. \end{array}$$

Clearly $\psi^{-1}(V' - V) \subset W' - W$, which implies (Theorem 2.1(f, h)) that

$$\lambda_{\partial V} \circ \varphi \leqq c \lambda_{\partial W} .$$

We now prove the other inclusion, $\psi^{-1}(V' - V) \supset W' - W$, which will complete the proof. Thus let $P \in W'$ satisfy $\psi(P) \in V$. We must show that $P \in W$. Let $C \subset W'$ be a curve with $P \in C$ and $C \cap W \neq \emptyset$. Let R be the normalization of the local ring $\mathcal{O}_{C,P}$, and let L be its quotient field. Then the map $\operatorname{Spec}(L) \to W \to V$ extends to a map $\operatorname{Spec}(R) \to V$, because the closed point of $\operatorname{Spec}(R)$ can be sent to $\psi(P)$. Since φ is proper, it follows that we can map $\operatorname{Spec}(R) \to W$ (valuative criterion of properness, [3, II.4.7]), and so $P \in W$ as desired.

We are now ready to assign a local height function to each closed subscheme of any quasi-projective variety.

Theorem 5.4. *There are maps*

$$\lambda = \lambda_V : Z(V) \to \mathscr{H}(V) ,$$

one for each quasi-projective variety V, with the following properties:

(a) *If V is projective, then λ_V is the map defined in Theorem 2.1.*

(b)–(g) *λ_V satisfies properties* (b)–(g) *of Theorem 2.1 provided $O(\lambda_{\partial V})$ is added onto each equation.*

(h) *Let* $\varphi: W \to V$ *be a morphism of quasi-projective varieties, and let* $X \in Z(V)$. *Then*

$$\lambda_{V,X} \circ \varphi = \lambda_{W,\varphi^* X} + O(\lambda_{\partial W}).$$

The function $\lambda_{V,X}$ *is determined up to* $O(\lambda_{\partial V})$ *(in* $\mathscr{H}(V)$*) by properties* (a) *and* (h).

Proof. For each V we fix a smooth projective closure V'; and for $X \in Z(V)$, we denote by $X' \in Z(V')$ the closure of X in V'. We then define

$$\lambda_{V,X} = \lambda_{V',X'}|_V,$$

where $\lambda_{V',X'}$ is given by Theorem 2.1.

Clearly (a) is satisfied, since if V is projective, then $V' = V$. The verification of (b)–(g) is also easy using Theorem 2.1. We illustrate by proving (b). Let $X, Y \in Z(V)$. Then clearly $(X \cap Y)'|_V = (X' \cap Y')|_V$. [Notice that $(X \cap Y)'$ and $X' \cap Y'$ may not agree on the boundary $V' - V$.] Now using the definitions of λ_V and $\lambda_{\partial V}$, and applying Theorem 2.1 (b, g), we see that

$$\lambda_{V,X \cap Y} = \min\{\lambda_{V,X}, \lambda_{V,Y}\} + O(\lambda_{\partial V}).$$

It remains to check (h). The morphism $\varphi: W \to V$ induces a rational map $\psi: W' \to V'$ with $\psi|_W = \varphi$. From Lemma 5.1 we conclude that

$$\lambda_{V',X'} \circ \psi = \lambda_{W',\psi^* X'} + O(\lambda_Z),$$

where $Z \in Z(W')$ is chosen with support equal to $W' - W$. Now $\lambda_Z|_W = \lambda_{\partial W}$ (Lemma 5.2), and $\psi^* X'$ is the closure in W' of $\varphi^* X$ (Lemma 5.1). Therefore

$$\lambda_{V',X'} \circ \psi = \lambda_{W',(\varphi^* X)'} + O(\lambda_{\partial W}),$$

which is exactly the desired relation.

Finally, the up-to-$O(\lambda_{\partial V})$ uniqueness of λ_V is clear from (a) and (h); simply embed V in some projective closure.

As in Sect. 3, we define a distance function by using the local height function relative to the diagonal.

