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On non linear perturbations of isometries*

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1 Introduction

Let X and Y be real Banach spaces and $\varepsilon > 0$. Following Hyers and Ulam [12] a mapping $f: X \to Y$ is called an ε-isometry if $|\|f(x) - f(y)\| - \|x - y\|| \leqq \varepsilon$ whenever $x, y \in X$. The stability problem for isometries raised by Hyers and Ulam can be formulated as follows:

(a) Does there exist a constant $K(X, Y)$ depending only on X and Y such that for each surjective ε-isometry $f: X \to Y$ there exists an isometry $U: X \to Y$ with $\|f(x) - U(x)\| \leqq K(X, Y)\varepsilon$ for all $x \in X$?

(b) If the answer to the previous question is affirmative, then find the best possible value of $K(X, Y)$.

It was observed by Hyers and Ulam that the surjectivity assumption on f is essential [12]. Note that when studying ε-isometries there is no loss of generality in assuming that $f(0) = 0$. Indeed, if mapping f is an ε-isometry then the same must be true for $f - f(0)$ and $f - f(0)$ can be approximated by an isometry U if and only if f is close to isometry $U + f(0)$. So, it will be assumed throughout the paper that each ε-isometry sends the origin of X into the origin of Y.

The above problem may be viewed as a part of a more general stability problem: Assume that a mathematical object satisfies a certain property approximately according to some convention. Is it then possible to find near this object some objects satisfying the property accurately? A general discussion of this kind of problems can be found in interesting papers of Ulam [17] and Hyers [11].

The first positive answer towards the solution of problem (a) was given by Hyers and Ulam [12] who proved that $K(X, Y) \leqq 10$ in case $X = Y$ is

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a real Hilbert space. D.G. Bourgin [5] proved that $K(X, Y) \leqq 12$ whenever X and Y belong to the class of uniformly convex real Banach spaces. The next partial result was given by Hyers and Ulam [13] who showed that the distance of an ε-isometry f to the set of all surjective isometries is at most 21ε in the case that X and Y are Banach spaces of real continuous functions on compact Hausdorff spaces with the supremum norm, provided that f is a homeomorphism. D.G. Bourgin [6] proved that the same result holds without the continuity and injectivity assumptions. This study was continued by R.D. Bourgin [8] who investigated the finite dimensional case under the additional assumption that the set of extreme points of the unit ball in X is totally disconnected. The two-dimensional case was considered by D.G. Bourgin [7].

Gruber [10] went very far towards a general solution of problem (a) by showing that all surjective asymptotically isometric ε-isometries can be uniformly approximated by isometries. More precisely, he proved that if $f: X \to Y$ is a surjective ε-isometry and if $U: X \to Y$ is an isometry with $U(0) = 0$ for which $\|f(x) - U(x)\|/\|x\| \to 0$ uniformly as $\|x\| \to \infty$, then U is surjective and $\|f(x) - U(x)\| \leqq 5\varepsilon$ for all $x \in X$. In particular, if $K(X, Y)$ exists for a certain pair of spaces X and Y, then $K(X, Y) \leqq 5$. He also proved that this is true for the case that X and Y are finite dimensional. Problem (a) was finally solved by Gevirtz [9] who succeeded to show that every surjective ε-isometry asymptotically behaves like a linear surjective isometry, thus showing that there exists a universal constant (not depending on the special properties of X and Y) $K \leqq 5$ such that the distance of every surjective ε-isometry to the set of all surjective isometries is no greater than $K\varepsilon$.

We now conclude these historical remarks by mentioning a result of Lindenstrauss and Szankowski [15] who studied a wider concept of approximate isometries. For a given surjective mapping $f: X \to Y$ they defined a function

$$\varphi_f(t) = \sup\{|\|f(x) - f(y)\| - \|x - y\|| : \|x - y\| \leqq t \text{ or } \|f(x) - f(y)\| \leqq t\}$$

for $t \geqq 0$ and investigated the order of growth of $\varphi_f(t)$ as a function of t for which an asymptotical stability result can be obtained. They developed a sharp result on the allowable order of growth and this result does not depend on X and Y.

It is the aim of this paper to obtain a sharp stability result for ε-isometries, thus answering question (b). We will prove that for every surjective ε-isometry $f: X \to Y$ satisfying $f(0) = 0$ there exists a unique surjective linear isometry $U: X \to Y$ for which $\|f(x) - U(x)\| \leqq 2\varepsilon$, $x \in X$. This inequality is sharp even in the "nicest" spaces, e.g., in the case that $X = Y$ is an n-dimensional real Hilbert space, $n = 1, 2, \ldots$.

The proof of our result falls into two parts. In the first part we prove that every surjective ε-isometry behaves asymptotically like a linear surjective isometry. This fact was already known to Gevirtz [9] who introduced the notion of a δ-onto ε-isometry and then used some ideas similar to those of Vogt [18] who proved a generalization of the Mazur–Ulam theorem on isometries. However, using an idea of Lindenstrauss and Szankowski [15] one can approximate every surjective ε-isometry by a bijective approximate isometry. This makes it possible to avoid the use of δ-onto ε-isometries in the first step of the proof. However, the main idea remains the same as in [9], and we will therefore give just an outline of this part of the proof. The main contribution of the paper is the second part of the proof in which we modify the Gruber's [10] approach using some new geometric ideas in order to achieve the best estimate on the distance between surjective ε-isometries and isometries.

In [18] Vogt considered a similar problem, where mappings which preserve equality of distance were studied, instead of isometries. When we say for a mapping $f: X \to Y$ that it preserves the equality of distance, we mean, of course, that there exists a function $p_f: [0,\infty) \to [0,\infty)$ such that $\|f(x) - f(y)\| = p_f(\|x - y\|), x, y \in X$. The property of a map f that it preserves equality of distance may be characterized equivalently by the requirement that for all $x, y, u, v \in X$, relation $\|x - y\| = \|u - v\|$ implies $\|f(x) - f(y)\| = \|f(u) - f(v)\|$. If such a mapping is surjective and satisfies $f(0) = 0$, then it is a scalar multiple of a surjective linear isometry, provided that the dimension of X is greater than one. Vogt's ideas proved to be very useful in the study of the stability problem for isometries. In this paper we give a new proof of his result which is shorter than the original one.

The question that arises naturally in view of the result of Vogt is whether a weaker assumption than preserving equality of distance may imply that a surjective mapping is a scalar multiple of an isometry. Consider again a surjective mapping f from a real Banach space X onto a real Banach space Y and assume that $\|f(x) - f(y)\| = \|f(u) - f(v)\|$ whenever $x, y, u, v \in X$ and $\|x - y\| = \|u - v\| = t$ for a given positive real number t. The question is, of course, whether this assumption assures that f is an isometry. Introducing a new mapping $g: X \to Y$ defined by $g(x) = s(f(tx) - f(0))$, where s is a suitable real constant, one can see that there is no loss of generality in assuming that such mappings preserve distance one, that is, $\|x - y\| = 1$ implies $\|g(x) - g(y)\| = 1$. An interested reader can find results on such mappings in [1–4, 14, 16]. In general, such mappings need not be isometries.

When seeking for applications of our main result, the question of distance-preservers is a natural candidate. Recall that a mapping $f: X \to Y$ preserves a given distance t in both directions if for all $x, y \in X$ with $\|x - y\| = t$ it follows that $\|f(x) - f(y)\| = t$ and conversely. We shall give an example of a homeomorphism f of $C[0,1]$ onto itself which preserves distance n in both directions for any positive integer n and is not an isometry (cf. [16]). However, it turns out that such mappings are always approximate isometries [16]. More precisely, if X and Y are real normed spaces such that one of them has dimension greater than one and if $f: X \to Y$ is a surjective mapping preserving distance one in both directions, then

$$|\|f(x) - f(y)\| - \|x - y\|| < 1$$

for all $x, y \in X$. Once again, there is no loss of generality in assuming that $f(0) = 0$. According to our main result we can find then a surjective linear isometry $U: X \to Y$ such that $\|f(x) - U(x)\| \leqq 2$ for all $x \in X$. However, in this particular case we shall prove that the upper bound 2 can be replaced by 1 in this inequality and that the new inequality is sharp in this situation again.