Definition. The *(arithmetic) distance function* on a quasi-projective variety V is the local height function $\delta_V \in \mathscr{H}(V \times V)$ given by

$$\delta_V = \lambda_{\Delta(V)}.$$

In view of Lemma 5.3(a), we see that $\delta_V(P, Q; v)$ is well-defined up to $O(\lambda_{\partial V}(P; v) + \lambda_{\partial V}(Q; v))$-equivalence. To ease notation, we will also write this as $O(\lambda_{\partial V}(P, Q; v))$.

Proposition 5.5. *Let* V *be a quasi-projective variety.*

(a) *(Symmetry)*

$$\delta_V(P, Q; v) = \delta_V(Q, P; v) + O(\lambda_{\partial V}(P, Q; v)) \quad \text{for all } (P, Q; v) \in V^2 \times M.$$

(b) *(Triangle Inequality 1)*

$$\delta_V(P, R; v) \geqq \min\{\delta_V(P, Q; v), \delta_V(Q, R; v)\} + O(\lambda_{\partial V}(P, Q, R; v))$$
$$\text{for all } (P, Q, R; v) \in V^3 \times M.$$

(c) *(Triangle Inequality 2) Let* $X \in Z(V)$. *Then*

$$\lambda_X(Q;v) \geqq \min\{\lambda_X(P;v), \delta_V(P,Q;v)\} + O(\lambda_{\partial V}(P,Q;v)) \quad \textit{for all } (P,Q;v) \in V^2 \times M.$$

(d) *Fix a point* $P \in V$. *Then*

$$\delta_V(P,Q;v) = \lambda_P(Q;v) + O(\lambda_{\partial V}(P,Q;v)) \quad \textit{for all } (Q,v) \in V \times M.$$

Proof. Let V' be a projective closure of V. Since the closure of $\Delta(V)$ in $V' \times V'$ is precisely $\Delta(V')$, we have

$$\delta_V = \delta_{V'}|_{V \times V} + O(\lambda_{\partial V}).$$

Further, letting $X' \in Z(V')$ be the closure in V' of $X \in Z(V)$, we have [by applying Theorem 5.4(h) to the inclusion map $V \subset V'$]

$$\lambda_X = \lambda_{X'}|_V + O(\lambda_{\partial V}).$$

Now all of Proposition 5.5 follows from the corresponding statements in Proposition 3.1.

6. The Inverse Function Theorem

Our goal in this section is to prove the following quantitative version of the inverse function theorem.

Theorem 6.1. *Let* $\varphi: W \to V$ *be a finite map of degree* d *between smooth quasi-projective varieties. Let* $R(\varphi) \in \mathrm{Div}^+(W)$ *be the ramification divisor of* φ. *If* $(P,q,v) \in W \times V \times M$ *satisfies*

$$\delta_V(\varphi P, q; v) > d\lambda_{W,R(\varphi)}(P;v) + O(\lambda_{\partial W}(P;v)),$$

then there exists a unique $Q \in W$ *such that*

$$\varphi(Q) = q$$

and

$$\delta_W(P,Q;v) \geqq \delta_V(\varphi P, q; v) - (d-1)\lambda_{W,R(\varphi)}(P;v) - O(\lambda_{\partial W}(P;v)).$$

The statement of this theorem requires some explanation; specifically, what is the meaning of the condition that $\delta_V(\varphi P, q; v)$ be (strictly) greater than $d\lambda_{W,R(\varphi)}(P;v) + O(\lambda_{\partial W}(P;v))$. After all, these functions are only determined up to addition of an M_K-bounded function (not to mention the big-O constant). What is meant is the following. Choose specific functions in the equivalence classes of the local height functions δ_V, δ_W, $\lambda_{W,R(\varphi)}$, and $\lambda_{\partial W}$, which we will denote with the same symbols. Then there is a constant $c > 0$ and an M_K-constant γ, *depending on* V, W, φ, *and the particular choice of functions made above*, so that if $(P,q,v) \in W \times V \times M$ satisfies

$$\delta_V(\varphi P, q; v) > d\lambda_{W,R(\varphi)}(P;v) + c\lambda_{\partial W}(P;v) + \gamma(v),$$

then there is a unique $Q \in W$ such that $\varphi(Q) = q$ and

$$\delta_W(P,Q;v) \geqq \delta_V(\varphi P, q; v) - (d-1)\lambda_{W,R(\varphi)}(P;v) - c\lambda_{\partial W}(P;v) - \gamma(v).$$

(We will use similar shorthand notation throughout this section.)

We mention that for any given points $P\in W$ and $q\in V$, unless $\varphi P=q$, there can be only finitely many absolute values $v\in M_K$ such that (P,q,v) satisfies the above strict inequality. This is because the right hand side is non-negative, while $\delta_V(\varphi P,q;v)=0$ for all but finitely many such v. It is also worth pointing out that even if (P,q,v) and (P,q,v') both satisfy the given inequality, then the corresponding inverse points $Q,Q'\in\varphi^{-1}(q)$ need not be the same. (One sees a similar phenomenon in the fact that a sequence of rational numbers may converge in different v-adic topologies to distinct rational numbers!)

Remark. The inverse function theorem says (qualitatively) that a finite map is a local isomorphism locally around any point not on the ramification locus. Theorem 6.1 gives a quantitative estimate for how far away from the ramification locus one must stay in order to define an inverse function, and says that the distance between the inverse points is then about the same as the distance between their images.

Remark. We mention now that finite maps are always proper [3, Example II.4.1]. We will thus frequently make use of Lemma 5.3(b) without additional comment.

We start with the following preliminary result.

Proposition 6.2. *Let $\varphi: W\to V$ be a finite map of smooth quasi-projective varieties.*

(a) *(Separation) Let $Q,Q'\in W$ be distinct points with $\varphi(Q)=\varphi(Q')$. Then*

$$\delta_W(Q,Q';v)\leqq\lambda_{R(\varphi)}(Q;v)+O(\lambda_{\partial V}(\varphi Q;v)).$$

(b) *(Distribution Relation) Let $P\in W$ and $q\in V$. Then*

$$\delta_V(\varphi P,q;v)=\sum_{Q\in\varphi^{-1}(q)}e_\varphi(Q/q)\delta_W(P,Q;v)+O(\lambda_{\partial W\times V}(P,q;v)).$$

(Here $e_\varphi(Q/q)$ is the ramification index of φ at Q. For example, if $Q\notin$ Support$(R(\varphi))$, then $e_\varphi(Q/q)=1$.)

Proof. (a) Let $\pi_1: W\times W\to W$ be projection on the first factor. We start with the ideal sheaf relation

$$(\varphi\times\varphi)^*\mathscr{I}_{\Delta(V)}+\mathscr{I}^2_{\Delta(W)}=\pi_1^*\mathscr{I}_{R(\varphi)}\cdot\mathscr{I}_{\Delta(W)}+\mathscr{I}^2_{\Delta(W)}.$$

This says nothing more than the fact that up to quadratic terms, a polynomial map is given by its differential. In terms of subschemes, this gives

$$(\varphi\times\varphi)^*\Delta(V)\cap 2\Delta(W)=(\pi_1^*R(\varphi)+\Delta(W))\cap 2\Delta(W).$$

Now applying Theorem 5.4 yields

$$\min\{\delta_V\circ(\varphi\times\varphi),2\delta_W\}=\min\{\lambda_{R(\varphi)}\circ\pi_1+\delta_W,2\delta_W\}+O(\lambda_{\partial W\times W}).$$

Next we evaulate this equation at the point $(Q,Q';v)\in W\times W\times M$. (From here on, we will omit reference to v in the notation.) Since $\varphi(Q)=\varphi(Q')$, $\delta_V\circ(\varphi\times\varphi)(Q,Q')=\infty$. Since further $Q\neq Q'$, $\delta_W(Q,Q')\neq\infty$, so we may subtract $\delta_W(Q,Q')$ from both sides. We then find

$$\delta_W(Q,Q')=\min\{\lambda_{R(\varphi)}(Q),\delta_W(Q,Q')\}+O(\lambda_{\partial W\times W}(Q,Q')).$$

This implies that

$$\delta_W(Q,Q')\leqq\lambda_{R(\varphi)}(Q)+O(\lambda_{\partial W\times W}(Q,Q')).$$