2 Results and examples

Main Theorem. *Let X and Y be real Banach spaces. Suppose that $\varepsilon > 0$ and that $f: X \to Y$ is a surjective ε-isometry satisfying $f(0) = 0$. Then there exists a unique surjective linear isometry $U: X \to Y$ such that*

$$\|f(x) - U(x)\| \leqq 2\varepsilon \tag{1}$$

for every $x \in X$.

We will postpone all the proofs until the next section. Our goal now is to show that inequality (1) is sharp. Let us first give a counterexample for the case of Hilbert, or equivalently Banach, spaces $X = Y = \mathbb{R}$. Define a surjective function $f: \mathbb{R} \to \mathbb{R}$ by $f(t) = -3t$ for $t \in [0, 1/2]$ and $f(t) = t - 1$ elsewhere. Clearly, f is a 1-isometry and therefore, it can be approximated by a linear isometry $U: \mathbb{R} \to \mathbb{R}$. There are only two such mappings, namely, $U(t) = t$ and $U(t) = -t$. Obviously, the second one does not approximate f uniformly. One can easily verify that $\max_{t\in\mathbb{R}}|f(t) - t| = 2$, which proves that inequality (1) is sharp in the one-dimensional case.

Using this example it is not difficult to construct a surjective ε-isometry $f: \mathbb{R}^2 \to \mathbb{R}^2$ at which the upper bound 2ε in inequality (1) is attained. However, in higher dimensions we can find even a stronger counterexample. Namely, because of some possible applications of our results on approximate isometries (see [12]) it seems natural to study a smaller class of ε-isometries that are also homeomorphisms. It is clear that the inequality under consideration cannot be sharp in one dimension for homeomorphisms which have to be monotone mappings of the real line. However, we will now give a two-dimensional example showing that the sharpness is still true within this smaller set of mappings.

Let $X = Y = \mathbb{R}^2$ be a two-dimensional Hilbert space and define $X = A \cup B \cup C$ where A, B, and C are pairwise disjoint sets defined by

$$A = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| \leqq 2\},$$

$$B = \{(x, y) \in \mathbb{R}^2 : \|(x - 14, y)\| \leqq 2\},$$

$$C = \mathbb{R}^2 \backslash (A \cup B).$$

Observe that we can find a continuous function $u: \mathbb{R}^2 \to \mathbb{R}$ satisfying

(a) $u(0, 0) = 0$,

(b) $0 \leqq u(x, y) \leqq 1$ for all $(x, y) \in A$,

(c) $u(14,0) = 2$,

(d) $1 \leqq u(x,y) \leqq 2$ for all $(x,y) \in B$,

(e) $u(x,y) = 1$ for all $(x,y) \in C$,

(f) for every fixed $x \in \mathbb{R}$ the mapping $y \to y + u(x,y)$ is strictly increasing. For instance, we could set $u(x,y)$ to be equal to $(x^2+y^2)/4$ on A, then $2 - ((x-14)^2 + y^2)/4$ on B, and finally 1 on C in order to fulfill the wanted conditions. Now, define a mapping $f: X \to Y$ by

$$f(x,y) = (x, y + u(x,y)) .$$

It is easy to see that f is a 1-isometry. For any fixed $x \in \mathbb{R}$ the mapping $y \mapsto y + u(x,y)$ is continuous and bijective. Consequently, f is a homeomorphism. Since by (a) we have that $f(0,0) = (0,0)$ there exists according to our main result a surjective linear isometry $U: X \to Y$ such that

$$\|f(x,y) - U(x,y)\| \leqq 2 \tag{2}$$

for all $(x,y) \in \mathbb{R}^2$. We want to show that 2 cannot be replaced by a smaller constant. Insert (tx,ty) in (2) instead of (x,y) for some positive t, say, divide it by t and send t to infinity to conclude that

$$U(x,y) = \lim_{t\to\infty} \frac{f(tx,ty)}{t} = (x,y) .$$

Condition (c) now gives us the point at which the estimate is sharp: $\|f(14,0) - U(14,0)\| = 2$.