Finally, using Lemma 5.3, we have

$$\lambda_{\partial W\times W}(Q,Q')\gg\ll\lambda_{\partial W}(Q)+\lambda_{\partial W}(Q')\gg\ll\lambda_{\partial V}(\varphi Q).$$

(b) *Step I. $\varphi: W\to V$ is a Galois covering.*

Let $\{\tau_1,\dots,\tau_n\}$ be the set of automorphisms of W over V. (We note that since W and V are integral schemes and φ is finite, every rational map $\tau: W\to W$ satisfying $\varphi\circ\tau=\varphi$ is actually a morphism.) Now under our assumption that φ be Galois, we have the equality

$$(\varphi\times\varphi)^*\Delta(V)=\Sigma(1\times\tau_i)^*\Delta(W).$$

Therefore, choosing any point $Q\in\varphi^{-1}(q)$, we have

$$\begin{aligned}\delta_V(\varphi P,q)&=\lambda_{\Delta(V)}(\varphi P,\varphi Q)\\&=\lambda_{(\varphi\times\varphi)^*\Delta(V)}(P,Q)+O(\lambda_{\partial W\times V}(P,q))\\&=\Sigma\,\lambda_{(1\times\tau_i)^*\Delta(W)}(P,Q)+O(\lambda_{\partial W\times V}(P,q))\\&=\Sigma\,\lambda_{\Delta(W)}(P,\tau_iQ)+O(\lambda_{\partial W\times V}(P,q))\\&=\sum_{Q\in\varphi^{-1}(q)}e_\varphi(Q/q)\delta_W(P,Q)+O(\lambda_{\partial W\times V}(P,q)).\end{aligned}$$

Here the last line follows from the fact that $\{\tau_iQ\}$ is equal to $\varphi^{-1}(q)$ counted with appropriate multiplicities.

Step II. $\varphi: W\to V$ arbitrary.

Let T be a smooth model for the Galois closure of W over V so that the map $\psi: T\to W$ is a finite morphism. We apply step I to the two Galois covers $\psi: T\to W$ and $\varphi\circ\psi: T\to V$. Thus letting $t\in T$ be any point with $\psi(t)=P$, we find

$$\begin{aligned}\sum_{Q\in\varphi^{-1}(q)}e_\varphi(Q/q)\delta_W(P,Q)&=\sum_{Q\in\varphi^{-1}(q)}e_\varphi(Q/q)\delta_W(\psi t,Q)+O(\lambda_{\partial W\times V}(P,q))\\&=\sum_{Q\in\varphi^{-1}(q)}e_\varphi(Q/q)\sum_{s\in\psi^{-1}(Q)}e_\psi(s/Q)\delta_T(t,s)+O(\lambda_{\partial W\times V}(P,q))\\&=\sum_{s\in(\varphi\circ\psi)^{-1}(q)}e_{\varphi\circ\psi}(s/q)\delta_T(t,s)+O(\lambda_{\partial W\times V}(P,q))\\&=\delta_V(\varphi\circ\psi(t),q)+O(\lambda_{\partial W\times V}(P,q))\\&=\delta_V(\varphi P,q)+O(\lambda_{\partial W\times V}(P,q)).\end{aligned}$$

We are now ready to prove the inverse function theorem. The key idea in the proof is to combine the two pieces of Proposition 6.2. From the distribution relation, if φP is close to q, then some of the points in $\varphi^{-1}(q)$ must be close to P; while the separation result says that no more than one of them can be close. We now make this intuition precise.

Proof of Proposition 6.1. Choose $Q_0\in\varphi^{-1}(q)$ so that

$$\delta_W(P,Q_0)=\max\{\delta_W(P,Q):\varphi Q=q\}.$$

First we show that Q_0 is essentially no closer to the ramification locus than P is. To explain matters most clearly, we assume that actual functions have been chosen to represent the various local height function classes, and write in explicit big-O

and M_K constants. (See the discussion above following the statement of Theorem 6.1.)

$$\begin{aligned}
\delta_W(P,Q_0) &\geqq (1/d) \sum_{Q\in\varphi^{-1}(q)} \delta_W(P,Q) && \text{choice of } Q_0 \\
&= (1/d)\delta_V(\varphi P,q) - c_1\lambda_{\partial W\times V}(P,q) - \gamma_1 && \text{Proposition 6.2(b)} \\
&> \lambda_{R(\varphi)}(P) + (c_2-c_1)\lambda_{\partial W\times V}(P,q) + (\gamma_2-\gamma_1) && \text{by assumption} \\
&\geqq \min\{\delta_W(P,Q_0), \lambda_{R(\varphi)}(Q_0)\} + (c_2-c_1-c_3)\lambda_{\partial W\times V}(P,q) \\
&\quad + (\gamma_2-\gamma_1-\gamma_3) && \text{Proposition 5.5(c)}.
\end{aligned}$$

Now the constants c_1, γ_1, and c_3, γ_3 are given, a priori, from Propositions 6.2 and 5.5 respectively. In the statement of Theorem 6.1 we can thus choose $c_2 = c_1 + c_3$ and $\gamma_2 = \gamma_1 + \gamma_3$, this choice being completely independent of P and q. Having done so, we conclude that

$$\delta_W(P,Q_0) > \lambda_{R(\varphi)}(P) \geqq \min\{\delta_W(P,Q_0), \lambda_{R(\varphi)}(Q_0)\}.$$

Now the strict inequality forces the right-hand minimum to be $\lambda_{R(\varphi)}(Q_0)$, so we conclude that

$$\lambda_{R(\varphi)}(P) \geqq \lambda_{R(\varphi)}(Q_0).$$

Next, for any $Q\in\varphi^{-1}(q)$ with $Q \neq Q_0$, we have

$$\begin{aligned}
\delta_W(P,Q) &= \min\{\delta_W(P,Q_0), \delta_W(P,Q)\} && \text{choice of } Q_0 \\
&\leqq \delta_W(Q_0,Q) + O(\lambda_{\partial W}(P,Q,Q_0)) && \text{Proposition 5.5(b)} \\
&\leqq \lambda_{R(\varphi)}(Q_0) + O(\lambda_{\partial W}(P,Q,Q_0)) && \text{Proposition 6.2(a)} \\
&\leqq \lambda_{R(\varphi)}(P) + O(\lambda_{\partial W}(P,Q,Q_0)) && \text{from above} \\
&= \lambda_{R(\varphi)}(P) + O(\lambda_{\partial W\times V}(P,q)) && \text{Lemma 5.3(b)}.
\end{aligned}$$

In other words, since P is close to Q_0, it is not too close to any of the other points of $\varphi^{-1}(q)$.

We now substitute this inequality in the distribution relation (Proposition 6.2(b)) to complete the existence part of the argument.

$$\begin{aligned}
\delta_V(\varphi P,q) &= \sum_{Q\in\varphi^{-1}(q)} \delta_W(P,Q) + O(\lambda_{\partial W\times V}(P,q)) \\
&\leqq \delta_W(P,Q_0) + (d-1)\lambda_{R(\varphi)}(P) + O(\lambda_{\partial W\times V}(P,q)).
\end{aligned}$$

To show that Q_0 is the only point in $\varphi^{-1}(q)$ satisfying this inequality, we use the inequality

$$\delta_W(P,Q) \leqq \lambda_{R(\varphi)}(P) + O(\lambda_{\partial W\times V}(P,q))$$

proven above for $Q\in\varphi^{-1}(q)$, $Q \neq Q_0$. If such a Q satisfied the conclusion of Theorem 6.1, then we would have

$$\delta_V(\varphi P,q) \leqq d\lambda_{R(\varphi)}(P) + O(\lambda_{\partial W\times V}(P,q)).$$

But this exactly contradicts the initial assumption on $\delta_V(\varphi P,q)$. Therefore Q_0 is unique, which completes the proof of Theorem 6.1.

7. Families of Varieties

Let $\pi: V \to T$ be a surjective proper morphism of quasi-projective varieties. We may think of π as giving an algebraic family of varieties, namely the fibers V_t as t ranges over T. Our first result says that the closest a point $P \in V$ can come to a section of π occurs for that point of the section on the same fiber as P.