A similar construction produces a homeomorphic 1-isometry f with $f(0) = 0$ at which the upper bound in (1) is attained acting either on an n-dimensional real Hilbert space, $n \geqq 3$, or on an infinite-dimensional real Hilbert space.

The proof of our main result depends on the following lemma which may deserve to be stated separately. Observe that this result is an extension of [15, Lemma 3].

Lemma. *Let X and Y be real Banach spaces. Suppose that $\varepsilon > 0$ and that $f: X \to Y$ is a surjective ε-isometry. Assume that n is a positive integer and that vectors $x_1, \ldots, x_n \in X$ satisfy $f(x_i)$-$f(x_j)$ for i-j. Then for every positive real number η there exists a bijective mapping $g: X \to Y$ such that $\|f(x) - g(x)\| \leqq \eta$, $x \in X$, and $f(x_i) = g(x_i)$, $i = 1, \ldots, n$.*

As a corollary of our main result and its proof we will get item (c) of the following assertion.

Corollary. *Let X and Y be real Banach spaces such that one of them has dimension greater than one. Suppose that $f: X \to Y$ is a surjective mapping that preserves distance one in both directions and satisfies $f(0) = 0$. Then:*

(a) *f preserves distance n in both directions for all nonnegative integers n,*

(b) *f is a 1-isometry,*

(c) *there exists a surjective linear isometry $U: X \to Y$ such that $\|f(x) - U(x)\| \leqq 1$ for all $x \in X$.*

Note that condition (a) with $n = 0$ implies that f is injective. It was shown in [16] that the assumptions of this theorem imply (a) and (b). Applying our main result one can get the existence of a surjective linear isometry $U: X \to Y$ such that $\|f(x) - U(x)\| \leqq 2$ for all $x \in X$. Condition (c) is an improvement of this statement and we will have to go into particularities of the proof of our Main Theorem if we want to get it. At the end of this section we also give an example showing that inequality $\|f(x) - U(x)\| \leqq 1$ is sharp.

One of the milestones on the way to finding isometries uniformly approximating approximate isometries was the method that yield the following result due to Vogt [18] called here Proposition for the sake of easier distinction. Namely, some of his ideas were helping Gevirtz when he was solving the main problem. Although the reader may still find some parts of these ideas in the proof of our Main Theorem, we have made the two theorems completely independent of each other.

Proposition. [18] *Let X and Y be real Banach spaces, $\dim X \geqq 2$, and let $f: X \to Y$ with $f(0) = 0$ be a surjective mapping which preserves equality of distance. Then, $f = tU$ where t is a non-zero real number and U is an isometry of X onto Y.*

Let us now observe some connections between the results, presented in this section. The Proposition raises a natural question whether it is enough to suppose that a surjective mapping f preserves only one distance in order to get its conclusions. The answer is positive in some special cases [1, 2, 3, 4, 14, 16], while in general it is negative. In order to see this choose $\eta \in (0, 1/2)$ and define $g_\eta: [0,1] \to [0,1]$ by $g_\eta(t) = \eta(1-\eta)^{-1}t$ for $t \in [0, 1-\eta]$ and $g_\eta(t) = (1-\eta)\eta^{-1}t + 2 - \eta^{-1}$ elsewhere. Define also $h_\eta: \mathbb{R} \to \mathbb{R}$ by $h_\eta(t) = [t] + g_\eta(t - [t])$. Here, $[t]$ denotes the integer part of t. Let $C[0,1]$ be the Banach space of all real-valued continuous functions defined on [0,1] equipped with norm $\|x\| = \max_{t \in [0,1]} |x(t)|$. One can verify (see [16]) that a homeomorphism $f_\eta: C[0,1] \to C[0,1]$ defined by $(f_\eta(x))(t) = h_\eta(x(t))$ satisfies much stronger condition than preserving just one distance, namely, for every $x, y \in C[0,1]$ and each nonnegative integer n we have that $\|x - y\| = n$ if and only if $\|f_\eta(x) - f_\eta(y)\| = n$. In spite of that, f_η is clearly not an isometry. On the other hand, it is easy to see that it is an approximate isometry. Actually, this fact follows from our Corollary. Fortunately, the same example also provides the sharpness of inequality (c) of the Corollary which we have promised to give. Namely, let $f_\eta: C[0,1] \to C[0,1]$ be defined as above and let U be a linear isometry of $C[0,1]$ onto itself such that $\|f_\eta(x) - U(x)\|$ is uniformly bounded. It is easy to see that U must be the identity operator on $C[0,1]$. Choose $x(t) \equiv 1 - \eta \in C[0,1]$. Then $\|f_\eta(x) - x\| = |\eta - 1 + \eta| = 1 - 2\eta$, and since η can be arbitrary small, the upper bound 1 in condition (c) of the Corollary is the best possible.