Proposition 7.1. *Let $\pi: V \to T$ be as above, and let $\varphi \in V(T)$ be a section relative to π. [I.e. $\varphi: T \to V$ is a morphism satisfying $\pi \circ \varphi = 1_T$.] Let $\varphi(T)$ denote the image of φ (with the induced-reduced subscheme structure.) Then*

$$\lambda_{\varphi(T)}(P; v) = \delta_V(P, \varphi \circ \pi(P); v) + O(\lambda_{\partial T}(\pi P; v)) \quad \textit{for all } (P, v) \in V \times M.$$

Proof. Consider the map

$$\psi: V \to V \times V, \qquad \psi(P) = (P, \varphi \circ \pi(P)).$$

Then

$$\psi * \Delta(V) = \varphi(T),$$

so the desired result follows from Theorem 5.4(h) and Lemma 5.3(b).

Suppose now that $\pi: A \to T$ makes A into an abelian scheme over T. In other words, we have morphisms

$$+: A \times_T A \to A \quad \text{and} \quad -: A \to A,$$

satisfying the usual group axioms, which make every fiber A_t into an abelian variety. (Note π is a proper morphism.) There is also a distinguished section $O \in A(T)$; we will often use O also to denote its image as an element of $Z(A)$. The next result shows that the distance function on A is translation invariant.

Proposition 7.2. *Let T be a quasi-projective variety and $\pi: A \to T$ an abelian scheme over T. Then*

$$\delta_A(p, q; v) = \delta_A(p + r, q + r; v) + O(\lambda_{\partial T}(\pi p; v))$$
$$\textit{for all } (p, q, r; v) \in (A \times_T A \times_T A) \times M.$$

(Note that p, q, r must all lie on the same fiber of A, since the group law is only defined fiber by fiber. Thus $\pi p = \pi q = \pi r$.)

Proof. We remark that the fibered product $A \times_T A$ sits naturally as a subvariety of $A \times A$, and we have $\Delta(A) \subset A \times_T A \subset A \times A$. [Precisely, $A \times_T A = (\pi \times \pi)^{-1} \Delta(T)$.] Now consider the subtraction map

$$\sigma: A \times_T A \to A, \qquad \sigma(p, q) = p - q.$$

Then $\sigma^* O = \Delta(A)$, so for all $(p, q) \in A \times_T A$,

$$\delta_A(p, q; v) = \lambda_O(p - q; v) + O(\lambda_{\partial T}(\pi p; v)).$$

The right-hand side of this equation clearly does not change if we replace p and q by $p + r$ and $q + r$, so the left-hand side is similarly invariant.

We conclude this section by reproving a theorem of Lang-Silverman-Tate [4, Chap. 12, Sect. 1]. Our reasons for doing so are two-fold. First, we can now eliminate the restriction that K have characteristic 0. Second, and of more importance, the formalism of local height functions on quasi-projective varieties makes both the statement and proof far more transparent.

Thus let T be a quasi-projective variety and $\pi: A \to T$ an abelian scheme over T as above. Let $D \in \mathrm{Div}^+(A)$ be a flat family of divisors over T. (This means that for each $t \in T$, the restriction of D to the fiber A_t is a divisor $D_t \in \mathrm{Div}^+(A_t)$. Cf. [3, III, Sect. 9].) Then for each $t \in T$, the equivalence class of local height functions λ_{A_t, D_t} on the fiber A_t contains a particular function, the *Néron function*, which is well-defined up to addition of an M_K-constant. (See [4, Chap. 11, Sect. 1.]) We piece these Néron functions together fiber-by-fiber to obtain a map

$$\hat{\lambda}_{A,D}: A \to [0, \infty],$$

well-defined up to an M_K-constant *on each fiber*. Of course, we also have the usual local height

$$\lambda_{A,D}: A \to [0, \infty],$$

which is only defined up to an M_K-bounded function.

Theorem 7.3. *With notation as above,*

$$\hat{\lambda}_{A,D} = \lambda_{A,D} + O(\lambda_{\partial T} \circ \pi).$$

(As usual, what this means is that it is possible to adjust the fiber-by-fiber M_K-constants in $\hat{\lambda}_{A,D}$ so that $|\hat{\lambda}_{A,D} - \lambda_{A,D}|$ is bounded by $c\lambda_{\partial T} \circ \pi$.)

Corollary 7.4. *Let $\hat{h}_{A,D}: A \to [0, \infty)$ be the canonical height on A relative to D (cf. [4, Chap. 5]). Then*

$$\hat{h}_{A,D} = h_{A,D} + O(h_{\partial T} \circ \pi) + O(1).$$

Proof. The corollary follows from the theorem by addition using [4, Chap. 11, Theorem 1.6]. To prove the theorem, we look at the multiplication-by-2 map, $[2]: A \to A$. (Note it is a morphism.) By linearity, it suffices to consider the case when D is either even or odd. Let $m = 4$ if D is even, $m = 2$ if D is odd. Then on each fiber, the theorem of the square says that $[2]^* D_t \sim mD_t$ (linear equivalence). Hence there is a function f on A and a divisor $E \in \mathrm{Div}(A)$ whose components are all fibral such that

$$[2]^* D = mD + \mathrm{div}(f) + E.$$

Since every fiber is irreducible, it follows that $E = \pi^* B$ for some $B \in \mathrm{Div}(T)$. Now this divisorial relation yields the height relation

$$\lambda_{A,D}(2P; v) = m\lambda_{A,D}(P; v) + v \circ f(P) + \lambda_{T,B}(\pi P; v) + O(\lambda_{\partial T}(\pi P; v)).$$

Applying [4, Chap. 11, Proposition 1.4 and the bound in Lemma 1.2], we conclude that

$$\hat{\lambda}_{A,D} = \lambda_{A,D} + O(\lambda_{T,B} \circ \pi) + O(\lambda_{\partial T} \circ \pi).$$

Finally, since $[2]^* D \sim mD$ on *every* fiber, we can repeat the above argument with functions $f_1, \ldots, f_n$ and divisors $B_1, \ldots, B_n \in \mathrm{Div}(T)$ having the property that

$\cap B_i = \emptyset$. Then

$$\min\{\lambda_{T,B_i}\} = \lambda_{T,\cap B_i} = \lambda_{T,\emptyset}$$

is M_K-bounded, so taking the minimum over i we obtain the desired relation

$$\hat{\lambda}_{A,D} = \lambda_{A,D} + O(\lambda_{\partial T} \circ \pi).$$

Appendix. Metrized Vector Bundles

One can reformulate the theory of local height functions corresponding to divisors in terms of sections to metrized line bundles in the following manner. Let $(\mathscr{L}, \|\cdot\|)$ be a metrized line bundle on a projective variety V. (For details, see [1, 2, or 5].) Let s be a global section to $\mathscr{L}$ and $D \in \mathrm{Div}^+(V)$ the corresponding divisor. Then as functions in $\mathscr{H}(V)$,

$$\lambda_D(P; v) = -\log\|s(P)\|_v.$$

In a similar manner, one can take the theory of local height functions corresponding to subschemes which we have developed above and include it in a theory of *metrized vector bundles* and their sections. Since our interest lies principally in the local height functions themselves, we have chosen to make them the centerpiece of this article. To round matters out, we will now briefly indicate, without proof, the salient facts concerning metrized vector bundles and their sections.

Definition. Let V be a projective variety and $\mathscr{E}$ a vector bundle (always assumed finite dimensional) on V. Let $v \in M$. A *v-adic metric on $\mathscr{E}$* consists of a (non-trivial) v-adic norm $\|\cdot\|_v$ on each fiber $\mathscr{E}_P \otimes K^a$ with the property that the norms "vary continuously with $P \in V$." [I.e. If $s \in \Gamma(U, \mathscr{E})$ is a section to $\mathscr{E}$ on some open set U, then the map