3 Proofs

The proof of the Lemma is a modification of the proof of [15, Lemma 3]. We will therefore only give the part of our proof that differs from the other one.

Proof of the Lemma. The proof that $\operatorname{card} X = \operatorname{card} Y$ goes exactly as in the quoted proof. Now, let $Y = Y_1 \cup \ldots \cup Y_n \cup (\cup Y_\alpha)$, where the sets $Y_1, \ldots, Y_n, Y_\alpha$ are pairwise disjoint and such that: (a) the diameter of each of them is no greater than the given positive real number η, (b) their cardinalities are equal to the cardinality of Y, and (c) $f(x_i) \in Y_i, i = 1, \ldots, n$. Define $X_i = f^{-1}(Y_i)$ and $X_\alpha = f^{-1}(Y_\alpha)$ to get from (b) that $\operatorname{card} X_i = \operatorname{card} Y_i, i = 1, \ldots, n$, and $\operatorname{card} X_\alpha = \operatorname{card} Y_\alpha$ for all α. Consequently, we can find bijective mappings g_i from X_i onto Y_i and g_α from X_α onto Y_α such that $g_i(x_i) = f(x_i)$. Conditions (a) and (c) now yield the desired properties for mapping $g: X \to Y$ defined by $g(x) = g_i(x)$ for $x \in X_i$ and $g(x) = g_\alpha(x)$ for $x \in X_\alpha$.

Proof of the Main Theorem. Choose $x, y \in X$ and assume for now that vectors $f(x), f(y)$, and $f((x+y)/2)$ are different from each other. Next, fix a positive real number δ greater than ε and find, using the Lemma, a bijective mapping $g: X \to Y$ such that $g(x) = f(x)$, $g(y) = f(y)$, $g((x+y)/2) = f((x+y)/2)$, and $\|f(u) - g(u)\| \leqq (\delta - \varepsilon)/2$ for all $u \in X$. Clearly, g is a δ-isometry.

Let us introduce:

(a) a sequence of bijective mappings h_k: $Y \to Y$, defined inductively by

$$h_0(u) = g(x + y - g^{-1}(u)), \quad h_1(u) = g(x) + g(y) - u, \quad \text{and}$$
$$h_n = h_{n-2} h_{n-1} h_{n-2}^{-1} \quad \text{for all } u \in Y \text{ for } n \geqq 2 \, ;$$

(b) a sequence of bijective mappings $k_n = h_n h_{n-1} \ldots h_0$, for $n \geqq 0$; and

(c) a sequence of vectors $a_n \in Y$, defined inductively by

$$a_1 = \frac{f(x) + f(y)}{2} \quad \text{and} \quad a_{n+1} = h_{n-1}(a_n) = k_{n-1}(a_1) \quad \text{for } n \geqq 1 \, .$$

Following Gevirtz's ideas [9] one can prove that there exist sequences (p_n), $(q_n), (r_n)$ and (s_n) of nonnegative real numbers for $n \geqq 0$ such that:

(A) h_n is a $(p_n \delta)$-isometry for $n \geqq 0$;

(B) k_n is a $(q_n \delta)$-isometry for $n \geqq 0$;

(C) $\|h_n(u) - u\| \geqq 2\|a_n - u\| - r_n \delta$ for all $u \in Y$ and all positive integers n; and

(D) $$\|a_2 - a_1\| \leqq \frac{1}{2^{n-2}} \|a_n - a_{n-1}\| + s_n \delta, \quad \text{for } n \geqq 2 \, .$$

As in [9] one can use (A), (B), (C) and (D) to prove that

(E) $$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x) + f(y)}{2} \right\| \leqq \frac{\|x - y\|}{2^{n-1}} + (n+2)\varepsilon$$

holds for all positive integers n. It is easy to see that (E) is valid also when vectors $f(x), f(y)$, and $f((x+y)/2)$ are not different from each other.

Inequality (E) forces that

$$U(x) = \lim_{p\to\infty} \frac{f(2^p x)}{2^p}$$

exists for all $x \in X$ and that U is a linear isometry. It also implies that

$$\|f(x) - U(x)\| \leqq 2(r+3)\varepsilon + 2^{-r+3}\|x\| \tag{3}$$

for all $x \in X$ and all positive integers r.

Let us now prove that U is surjective. Suppose to the contrary that the range of U is a proper subspace in Y. As it is closed, there exists a vector $z \in Y$ of norm one such that its distance to the range of U is greater than 1/2. The surjectivity of f implies existence of a vector $x_t \in X$ such that $f(x_t) = tz$ for any positive real number t. Recall again that mapping f is an ε-isometry with $f(0) = 0$ and observe that $|\|f(x_t)\| - \|x_t\|| \leqq \varepsilon$ yields $t - \varepsilon \leqq \|x_t\| \leqq t + \varepsilon$. Thus, on one hand

$$\|f(x_t) - U(x_t)\| = t\|z - t^{-1}U(x_t)\| > \frac{t}{2}$$

for every positive real number t, while on the other one, (3) implies that $\|f(x_t) - U(x_t)\| \leqq 2(r+3)\varepsilon + 2^{-r+3}\|x_t\|$ for every positive real number t. Choose here first $t = r^2$ and then send r to infinity to get a contradiction with the above displayed inequality thus proving that U is surjective.

Let us define a mapping $T\colon X \to X$ by $T = U^{-1}f$. Clearly, T is a surjective ε-isometry with $T(0) = 0$. It follows that

$$|\|T(x)\| - \|x\|| \leqq \varepsilon \tag{4}$$

for all $x \in X$. Relation (3) yields

$$\|T(x) - x\| \leqq 2(r+3)\varepsilon + 2^{-r+3}\|x\| \tag{5}$$

for all $x \in X$ and positive integers r. The second part of the proof will be devoted to verification of the estimate

$$\|T(x) - x\| \leqq 2\varepsilon \tag{F}$$

for all $x \in X$. The proof of existence will then follow by a simple observation that U then satisfies $\|f(x) - U(x)\| \leqq 2\varepsilon$ for all $x \in X$. Note that if we interchange here U by any linear isometry V, the estimate produces inequality

$$\left\|\frac{f(2^m x)}{2^m} - V(x)\right\| \leqq \frac{\varepsilon}{2^{m-1}}$$

which yields $U = V$ in the limit by the definition of U, thus proving its uniqueness and completing the proof of the Main Theorem.

To prove (F) consider any $x \in X$ satisfying $\|T(x) - x\| = a > 0$ and set $y = a^{-1}(T(x) - x)$. Clearly, we have $\|y\| = 1$. As T is surjective, we can find for every positive integer n a vector $z_n \in X$ such that $T(x + z_n) = x + ay + ny$. From $n = \|T(x + z_n) - T(x)\|$ and the fact that T is an ε-isometry we conclude that

$$n - \varepsilon \leqq \|z_n\| \leqq n + \varepsilon \tag{6}$$

holds for all positive integers n. Interchange vector x in (5) by $x + z_n$ and divide the so obtained inequality by n to obtain

$$\|n^{-1}ay + y - n^{-1}z_n\| \leqq n^{-1}[2(r+3)\varepsilon + 2^{-r+3}(\|x\| + n + \varepsilon)]$$