$$U \to [0, \infty) \qquad P \to \|s(P)\|_v$$

is continuous when U is given the v-adic topology.] A *metric on $\mathscr{E}$* is a collection of v-adic metrics $\|\cdot\|_v$ on $\mathscr{E}$, one for each $v \in M$. Two metrics $\|\cdot\|$ and $\|\cdot\|'$ on $\mathscr{E}$ are *M_K-equivalent*, denoted $\|\cdot\| \sim \|\cdot\|'$, if there is an M_K-constant γ such that

$$e^{-\gamma(v)}\|\cdot\|_v \leqq \|\cdot\|'_v \leqq e^{\gamma(v)}\|\cdot\|_v.$$

(N.B. If two metrics are M_K-equivalent, then for almost all $v \in M_K$ they are equal.)

Theorem A.1. *Let V be a projective variety. It is possible to assign to each vector bundle $\mathscr{E}$ over V a metric $\|\cdot\|_{\mathscr{E}}$, unique up to $\sim$ equivalence, so that the following properties hold:*

(a) *If $\mathscr{L}$ is a line bundle over V which is generated by the global sections $s_1, \ldots, s_n$, then for any global section s,*

$$\|s(P)\|_{\mathscr{L},v} \sim \min\{|(s/s_i)(P)|_v : s_i(P) \neq 0\}.$$

(Note each s/s_i is a function on V. The M_K-constant inherent in the $\sim$ equivalence will depend on $s_1, \ldots, s_n$, but not *on s or P.)*

Let $\mathscr{E}$ and $\mathscr{F}$ be vector bundles on V, and let $\mathscr{L}$ be a line bundle.

(b) $$\|\cdot\|_{\mathscr{E}\oplus\mathscr{F}} \sim \max\{\|\cdot\|_{\mathscr{E}}, \|\cdot\|_{\mathscr{F}}\}.$$

(c) $$\|\cdot\|_{\mathscr{E}\otimes\mathscr{L}} \sim \|\cdot\|_{\mathscr{E}}\|\cdot\|_{\mathscr{L}}.$$

(d) *Suppose that $\mathscr{E}\subset\mathscr{F}$. Then*

$$\|\cdot\|_{\mathscr{E}} \sim \textit{restriction of } \|\cdot\|_{\mathscr{F}} \textit{ to } \mathscr{E}.$$

(e) *Let $\varphi: W \to V$ be a morphism of projective varieties. Then*

$$\|\cdot\|_{\varphi^*\mathscr{E}} \sim \|\cdot\|_{\mathscr{E}} \circ \varphi.$$

[The uniqueness is clear. For after choosing an ample line bundle $\mathscr{O}(1)$ on V, one can find integers $d, n > 0$ so that $\mathscr{E}$ sits as a subbundle of $\mathscr{O}(n)^d$. Then $\|\cdot\|_{\mathscr{E}}$ is determined (up to $\sim$ equivalence) by properties (a), (b), and (d). We do not include a proof of the existence.]

Definition. A *metrized vector bundle* on V is a pair $(\mathscr{E}, \|\cdot\|)$, where $\mathscr{E}$ is a vector bundle on V and $\|\cdot\|$ is a metric on $\mathscr{E}$ satisfying the conditions of theorem A.1.

Definition. Let $\mathscr{E}$ be a vector bundle on V and let s by a global section to $\mathscr{E}$. Then s determines a closed subscheme $Z(s) \in Z(V)$, its *scheme of zeros.* (Cf. [3, Appendix A.3].)

Proposition A.2. *Let V be a projective variety, $(\mathscr{E}, \|\cdot\|)$ a metrized vector bundle on V, and s a global section to $\mathscr{E}$. Then as functions in $\mathscr{H}(V)$,*

$$\lambda_{Z(s)}(P; v) = -\log\|s(P)\|_v.$$

(Proof omitted.)

Now it is a standard fact that every closed subscheme is the scheme of zeros of some global section to some vector bundle. Thus once one has developed the theory of metrized vector bundles, then λ_X can be defined as $-\log\|s\|$ for one (any) s satisfying $Z(s) = X$. Of course, one is still left with the task of showing that λ_X is well-defined and verifies all of the properties given in Theorem 2.1.

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