Letting r equal to the integer part of the square root of n, sending n to infinity and using (6) gives

$$\lim_{n \to \infty} \frac{z_n}{\|z_n\|} = y\,. \tag{7}$$

Introduce $\lambda = \limsup_{n \to \infty}(\|x + z_n\| - \|z_n\|)$ and let q be a real number. It follows from (6) that $1 - q\|z_n\|^{-1}$ is positive if n is large enough. For such integers n it holds that

$$\begin{aligned}\|x + z_n\| - \|z_n\| + q = \|(x + q\|z_n\|^{-1}z_n) + (1 - q\|z_n\|^{-1})z_n\| \\ - \|(1 - q\|z_n\|^{-1})z_n\| \leqq \|x + q\|z_n\|^{-1}z_n\|\,.\end{aligned}$$

This implies together with (7) that

$$\lambda + q \leqq \|x + qy\|$$

for all $q \in \mathbb{R}$. Similarly,

$$\lambda + q \geqq \limsup_{n \to \infty}(-\|x + q\|z_n\|^{-1}z_n\|) = -\|x + qy\|, \quad q \in \mathbb{R}\,.$$

Consequently, we have that

$$|\lambda + q| \leqq \|x + qy\|, \quad q \in \mathbb{R}\,. \tag{8}$$

In the rest of the proof of (F) we will have to consider two special cases. Let us first treat the situation when x and y are linearly independent. Define a functional φ on the linear span of vectors x, y by $\varphi(x) = \lambda$ and $\varphi(y) = 1$. Inequality (8) implies that $\|\varphi\| = 1$. Inserting $x + z_n$ instead of x into (4) gives

$$\varepsilon \geqq \|x + ay + ny\| - \|x + z_n\| \geqq \varphi(x) + n + a - \|x + z_n\|$$

for every positive integer n. Applying (6) we see that

$$\varepsilon \geqq \lambda - (\|x + z_n\| - \|z_n\|) + a - \varepsilon$$

which further implies $2\varepsilon - a \geqq 0$. This completes the proof in case that x and y are linearly independent.

Finally, assume that $x = \mu y$ for some real number μ. It follows from (8) that $|\lambda + q| \leqq |\mu + q|$ for an arbitrary real q. Substituting $q = -\mu$ we see that

$\lambda = \mu$. If a positive integer n is large enough, then $\|x + ay + ny\| = \lambda + a + n$. For such integers n we have

$$\varepsilon \geqq \|x + ay + ny\| - \|x + z_n\| = a + \lambda - (\|x + z_n\| - \|z_n\|) + n - \|z_n\|$$

which yields (F) in this case as well. This completes the proof.

Proof of the Corollary. Assertions (a) and (b) were proved in [16]. According to our Main Theorem there exists a surjective linear isometry $U: X \to Y$ such that $\|f(x) - U(x)\| \leqq 2$ for all $x \in X$. In the same way as in the proof of the Main Theorem we define a mapping $T: X \to X$, a sequence of vectors z_n and a real number λ. But now, we have an additional property of T, that is, it preserves distance n in both directions for every nonnegative integer n. So, it follows from $n = \|T(x + z_n) - T(x)\|$ that $\|z_n\| = n$ for all positive integers n. Similar reasoning as in the proof of the Main Theorem now leads us to $\|T(x) - x\| \leqq 1$ for all $x \in X$. Consequently, inequality $\|f(x) - U(x)\| \leqq 1$ holds true for all $x \in X$ as we wanted to show.

Proof of the Proposition. Let us start by formulating a property of a Banach space X having dimension no smaller than two that we will need a few times in the course of the proof: If x and y belong to such a space and if $\|x - y\| \leqq na$ for some integer $n \geqq 2$ and some positive real number a, then there exists a sequence of vectors $x = x_0, x_1, \ldots, x_n = y$ such that $\|x_i - x_{i-1}\| = a$ for $i = 1, \ldots, n$. In case $n = 2$ this is an easy consequence of the fact that in such a space X any sphere is connected and the general case follows by induction.

According to our assumptions there exists a function $p: [0, \infty) \to [0, \infty)$ such that $\|f(x) - f(y)\| = p(\|x - y\|), x, y \in X$. We have to show that $p(t) = tp(1)$ for all nonnegative real numbers t. First, we will show that f is uniformly continuous. Given $\varepsilon > 0$ we can choose $x, y \in X$ such that $x \neq y$ and $\|f(x) - f(y)\| < \varepsilon/2$. Set $\delta = 2\|x - y\|$, choose a pair of vectors $v, w \in X$ satisfying $\|v - w\| < \delta$, and find $z \in X$ with $\|v - z\| = \|w - z\| = \delta/2$ which gives $\|f(v) - f(w)\| \leqq \|f(v) - f(z)\| + \|f(z) - f(w)\| = 2p(\|x - y\|) < \varepsilon$.

Next, we will show that

$$k(a) = \inf\{p(t): t \geqq a\} > 0 \tag{9}$$

for every positive real number a. Assume to the contrary that there exists $a > 0$ with $k(a) = 0$. Let x be any vector from X and choose an integer $n > a^{-1}\|x\|$. Because $k(a) = 0$ we can find $t_0 \geqq a$ with $p(t_0) < n^{-1}$. We have $nt_0 \geqq na \geqq \|x\|$, and, therefore, we can find a sequence of vectors $0 = x_0, x_1, \ldots, x_n = x$ such that $\|x_i - x_{i-1}\| = t_0, i = 1, \ldots, n$. On account of these considerations we must have $\|f(x)\| = \|f(x) - f(0)\| \leqq np(t_0) < 1$, contradicting the surjectivity assumption on f.

Property (9) implies that f is injective and, thus, bijective. It also yields uniform continuity of f^{-1}. Indeed, let $\varepsilon > 0$ and let $x, y \in Y$ satisfy $\|x - y\| < k(\varepsilon)$. Observation $\|x - y\| = \|f(f^{-1}(x)) - f(f^{-1}(y))\| = p(\|f^{-1}(x) - f^{-1}(y)\|)$ gives $\|f^{-1}(x) - f^{-1}(y)\| < \varepsilon$ and this implies that f is a homeomorphism. This means, in particular, that $\dim Y \geqq 2$.

In our next step we will prove that p is a strictly increasing function. For any $t > 0$ set $S_t = \{x \in X: \|x\| = t\}$ and $Z_{p(t)} = \{y \in Y: \|y\| = p(t)\}$. Obviously, we have $f(S_t) \subset Z_{p(t)}$. The complement of S_t in X consists of two connected components for every $t > 0$. The bounded component contains 0. Since f is a homeomorphism satisfying $f(0) = 0$ we have necessarily that $f(S_t) = Z_{p(t)}$ and that p is strictly increasing.

We claim that $p(nt) = np(t)$ for all positive integers n and positive real numbers t. Let us prove it by induction on n. There is nothing to prove in case $n = 1$. So, assume that $p(nt) = np(t)$ and choose $x \in X$ with $\|x\| = (n+1)t$. Then

$$\|f(x)\| \leqq \sum_{k=1}^{n+1} \|f(k(n+1)^{-1}x) - f((k-1)(n+1)^{-1}x)\| = (n+1)p(t)\,. \quad (10)$$

Denote by v the vector in X satisfying $f(v) = u = (1 - p(t)\|f(x)\|^{-1})f(x)$. Note that $\|u - f(x)\| = p(t)$ and therefore $\|v - x\| = t$. In order to show that $\|u\| \geqq np(t)$, suppose the contrary: $\|u\| < np(t) = p(nt)$ and observe that this would imply $\|v\| < nt$ which would yield $\|x\| \leqq \|v\| + \|v - x\| < (n+1)t$ contradicting the above. Hence, we have

$$np(t) \leqq \|u\| = |\|f(x)\| - p(t)|$$

which implies together with (10) that $\|f(x)\| = (n+1)p(t)$, and finally, $p((n+1)t) = (n+1)p(t)$.

One can now easily see that $p(n^{-1}t) = n^{-1}p(t)$ and this means that $p(rt) = rp(t)$ for all positive rational numbers r and positive real numbers t. Applying the fact that p is increasing we finally get the desired relation $p(t) = tp(1)$ for all $t \geqq 0$.

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